Circular Colourings of Infinite Graphs

Bruce Bauslaugh
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada

Xuding Zhu
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan

April 17, 1997

Abstract

We show that for an infinite graph G, if every finite subgraph of G is r-circular colourable, then G is r-circular colourable; if G has circular-chromatic number r then G has an r-circular colouring.

The circular chromatic number of a graph is a natural generalization the chromatic number of a graph, which was introduced by Vince [4] in 1988 (under a different name "the star chromatic number") and which has attracted considerable attention since then.

Given a pair of integers (k, d) with $k \geq 2d$. A (k, d)-colouring of a graph G is a mapping c of $V(G)$ to $\{0, 1, \cdots, k-1\}$ such that for any edge xy of G, $d \leq |c(x) - c(y)| \leq k - d$. The circular chromatic number $\chi_c(G)$ of G is the infimum of the ratios k/d for which there exists a (k, d)-colouring of G.

It was shown by Vince [4] and by Bondy and Hell [2] that for finite graphs, the infimum in the definition above is attained, and hence $\chi_c(G)$ is rational for any finite graph G. However, as observed in [5], the circular chromatic number of an infinite graph could be any irrational number greater than 2. Therefore for infinite graphs, the infimum in the definition above may not be attained, simply for the reason that any (k, d)-colouring corresponds to the ratio k/d, which is rational.

An alternate definition of the circular chromatic number was given in [5]. Let C be a circle of unit circumference, and let $r \geq 2$ be any real number. An r-circular colouring of a graph G is a mapping c which assigns each vertex x of G an open arc of C of length $1/r$ in such a way that arcs assigned to adjacent vertices are disjoint. It was shown in [5] that if $r = k/d$ is rational, then an r-circular colouring of a graph G corresponds to a (k, d)-colouring of G, and vice versa. Therefore the circular chromatic number of a graph is equal to the infimum of the numbers r such that there exists an r-circular colouring of G. Then a natural question is that whether or not the infimum of the circular colourings can be attained. To be precise, we have the following question:

Question 1 *If $\chi_c(G) = r$, is it true that there is an r-circular colouring of G?*

If G is finite, then $\chi_c(G) = r = k/d$ is rational. Since an r-circular colouring corresponds to a (k, d)-colouring, we conclude that the answer to the above question is affirmative for finite graphs G. If G is infinite but $r = k/d$ is rational, then we can also conclude that G is r-circular colourable by applying the compactness theorem. Because the constraints in a (k, d)-colouring of a graph can be written in a finite number of first order sentences, we know that G is (k, d)-colourable if and only if every finite subgraph of G is (k, d)-colourable. As $\chi_c(G) = r = k/d$, then of course every finite subgraph of G has circular chromatic number at most k/d and hence is (k, d)-colourable. Therefore G is (k, d)-colourable.

If r is irrational, the situation is quite different, and we cannot apply the compactness theorem directly. Indeed, the following question is non-trivial:

Question 2 *If every finite subgraph of G is r-circular colourable, then is it true that G is also r-circular colourable?*

In this note we shall answer these two questions simultaneously. For any real number $r \geq 2$, we define G_r^* to be the graph whose vertex set is the collection of all open arcs of length $1/r$ on a circle of unit circumference, and with two vertices adjacent if their corresponding intervals are disjoint. We define a homomorphism of a graph G to a graph H as an edge preserving mapping from the vertex set of G to the vertex set of H. Then it is easy to see that an r-circular colouring of a graph G is just a homomorphism of G to G_r^*.

An infinite graph H is said to be *homomorphically compact* if for all graphs G the following are equivalent:

- G admits a homomorphism to H, and

- every finite subgraph of G admits a homomorphism to H.

Question 2 asks whether or not G_r^* is homomorphically compact. Theorem 1 below answers this question in affirmative.

Theorem 1 *The graph G_r^* is homomorphically compact.*

Proof. The compactness of infinite directed graphs was studied in [1]. Theorem 1 can be obtained from results in [1], by regarding an undirected graph as a symmetric directed graph. However, for the completeness of this note, we shall give a direct and short proof.

Suppose G is a graph with $\chi_c(G) = r$. Consider the set $\mathcal{S}$ of all the mappings $f : V(G) \to V(G_r^*)$. Define a metric on $V(G_r^*)$ by letting $d(A, B)$ be the length of the shorter arc between the center points of the intervals A and B. This is equivalent to the restriction of the usual metric topology on $\Re^2$ to a circle of unit circumference, and so this is a compact topology. Then consider the set $\mathcal{S}$ as endowed with the product topology, which is compact by Tychonoff Theorem. For each finite subgraph H of G, let X_H be the subset of $\mathcal{S}$ such that for each $f \in X_H$, the restriction of f to H is a homomorphism of H to G_r^*. Now for each finite subgraph H, the set X_H is non-empty, because $\chi_c(H) \leq \chi_c(G) = r$ and for finite graphs the circular chromatic number is attained. Also it is easy to see that the set X_H is closed. Indeed, an element $f \in \mathcal{S}$ is in X_H if and only if for every edge xy of H, $d(f(x), f(y)) \geq 1/r$. Therefore if $f_i \in X_H$ for $i = 1, 2, \cdots$, and $\lim_{i \to \infty} f_i = f$, then for any edge xy of H, $d(f_i(x), f_i(y)) \geq 1/r$ for all i, and hence $d(f(x), f(y)) \geq 1/r$. This implies that $f \in X_H$.

Consider the family $\mathcal{F} = \{X_H : H \text{ is a finite subgraph of } G\}$ of closed subsets of $\mathcal{S}$. It certainly has the finite intersection property, i.e., the intersection of any finite number of subsets of $\mathcal{F}$ is non-empty. As the space $\mathcal{S}$ is compact, we conclude that

$$X = \cap\{X_H : H \text{ is a finite subgraph of } G\} \neq \emptyset.$$

Now any element of X is a homomorphism of G to G_r^*. This completes the proof of Theorem 1. ■

The next theorem answers Question 1.

Theorem 2 *Given an infinite graph G, if G is homomorphic to G_r^* for every $r > r_0$, then G is homomorphic to $G_{r_0}^*$.*

Proof. If G is finite, the result was proved in [5]. Suppose G is an infinite graph and that G admits a homomorphism to G_r^* for every $r > r_0$. Then each finite subgraph of G admits a homomorphism to G_r^* for every $r > r_0$. As Theorem 2 is true for finite graphs [5], we conclude that each finite subgraph of G admits a homomorphism to $G_{r_0}^*$. By Theorem 1, $G_{r_0}^*$ is homomorphically compact. Therefore G admits a homomorphism to $G_{r_0}^*$. ■

The fractional chromatic number of a graph is another well known variation of the chromatic number of a graph. However, we note that the situation for fractional chromatic number is in sharp contrast to that for the circular chromatic number. It is well known that the fractional chromatic number of finite graphs are attained (i.e.,

if a finite graph G has fractional chromatic number r, then G has an r-fractional colouring). However, it was shown by Leader [3] that the fractional chromatic number of infinite graphs may not be attained. Leader also proved