Circular coloring and graph homomorphism

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Abstract

For any pair of integers p, q such that $(p, q) = 1$ and $p \geq 2q$, the graph G_p^q has vertices $\{0, 1, \cdots, p-1\}$ and edges $\{ij : q \leq |i-j| \leq p-q\}$. These graphs play the same role in the study of circular chromatic number as that played by the complete graphs in the study of chromatic number. The graphs G_p^q share many properties of the complete graphs. However, there are also striking difference between the graphs G_p^q and the complete graphs. We shall prove in this paper that for many pairs of integers p, q, one may delete most of the edges of G_p^q so that the resulting graph still has circular chromatic number p/q. To be precise, we shall prove that for any number $r > 2$, there exists a rational number p/q (where $(p, q) = 1$) which is less than r but arbitrarily close to r, and that G_p^q contains a subgraph H such that $\chi_c(H) = \chi_c(G_p^q) = p/q$ and that $|E(H)| = O(\sqrt{|E(G_p^q)|})$. This is in sharp contrast to the fact that the complete graphs are edge critical, i.e., the deletion of any edge will decrease its chromatic number and its circular chromatic number.

1 Introduction

All graphs considered in this paper are finite and simple. Suppose G and H are graphs. A *homomorphism* from G to H is a mapping f from $V(G)$ to $V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. Homomorphism of graphs are studied as a generalization of graph colorings. Indeed, a vertex

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coloring of a graph G with n-colors is equivalent to a homomorphism from G to K_n. We write $G \preceq H$ if there exists a homomorphism from G to H. Then $\preceq$ defines a partial order on the set of all finite graphs, which we denote by $(\mathcal{F}, \preceq)$. Two graphs G and H are *hom-equivalent*, written as $G \sim H$, if $G \preceq H$ and $H \preceq G$. Obviously $\sim$ defines an equivalence relation on the set $\mathcal{F}$. We shall denote by $[G]$ the equivalent class of $\mathcal{F}/\sim$ that contains the graph G.

The structure of the partial order $(\mathcal{F}/\sim, \preceq)$ has been extensively studied, [4, 7, 9, 11, 12]. For example, there are many density results concerning this partial order [7, 9, 11, 12], and it is known that $(\mathcal{F}/\sim, \preceq)$ forms a distributive lattice [4], etc. The study of the chromatic number of graphs can also be viewed as an investigation of the structure of this partial order $(\mathcal{F}/\sim, \preceq)$. We shall denote by $\mathcal{Z}_\mathcal{G}$ the set of complete graphs, i.e., $\mathcal{Z}_\mathcal{G} = \{K_1, K_2, \cdots, \}$. Then $\mathcal{Z}_\mathcal{G}$ forms an infinite increasing chain in $(\mathcal{F}/\sim, \preceq)$, which maybe viewed as a representation of the natural numbers. Any graph $G \in \mathcal{F}$ admits a homomorphism to some member of the set $\mathcal{Z}_\mathcal{G}$. The chromatic number $\chi(G)$ is the minimum n such that $G \preceq K_n$, i.e., the least element of the set $\mathcal{Z}_\mathcal{G}$ which is "above" G in the order $\preceq$ (as we use K_n to represent the integer n). In this sense, we may view the set $\mathcal{Z}_\mathcal{G}$ as a scale that measures a dimension of graphs.

Just as the set of natural numbers are extended to the set of positive rationals, we can "extend" the set $\mathcal{Z}_\mathcal{G}$ into a larger set. For those fractions p/q with $(p, q) = 1$ and $p \geq 2q$, we construct a graph G_p^q, which has vertices $\{0, 1, \cdots, p-1\}$ and edges $\{ij : d \leq |i-j| \leq p-d\}$. We shall denote by $\mathcal{Q}_\mathcal{G}$ the set $\{G_p^q : (p, q) = 1 \text{ and } p \geq 2d\} \cup \{K_1\}$. Note that $G_p^1 = K_p$, and hence $\mathcal{Q}_\mathcal{G}$ is indeed an extension of $\mathcal{Z}_\mathcal{G}$. Moreover, the set $\mathcal{Q}_\mathcal{G}$ is also linearly ordered. It was shown in [3, 10] that if $p'/q' \geq 2$ and $p/q \geq 2$, then $p'/q' \leq p/q$ if and only if $G_{p'}^{q'} \preceq G_p^q$. Thus the set $\mathcal{Q}_\mathcal{G}$ together with the order $\preceq$ may be viewed as a representation of those rationals $r \geq 2$ or $r = 1$. The *circular chromatic number* $\chi_c(G)$ of a graph is the infimum of the ratios p/q for which $G \preceq G_p^q$. It was shown in [10] that the infimum in this definition is always attained, and hence the infimum can be replaced by minimum. Therefore $\chi_c(G)$ is the least member of $\mathcal{Q}_\mathcal{G}$ which is above G in the order $\preceq$ (as we use G_p^q to represent the rational p/q).

If the set $\mathcal{Z}_\mathcal{G}$ is considered as a scale that measures a dimension of graphs, then the set $\mathcal{Q}_\mathcal{G}$ is a refinement of that scale, just as the set of rational numbers provides a finer scale that measures the length of an object than that of integers. In this sense, the chromatic number $\chi(G)$ of a graph G maybe regarded as an approximation of its circular chromatic number $\chi_c(G)$. The circular chromatic number of a graph was introduced by Vince [10] in 1988 under the name "the star chromatic number". The concept has enjoyed considerable attention [1, 3, 5, 8, 10, 13, 14, 15, 17, 18].

The graphs G_p^q share many properties of the complete graphs K_n. We say a graph G is *vertex critical* if for any vertex x of G, we have $\chi_c(G-x) < \chi_c(G)$. We say a graph G is *edge critical* if for any edge e of G, we have $\chi_c(G-e) < \chi_c(G)$. It was shown in [3, 10] that for any pair of integers p, q with $p \geq 2q$ and $(p, q) = 1$, the graph G_p^q is vertex critical. It follows that the graphs G_p^q, just like the complete graphs, have the least number of vertices among all the graphs G with $\chi_c(G) = p/q$. However, there are also striking differences between the graphs G_p^q and the complete graphs. It was noted in [13] that for some pairs of integers (p, q), the graphs G_p^q are not edge critical, while the complete graphs are obviously edge critical. We shall explore this observation further in this paper. We shall determine exactly which graphs G_p^q are edge critical, and we shall estimate the number of edges that can be deleted from a graph G_p^q so that the resulting graph still has the same circular chromatic number. Surprisingly, we shall prove that for many pairs of integers (p, q), we can delete most of the edges of G_p^q without decreasing its circular chromatic number. To be precise, we shall prove the following theorems:

Theorem 1.1 *Suppose $p \geq 2q$ and $(p, q) = 1$. The graph G_p^q is edge critical if and only if either $q = 1$ or $p = 2q + 1$.*

Theorem 1.2 *For any number $r > 2$, there is a sequence of rational numbers p_i/q_i (where $(p_i, q_i) = 1$) such that the following is true:*

- *$p_i/q_i < r$ and $\lim_{i\to\infty} p_i/q_i = r$;*
- *each of the graphs $G_{p_i}^{q_i}$ contains a subgraph H_i with $\chi_c(H_i) = \chi_c(G_{p_i}^{q_i}) = p_i/q_i$ and that $|E(H_i)| = O(\sqrt{|E(G_{p_i}^{q_i})|})$.*

2 The construction

We note that if $q = 1$ then $G_p^q = K_p$ is obviously edge critical. If $p = 2q + 1$, then $G_p^q = C_p$ is an odd cycle, which is also obviously edge critical. This proves one direction of Theorem 1.1. The rest of this paper is devoted to the proof the other direction of Theorem 1.1 and the proof of Theorem 1.2. We shall first present a systematic way of constructing, for any pair of integers $p \geq 2q$ and $(p, q) = 1$, a subgraph $M(p, q)$ of G_p^q which has the same circular chromatic number as G_p^q. Once the construction is finished, the proofs of Theorems 1.1 and 1.2 are just a comparison of the number of edges of G_p^q and $M(p, q)$. The method presented here is a generalization of the method used in [6] and [17] to construct planar graphs with circular chromatic number r for any rational $2 \leq r \leq 4$. In case $q = 1$, then the graph $M(p, q) = G_p^q = K_p$. In the following we only need to consider the case that p/q is not an integer.

For the remaining part of this section, we assume that $p/q > 2$ is a fixed rational number and that $(p,q) = 1$. Let $m = \lfloor p/q \rfloor$. Thus p/q is strictly between m and $m+1$. To construct the graph $M(p,q)$, we need to construct two auxiliary sequences of numbers: the Farey sequence and the alpha sequence, which are determined by the number p/q.

Let p', q' be the unique positive integers such that $p' < p, q' < q$ and $pq' - qp' = 1$. It is straightforward to verify that $p'/q' < p/q$ and that p'/q' is the largest fraction with the property that $p'/q' < p/q$ and $p' \leq p$. Similarly, we let p'', q'' be positive integers such that $p'' < p', q'' < q'$ and $p'q'' - p''q' = 1$. Then p''/q'' is the largest fraction with the property that $p''/q'' < p'/q'$ and that $p'' \leq p'$. Repeating this process of finding smaller and smaller fractions, we shall stop at the fraction $m/1$ in a finite number of steps. Thus we obtain a unique sequence of fractions

$$\frac{m}{1} = \frac{p_0}{q_0} < \frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n} = \frac{p}{q}.$$

We call the sequence $(p_i/q_i : i = 0, 1, \cdots, n)$ the *Farey sequence* of p/q.

For convenience, we let $p_{-1} = -1$ and $q_{-1} = 0$. As $p_i q_{i-1} - p_{i-1} q_i = 1$ and $p_{i-1} q_{i-2} - p_{i-2} q_{i-1} = 1$, it follows that $p_{i-1}(q_i + q_{i-2}) = q_{i-1}(p_i + p_{i-2})$ for $i \geq 1$. As p_{i-1}, q_{i-1} are co-prime,

$$\alpha_i = \frac{p_i + p_{i-2}}{p_{i-1}} = \frac{q_i + q_{i-2}}{q_{i-1}}$$

is an integer (for $i \geq 1$), which is greater than 1, and hence is at least 2. We call $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ the *alpha sequence* of p/q, which is obviously uniquely determined by p/q. The process of deducing the alpha sequence from the rational p/q can also be reversed. In other words, given the integer m, each sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\alpha_i \geq 2$ determines a rational p/q between m and $m+1$. Indeed, given the alpha sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, the fractions p_i/q_i can be easily determined by solving the difference equations

$$p_i = \alpha_i p_{i-1} - p_{i-2}, \quad q_i = \alpha_i q_{i-1} - q_{i-2}, \qquad (*)$$

with the initial condition that $(p_{-1}, q_{-1}) = (-1, 0)$ and $(p_0, q_0) = (m, 1)$.

Having determined the alpha sequence

$$(\alpha_1, \alpha_2, \cdots, \alpha_n)$$

and the Farey sequence

$$\frac{m}{1} = \frac{p_0}{q_0} < \frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n} = \frac{p}{q}$$

of p/q, we can start the construction of the graph $M(p,q)$.

We shall construct a sequence of graphs G_i such that $\chi_c(G_i) = p_i/q_i$. Then $M(p,q) = G_n$. Before constructing the graphs G_i, we shall recursively construct ordered graphs F_i, H_i, i.e., the vertices of F_i and H_i are linearly ordered. Let $f_i = |F_i|$ and $h_i = |H_i|$, then the vertices of F_i will be denoted by $(x_{i,1}, x_{i,2}, \cdots, x_{i,f_i})$ in that order, and the vertices of H_i will be denoted by $(y_{i,1}, y_{i,2}, \cdots, y_{i,h_i})$ in that order. For an edge $e = (x, y)$ of an ordered graph, we define the *order length* of e, denoted by $\ell(e)$, to be the positive difference between the positions of x and y.

Definition 2.1 *Suppose X and Y are disjoint ordered graphs whose vertex orderings are $(x_1, x_2, \cdots, x_s)$ and $(y_1, y_2, \cdots, y_t)$, respectively. When we say* hook X to Y, *it means to add the following edges between X and Y:*

$$x_1y_t, x_1y_{t-1}, \cdots, x_1y_{t-m+2}, x_sy_1, x_sy_2, x_sy_3, \cdots, x_sy_{m-1}.$$

The result of hooking a sequence $X_1, X_2, \cdots, X_\beta$ of ordered graphs is regarded as another ordered graph, the order of the vertices being: those of X_1 in order, followed by those of X_2 in order, etc.

For an integer t, we let Q_t be the $(m-1)$th power of the path of length $t-1$, i.e., Q_t has vertex set $\{v_1, v_2, \cdots, v_t\}$ in which two vertices v_i and v_j are adjacent if $|i-j| \le m-1$. The graph Q_t is considered as an ordered graph in the following, where the order of the vertices is $(v_1, v_2, \cdots, v_t)$.

First of all, we let F_1 be a singleton, let $H_1 = Q_{m\alpha_1}$, and let $F_2 = Q_{m(\alpha_1-1)}$.

For $i \ge 1$, to construct the graph H_{i+1}, we take α_{i+1} copies of F_i, denoted by $F_i^1, F_i^2, \cdots, F_i^{\alpha_{i+1}}$, and $\alpha_{i+1}-1$ copies of H_i, denoted by $H_i^1, H_i^2, \cdots, H_i^{\alpha_{i+1}-1}$. If i is odd, then for $j = 1, 2, \cdots, \alpha_{i+1}-1$, we hook F_i^j and F_i^{j+1} to H_i^j. If i is even, then for $j = 1, 2, \cdots, \alpha_{i+1}-1$, we hook H_i^j to F_i^j and F_i^{j+1}. The resulting graph is H_{i+1}. The graph F_{i+2} is constructed in the same way as the graph H_{i+1}, but with one less copy of F_i and H_i, i.e., F_{i+2} is constructed from $\alpha_{i+1}-1$ copies of F_i and $\alpha_{i+1}-2$ copies of H_i, by appropriately hooking them together.

Finally when i is even, we let G_i be the graph obtained by hooking H_i to F_i; when i is odd, we let G_i be the graph obtained by hooking F_i to H_i.

This finishes the construction of the graphs G_i. The graph $M(p,q)$ is equal to G_n. We note that when $m = 2$, the graphs constructed here is the same as constructed by Moser in [6].

3 The circular chromatic number

In this section, we shall prove that $M(p,q) = G_n$ is a subgraph of G_p^q and that $\chi_c(M(p,q)) = p/q$.

First we count the number of vertices of G_i.

Lemma 3.1 *The graph G_i has p_i vertices.*

Proof. From the construction of G_i, we know that G_i has $g_i = f_i + h_i$ vertices. From the construction of F_i, H_i, we know that

$$f_1 = 1, \quad f_2 = m\alpha_1 - m, \quad h_1 = m\alpha_1,$$

and for $i \geq 2$,

$$h_i = \alpha_i f_{i-1} + (\alpha_i - 1)h_{i-1},$$

for $i \geq 3$,

$$f_i = (\alpha_{i-1} - 1)f_{i-2} + (\alpha_{i-1} - 2)h_{i-2}.$$

Simple algebraic calculation shows that

$$h_i = \alpha_i g_{i-1} - h_{i-1}, \; f_i = (\alpha_{i-1} - 1)g_{i-2} - h_{i-2} = h_{i-1} - g_{i-2}.$$

Hence

$$g_i = \alpha_i g_{i-1} - g_{i-2}.$$

Since $g_1 = p_1, g_2 = p_2$, and g_i, p_i satisfy the same difference equation, we conclude that $|G_i| = g_i = p_i$. ∎

Next we shall show that $M(p, q)$ is a subgraph of G_p^q. For this purpose, we shall prove by induction that for each $i \leq n$, the graph G_i is a subgraph of $G_{p_i}^{q_i}$.

Before proving this, we need some preliminary results about the relation between the Farey sequence and the alpha sequence. We observed before that the Farey sequence is uniquely determined by the alpha sequence. The numbers p_i and q_i are obtained by solving the following difference equations:

$$p_i = \alpha_i p_{i-1} - p_{i-2}, \quad q_i = \alpha_i q_{i-1} - q_{i-2}, \qquad (*)$$

with the initial condition that $(p_{-1}, q_{-1}) = (-1, 0)$ and $(p_0, q_0) = (m, 1)$.

By repeatedly applying the equation $(*)$, we may express p_i (respectively q_i) in terms of p_j and p_{j-1} (respectively q_j and q_{j-1}) for any $0 \leq j \leq i - 2$. Lemma 3.2 below gives the explicit expressions.

For $1 \leq r \leq s \leq n$, we let

$$\Lambda_{r,s} = \det \begin{pmatrix} \alpha_r & 1 & 0 & \cdots & 0 & 0 \\ 1 & \alpha_{r+1} & 1 & \cdots & 0 & 0 \\ 0 & 1 & \alpha_{r+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{s-1} & 1 \\ 0 & 0 & 0 & \cdots & 1 & \alpha_s \end{pmatrix}$$

Lemma 3.2 *For $0 \le j \le i-2$, we have*

$$p_i = p_j \Lambda_{j+1,i} - p_{j-1}\Lambda_{j+2,i}, \quad q_i = q_j \Lambda_{j+1,i} - q_{j-1}\Lambda_{j+2,i} \quad (**)$$

Proof. It suffices to prove the first equality. We shall prove it by induction on i. When $i = j+2$, by applying $(*)$ twice, we obtain $(**)$. Suppose $i \ge j+3$, and that $(**)$ is true for any $i' < i$. Then by cofactor expansion,

$$\begin{aligned} p_j\Lambda_{j+1,i} - p_{j-1}\Lambda_{j+2,i} &= \alpha_i(p_j\Lambda_{j+1,i-1} - p_{j-1}\Lambda_{j+2,i-1}) \\ &\quad -(p_j\Lambda_{j+1,i-2} - p_{j-1}\Lambda_{j+2,i-2}) \\ &= \alpha_i p_{i-1} - p_{i-2} = p_i. \end{aligned}$$

The second equality uses the induction hypothesis. ∎

By letting $j = 0$ in $(**)$, and by using the initial condition, we have

$$p_i = m\Lambda_{1,i} + \Lambda_{2,i}, \quad q_i = \Lambda_{1,i}. \quad (***)$$

Lemma 3.3 *For $0 \le j \le i-2$, $p_j q_i = p_i q_j - \Lambda_{j+2,i}$.*

Proof. By applying Lemma 3.2, we have

$$\begin{aligned} p_i q_j - p_j q_i &= (p_j\Lambda_{j+1,i} - p_{j-1}\Lambda_{j+2,i})q_j - p_j(q_j\Lambda_{j+1,i} - q_{j-1}\Lambda_{j+2,i}) \\ &= \Lambda_{j+2,i}(p_j q_{j-1} - p_{j-1}q_j) \\ &= \Lambda_{j+2,i}. \end{aligned}$$

∎

Lemma 3.4 *For any $2 < t \le i$, $\Lambda_{t,i} < \Lambda_{t-1,i}$.* ∎

We omit the proof, which is an easy induction, by noting that $\alpha_j \ge 2$.

Let the vertices of $G_{p_i}^{q_i}$ be $\{0, 1, \cdots, p_i - 1\}$. We shall define a 1-1 mapping c from $V(G_i)$ to $V(G_{p_i}^{q_i})$ as follows:

We know that the vertex set of G_i is the union of the vertex set of F_i and the vertex set of H_i, while the vertice of F_i are $(x_{i,1}, x_{i,2}, \cdots, x_{i,f_i})$ and the vertice of H_i are $(y_{i,1}, y_{i,2}, \cdots, y_{i,h_i})$. We shall rename the vertices of G_i by letting $v_j = x_{i,j}$ for $j = 1, 2, \cdots, f_i$, and let $v_j = y_{i,j-f_i}$ for $j = f_i + 1, f_i + 2, \cdots, f_i + h_i (= p_i)$.

Let $c(v_j) = jq_i \pmod{p_i}$. As p_i and q_i are coprime, we know that c is a 1-1 mapping. Now we shall show that for any edge $(v_j, v_{j'})$ of G_i, $(c(v_j), c(v_{j'}))$ is an edge of $G_{p_i}^{q_i}$, i.e., $q_i \le |c(v_j) - c(v_{j'})| \le p_i - q_i$.

First we determine the order length of all edges of G_i. Recall that the order length $\ell(e)$ of an edge $e = (x, y)$ is the positive difference of the positions of x and y in the above ordering of the vertices of G_i, i.e., if $e = (v_i, v_j)$ then $\ell(e) = |i - j|$.

Lemma 3.5 *Let $L = \{p_j - s : 0 \leq j \leq i,\ 1 \leq s \leq m-1\}$. Then for any edge e of G_i, we have $\ell(e) \in L$.*

Proof. This is easily proved by induction. Each edge of H_1 or F_2 has order length $m - s = p_0 - s$ for some $1 \leq s \leq m-1$. Suppose the order length of edges in H_j and F_j are elements of L. When copies of F_j and H_j are hooked together to form H_{j+1} or F_{j+2}, then the order length of the edges in the copies of F_j and H_j are unchanged. For those edges e of the hooks, it is easy to verify that either $\ell(e) \in \{1, 2, \cdots, m-1\}$ or $\ell(e) \in \{p_j - 1, p_j - 2, \cdots, p_j - m + 1\}$. This completes the proof of Lemma 3.5. ∎

Lemma 3.6 *For each $i \leq n$, the graph G_i is a subgraph of $G^{q_i}_{p_i}$.*

Proof. It suffices to show that the mapping c defined above is edge preserving, i.e., for any edge (x, y) of G_i, $q_i \leq |c(x) - c(y)| \leq p_i - q_i$.

Suppose $e = (x, y)$ is an edge of G_i of order length $\ell(e)$. Then by the definition, we have $|c(x) - c(y)| = \ell(e)q_i \pmod{p_i}$. By Lemma 3.5, $\ell(e) \in L$.

By Lemma 3.3, for any $0 \leq j \leq i-2$, we have

$$p_j q_i = p_i q_j - \Lambda_{j+2,i}.$$

By Lemma 3.4, $(***)$ and the definition of $\Lambda_{r,s}$, and by noting that $p_0 = m, q_0 = 1$, we have

$$2 \leq \alpha_i = \Lambda_{i,i} \leq \Lambda_{j+2,i} \leq \Lambda_{2,i} = p_i - mq_i.$$

This implies that for any $0 \leq j \leq i-2$ and $1 \leq s \leq m-1$,

$$q_i = p_i - (m-1)q_i - \Lambda_{2,i} \leq p_i - sq_i - \Lambda_{j+2,i} \leq p_i - sq_i - 2 < p_i - q_i.$$

Now if $\ell(e) = p_j - s$ where $0 \leq j \leq i-2$ and $1 \leq s \leq m-1$, then by Lemma 3.3, we have

$$|c(x) - c(y)| \equiv (p_j - s)q_i \equiv p_i - sq_i - \Lambda_{j+2,i} \pmod{p_i},$$

and so $q_i \leq |c(x) - c(y)| \leq p_i - q_i$, as required.

If $\ell(e) = p_{i-1} - s$ for some $1 \leq s \leq m-1$, then since $p_i q_{i-1} - p_{i-1} q_i = 1$ by the definition of the Farey sequence, we also have

$$q_i \leq \ell(e)q_i \pmod{p_i} \leq p_i - q_i.$$

If $\ell(e) = p_i - s$ for some $1 \leq s \leq m-1$, then trivially we have

$$q_i \leq \ell(e)q_i \pmod{p_i} \leq p_i - q_i.$$

■

As $M(p,q) = G_n$, it follows that $M(p,q)$ is a subgraph of G_p^q. Next we shall prove that $\chi_c(M(p,q)) = p/q$. This is achieved by showing recursively that for each i, $\chi_c(G_i) = p_i/q_i$.

As G_i is a subgraph of $G_{p_i}^{q_i}$, it follows that $\chi_c(G_i) \leq p_i/q_i$. Thus it suffices to show that $\chi_c(G_i) \geq p_i/q_i$. First we need a few lemmas.

Lemma 3.7 below was proved in [5] and also implicitly used in [10, 13]. Suppose $k \geq 2d$ are integers and $(k,d) = 1$. We call a homomorphism of G to the graph G_k^d a (k,d)-coloring of G.

Given a (k,d)-coloring c of a graph G. We define a directed graph $D_c(G)$ on the vertex set of G by putting a directed edge from x to y if and only if (x,y) is an edge of G and that $c(x) - c(y) = d \pmod{k}$.

Lemma 3.7 *For any graph G, $\chi_c(G) = k/d$ if and only if G is (k,d)-colorable, and for any (k,d)-coloring c of G, the directed graph $D_c(G)$ contains a directed cycle.* ■

A simple calculation shows that the length of the directed cycle in $D_c(G)$ is a multiple of k, and hence is at least k.

Corollary 3.1 *For any graph G, if $\chi_c(G) = k/d$ where $(k,d) = 1$, then G has a cycle of length at least k. In particular $k \leq |V(G)|$.* ■

Suppose $\chi_c(G_i) = p_i/q_i$, and that Δ is an (p_i,q_i)-coloring of G_i. It follows from Lemma 3.7 that there is a directed cycle of $D_\Delta(G_i)$ of length at least p_i. Since $|G_i| = p_i$, we conclude that there is a Hamiltonian cycle, say $Q = (c_1, c_2, \cdots, c_{p_i}, c_1)$, of G_i such that $\Delta(c_j) - \Delta(c_{j-1}) = q_i \pmod{p_i}$.

We say a Hamiltonian cycle $Q = (c_1, c_2, \cdots, c_t, c_1)$ of the graph G is a *good Hamiltonian cycle* (with respect to p/q) if for any edge (c_k, c_s) of G we have $k - s \neq m, m+1$. Similarly a Hamiltonian path $P = (c_1, c_2, \cdots, c_t)$ of a graph G is a *good Hamiltonian path* if for any edge (c_k, c_ℓ) of G, we have $k - \ell \neq m, m+1$.

Now we shall show that if $\chi_c(G_i) = p_i/q_i$, then the Hamiltonian cycle induced by any (p_i,q_i)-coloring of G_i is a good Hamiltonian cycle.

Lemma 3.8 *Suppose $\chi_c(G_i) = p_i/q_i$ and that Δ is an (p_i,q_i)-coloring of G_i. Let $Q = (c_1, c_2, \cdots, c_{p_i}, c_1)$ be the Hamiltonian cycle of G_i such that $\Delta(c_j) - \Delta(c_{j-1}) = q_i \pmod{p_i}$. Then Q is a good Hamiltonian cycle of G_i.*

Proof. Assume to the contrary that there is an edge (c_k, c_s) of G_i such that $|k - s| = m$ or $m+1$. Then $|\Delta(c_k) - \Delta(c_s)| = mq_i \pmod{p_i}$ or $(m+1)q_i$

(mod p_i). However, $p_i - q_i < mq_i < p_i$ and $(m+1)q_i \pmod{p_i} < q_i$, because $m < p_i/q_i < m+1$. This is in contrary to the assumption that Δ is a (p_i, q_i)-coloring of G_i. ∎

Lemma 3.9 *If i is odd (resp. even), then for any good Hamiltonian path P of H_i (resp. F_i), the first $(m-1)$ vertices of P (as a set) is the first $(m-1)$ vertices of H_i (resp. F_i), and the last $(m-1)$ vertices of P (as a set) is the last $(m-1)$ vertices of H_i (resp. F_i). Here we may reverse the order of all the vertices of P, if needed.*

Proof. First we consider the graphs H_1 and F_2. Each of them is of the form Q_t for some positive integer t. We shall simply prove that for any positive integer t, the graph Q_t has a unique good Hamiltonian path, up to an isomorphism. When $m = 2$, then Q_t is simply a path, and there is nothing to be proved. When $t \leq m$, then Q_t is a complete graph, and there is also nothing to be proved. Assume now that $t \geq m+1$ and $m \geq 3$. Suppose the vertices of Q_t are $1, 2, \cdots, t$, where (x, y) is an edge if and only if $|x - y| \leq m - 1$. Let $P = (x_1, x_2, \cdots, x_t)$ be a good Hamiltonian path of Q_t. Then for any edge (x_i, x_j) of Q_t, we have $|i - j| \neq m, m+1$. This, in particular, implies that for any $i \leq t - m$, the pair (x_i, x_{i+m}) is not an edge of Q_t. In other words, for any $i \leq t - m$, $|x_i - x_{m+i}| \geq m$.

We shall assume that $x_1 < x_{m+1}$. The case that $x_1 > x_{m+1}$ can be treated similarly. (In that case, we need to reverse the order of all the vertices of P.) Since $|x_1 - x_{m+1}| \geq m$, we have $x_1 \leq x_{m+1} - m$. Because $x_2 \leq x_1 + m - 1$ and $x_{m+2} \geq x_{m+1} - m + 1$ (as (x_1, x_2) and (x_{m+1}, x_{m+2}) are edges of Q_t), we conclude that $x_2 \leq x_{m+2} + m - 2$. Since $|x_2 - x_{m+2}| \geq m$, we conclude that $x_2 \leq x_{m+2} - m$. Repeating this argument, we can prove that $x_i \leq x_{i+m} - m$ for all $i \leq t - m$. This implies that $\{x_1, x_2, \cdots, x_m\} = \{1, 2, \cdots, m\}$, for otherwise there would exist an $x \leq m$ and an $i \geq 1$ such that $x_{i+m} = x$ and hence $1 \leq x_i \leq x_{i+m} - m = x - m \leq 0$, an obvious contradiction.

Suppose $x_i = m+1$. Then $i \geq m+1$, by the previous paragraph. Since $x_{i-m} \leq x_i - m = 1$, we conclude that $x_{i-m} = 1$. If $i - m > 1$, then $2 \leq x_{i-m-1} \leq m$ and hence $|x_i - x_{i-m-1}| \leq m - 1$ and $x_{i-m-1}x_i$ is an edge of Q_t, contrary to the assumption that P is a good Hamiltonian path. Therefore we have $x_1 = 1$ and $x_{m+1} = m+1$.

Now we shall prove by induction that for all $1 \leq i \leq \min\{m, t-m\}$, we have $x_i = i$ and $x_{i+m} = i + m$. When $i = 1$, this has been proved in the previous paragraph. Assume that for any $1 \leq j < i$, we have $x_j = j$ and $x_{j+m} = j + m$. Let $i + m = x_k$. By the previous discussion and the induction hypotheses, we know that $k \geq i + m$. Since $x_{k-m} \leq x_k - m = i$, we conclude that $x_{k-m} = i$, because $x_{k-m} \neq j$ for any $j < i$ by the induction hypotheses. If $k > i + m$, then $i < x_{k-m-1} \leq m$ (by using the induction hypotheses). This implies that (x_{k-m-1}, x_k) is an edge of Q_t, contrary to the assumption that P is a good Hamiltonian path. Therefore we must have $x_i = i$ and $x_{i+m} = i+m$, for all $i \leq \min\{m, t-m\}$. If $t \geq 2m$, then we have $x_i = i$ for all $1 \leq i \leq t$. If $t = m+j$ for some $1 < j < m$, then $x_i = i$ except possibly where $j+1 \leq i \leq m$. But each vertex labelled $j+1, \cdots, m$ is adjacent to every vertex of Q_t except itself, so an arbitrary permutation of these vertices gives an automorphism of Q_t.

This finishes the proof that any good Hamiltonian path of H_1 (resp. F_2) have the same first $(m-1)$ vertices and the same $(m-1)$ last vertices as H_1 (resp. F_2).

Assume that the lemma is true for i. We shall show that it is true for $i+1$. First we consider the case that i is even. The graph H_{i+1} is obtained by approriately hooking copies of H_i to copies of F_i (cf. the construction in Section 2).

Since the first and the last vertex of each copy of H_i (in H_{i+1}) form a 2-vertex cut of H_{i+1}, we conclude that any good Hamiltonian path of H_{i+1} is the concatenation of good Hamiltonian paths of the copies of F_i and H_i. Therefore the first $(m-1)$ vertices of any good Hamiltonian path of H_{i+1} are the first $(m-1)$ vertices of a good Hamiltonian path of the first copy of F_i. By the induction hypothesis, these $(m-1)$ vertices are the first $(m-1)$ vertices of the first copy of F_i, which by definition are the first $(m-1)$ vertices of H_{i+1}. Similarly the last $(m-1)$ vertices of any good Hamiltonian path of H_{i+1} are the last $(m-1)$ vertices of H_{i+1}.

The case that i is odd can be treated similarly, and is omitted. ∎

Lemma 3.10 *Suppose $\chi_c(G_i) = p_i/q_i$ for some i. Let Δ be any (p_i, q_i)-coloring of G_i. If i is odd, then the colors of the first and last vertices of F_i uniquely determines the colors of the first and last $(m-1)$ vertices of H_i. Conversely, the colors of the first and last $(m-1)$ vertices of H_i uniquely determines the colors of the first and last vertices of F_i. If i is even, then the colors of the first and last vertices of H_i uniquely determines the colors of the first and last $(m-1)$ vertices of F_i. Conversely, the colors of the first and last $(m-1)$ vertices of F_i uniquely determines the colors of the first and last vertices of F_i.* ∎

Proof. We only consider the case that i is odd. Let Δ be a (p_i, q_i)-coloring of G_i. By Lemma 3.8, there is a good Hamiltonian cycle $Q = (c_1, c_2, \cdots, c_{p_i}, c_1)$ of G_i such that $\Delta(c_j) - \Delta(c_{j-1}) = q_i \pmod{p_i}$.

The graph G_i is obtained by hooking F_i to H_i. The first and the last vertex of F_i form a 2-vertex cut of G_i. Therefore the good Hamiltonian cycle Q is the union of a good Hamiltonian path P of H_i and a good Hamiltonian path P' of F_i. By Lemma 3.9, the first $(m-1)$ vertices of P are the first $(m-1)$ vertices of H_i, and the last $(m-1)$ vertices of P are the last $(m-1)$

vertices of H_i. As the first and the last vertex of F_i form a 2-vertex cut of G_i, the first and last vertex of P' must be the first and last vertex of F_i, respectively. Therefore the colors of the first and last vertex of F_i are uniquely determined by the colors of the first and last $(m-1)$ vertices of H_i, and that the first and last $(m-1)$ vertices of H_i are uniquely determined by the colors of the first and last vertices of F_i. ∎

To prove that $\chi_c(G_i) \geq p_i/q_i$ (and hence $\chi_c(G_i) = p_i/q_i$), we need another gadget. If $i \geq 2$ is even, let T_i be the graph obtained by hooking F_{i-1} to F_i. If $i \geq 2$ is odd, let T_i be the graph obtained by hooking F_i to F_{i-1}.

Theorem 3.1 *For each $i \geq 2$, $\chi_c(G_i) = p_i/q_i$ and $\chi_c(T_i) > p_{i-1}/q_{i-1}$. Moreover, $\chi_c(G_1) = p_1/q_1$.*

Proof. First we prove that $\chi_c(G_1) = p_1/q_1$. By Lemma 3.6, it suffices to show that $\chi_c(G_1) \geq p_1/q_1$. It is easy to verify that $\chi(G_1) = m+1$. Hence $\chi_c(G_1) > m$. Suppose $\chi_c(G_1) = k/d > m$, then $k \leq |V(G_1)| = p_1$ by Corollary 3.1. Therefore $k/d \geq p_1/q_1$, because it follows from the construction of the Farey sequence that any fraction a/b strictly between $m = p_0/q_0$ and p_1/q_1 must have numerator $a > p_1$.

Next we show that $\chi_c(T_2) > p_1/q_1$. Again it is easy to verify that $\chi(T_2) = m+1$. Suppose $\chi_c(T_2) = k/d > m$. As $|V(T_2)| < p_1$ (because $|V(F_2)| < |V(H_1)|$), we know that $k < p_1$. Therefore $k/d > p_1/q_1$, because by the construction of the Farey sequence, any fraction a/b strictly between m and p_1/q_1 has numerator $a > p_1$ (note that $k/d \neq p_1/q_1$).

Now assume that $i \geq 2$, $\chi_c(T_i) > p_{i-1}/q_{i-1}$ and that $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$. We shall prove that $\chi_c(G_i) = p_i/q_i$.

Assume to the contrary that $\chi_c(G_i) = k/d < p_i/q_i$ and $(k,d) = 1$. Then $k \leq p_i$ and hence $k/d \leq p_{i-1}/q_{i-1}$, because by the construction of the Farey sequence, any fraction a/b strictly between p_{i-1}/q_{i-1} and p_i/q_i has numerator $a > p_i$. Since $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$ and that G_{i-1} is a subgraph of G_i, it follows that $\chi_c(G_i) = p_{i-1}/q_{i-1}$.

Let Δ be a (p_{i-1}, q_{i-1})-coloring of G_i. The graph G_i is obtained by hooking H_i and F_i together, and H_i is constructed from α_i copies of F_{i-1} and $\alpha_i - 1$ copies of H_{i-1}. The union of the first copy of H_{i-1} the first copy of F_{i-1} induces a G_{i-1}. The union of the first copy of H_{i-1} and the second copy of F_{i-1} also induces a G_{i-1}.

Assume first that i is odd. By using the induction hypotheses that $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$, and by applying Lemma 3.10 to each of the two copies of G_{i-1}, we conclude that the last $(m-1)$ vertices of the first copy of F_{i-1} is colored the same way as the last $(m-1)$ vertices of the second copy of F_{i-1}. Similarly, the first and last vertices of the first copy of H_{i-1} are colored the same way as the first and last vertices of the second copy of H_{i-1}. Repeating the same argument, we conclude that the last $(m-1)$ vertices of the first copy F_{i-1} is colored the same way as the last $(m-1)$ vertices of the last copy of F_{i-1} of H_i.

This implies that the restriction of Δ to the union of F_i and the first copy of F_{i-1} in H_i is indeed a (p_{i-1}, q_{i-1})-coloring of T_i, contrary to our assumption that $\chi_c(T_i) > p_{i-1}/q_{i-1}$. The case that i is even can be treated similarly.

Finally, assuming that $i \geq 2$, $\chi_i(G_i) = p_i/q_i$ and that $\chi_c(T_i) > p_{i-1}/q_{i-1}$, we shall prove that $\chi_c(T_{i+1}) > p_i/q_i$.

Assume to the contrary that $\chi_c(T_{i+1}) = k/d \leq p_i/q_i$. Since $|F_{i+1}| < |H_i|$, hence $|T_{i+1}| < |G_i| = p_i$. It follows from Corollary 3.1 that $k < p_i$. As p_{i-1}/q_{i-1} is the largest fraction satisfying the property that $p_{i-1} < p_i$ and $p_{i-1}/q_{i-1} \leq p_i/q_i$, we conclude that $\chi_c(T_{i+1}) \leq p_{i-1}/q_{i-1}$.

We consider two cases:

Case 1: $\alpha_i = 2$. In this case $F_{i+1} = F_{i-1}$, and hence $T_{i+1} = T_i$. By induction hypothesis, $\chi_c(T_i) > p_{i-1}/q_{i-1}$.

Case 2: $\alpha_i > 2$. In this case F_{i+1} consists of $\alpha_i - 1$ copies of F_{i-1} and $\alpha_i - 2$ copies of H_{i-1}. The union of any copy of F_{i-1} and the consecutive copy of H_{i-1} induces a copy of G_{i-1}. Therefore we must have $\chi_c(T_{i+1}) = p_{i-1}/q_{i-1}$. Using the same argument as before (cf. the proof of the fact that $\chi_c(G_i) = p_i/q_i$), we conclude that for any (p_{i-1}, q_{i-1})-coloring Δ of T_{i+1}, the restriction of Δ to the union of F_i and the first copy of F_{i-1} in F_{i+1} is indeed a (p_{i-1}, q_{i-1})-coloring of T_i, contrary to our assumption that $\chi_c(T_i) > p_{i-1}/q_{i-1}$. This completes the proof of Theorem 3.1. ∎

4 Counting the number of edges

In this section, we shall prove Theorems 1.1 and 1.2, by counting the number edges of $M(p, q)$ for special values of p and q.

Proof of Theorem 1.1: It has been proved, at the beginning of Section 2, that if $q = 1$ or $p = 2q + 1$ then G_p^q is edge critical. It remains to show that if $q \neq 1$ and $p \neq 2q + 1$ then G_p^q is not edge critical. As $M(p, q)$ is a subgraph of G_p^q with $\chi_c(M(p, q)) = p/q$, it suffices to show that $M(p, q) \neq G_p^q$. This is obvious, because G_p^q is regular, but when $q \neq 1$ and $p \neq 2q + 1$, $M(p, q)$ is not regular. ∎

Proof of Theorem 1.2: First we consider the case that r is rational. Let $2 \leq m < r \leq m+1$ be any rational number. Let $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ be the alpha sequence of r. Note that when $r = m + 1$, then we let the alpha sequence be (1). This does not satisfy the definition of alpha sequence, however, all the argument below are still valid.

For each $i \geq 1$, let r_i be the rational number corresponds to the alpha sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n+1, 2, \cdots, 2)$, whose first $(n-1)$ entries coincide with the first $(n-1)$ entries of the alpha sequence of r, the nth entry is equal to 1 plus the nth entry of the alpha sequence of r, and the last i entries are equal to 2. In particular, when $r = m+1$, then the alpha sequence of r_i has $i+1$ entries, all equal to 2.

Let

$$m/1 = p_0/q_0 < p_1/q_1 < \cdots < p_{n-2}/q_{n-2} < p_{n-1}/q_{n-1} < p_n/q_n = r$$

be the Farey sequence of r, and let

$$m/1 = p'_0/q'_0 < p'_1/q'_1 < \cdots < p'_n/q'_n < p'_{n+1}/q'_{n+1} < \cdots < p'_{n+i}/q'_{n+i} = r_i$$

be the Farey sequence of r_i. Then $p'_j = p_j$ and $q'_j = q_j$ for $j = 0, 1, \cdots, n-1$. Moreover, by applying $(**)$, it is straightforward to verify that

$$p_n = \alpha_n p_{n-1} - p_{n-2}, \quad p'_n = (\alpha_n + 1)p_{n-1} - p_{n-2}$$

and that for $j = 1, 2, \cdots, i$, we have

$$p'_{n+j} = (j+1)((\alpha_n+1)p_{n-1} - p_{n-2}) - jp_{n-1} = (j+1)p_n + p_{n-1}$$

and similarly

$$q'_{n+j} = (j+1)q_n + q_{n-1}.$$

Therefore

$$r_i = \frac{(i+1)p_n + p_{n-1}}{(i+1)q_n + q_{n-1}} < p_n/q_n,$$

and that $\lim_{i\to\infty} r_i = r$.

As we counted before the number of vertices of $M(p'_{n+i}, q'_{n+i})$ is equal to the number of vertices of $G_{p'_{n+i}}^{q'_{n+i}}$, which is p'_{n+i}. Now we shall count the number of edges of the graphs $M(p'_{n+i}, q'_{n+i})$ and $G_{p'_{n+i}}^{q'_{n+i}}$.

In the graph $G_{p'_{n+i}}^{q'_{n+i}}$, each vertex has degree $p'_{n+i} - 2q'_{n+i} + 1 = (i+1)(p_n - 2q_n) + p_{n-1} - 2q_{n-1} + 1$. Hence the number of edges is equal to

$$\frac{1}{2}((i+1)p_n + p_{n-1})((i+1)(p_n - 2q_n) + p_{n-1} - 2q_{n-1} + 1)$$

which has order $\Theta((i+1)^2)$ as i goes to infinity.

Let F_j and H_j be the graphs constructed as described in Section 2, by using the alpha sequence of r_i. Let a, b and c be the numbers of edges of the graphs F_n, H_n and F_{n+1} respectively. Then for $j = 1, 2, \cdots, i$, F_{n+j} has

a edges if j is even, and has c edges if j is odd. Let e_j be the number of edges of H_{n+j}. Then $e_j = e_{j-1} + 2(a + 2m - 2)$ when j is odd, and $e_j = e_{j-1} + 2(c+2m-2)$ when j is even. The number of edges of $M(p'_{n+i}, q'_{n+i})$ is equal to $e_i + a + 2m - 2$ when i is even, and is equal to $e_i + c + 2m - 2$ when i is odd. An explicit formula for this number can be found by solving the difference equation above. However, we shall not bother to solve it, just to observe that $e_i - e_{i-1}$ is bounded by a constant. Therefore the number of edges of $M(p'_{n+i}, q'_{n+i})$ has order $O(i)$ when i goes to infinity. Hence

$$|E(M(p'_{n+i}, q'_{n+i}))| = O\left(\sqrt{|E(G^{q'_{n+i}}_{p'_{n+i}})|}\right).$$

This completes the proof of Theorem 1.2 for the case that r is rational. If r is irrational, then we let s_i be rationals less than r but approaching r. For each of the rationals s_i, we construct the corresponding sequence of graphs as above, then we use the diagonal method to choose one graph from each of these sequences of graphs. It is obvious that the resulting sequence of graphs gives a proof of Theorem 1.2 in this case. ∎

When $m \geq 3$, we know that some more edges can be deleted from $M(p, q)$ without decreasing the circular chromatic number. However, we do not know if the order could be reduced to be smaller than $O(\sqrt{|E(G^q_p)|})$. On the other hand, let H be a subgraph of G^q_p with the least number of edges such that $\chi_c(H) = p/q$, we do not know any non-trivial lower bound for the number of edges of H.

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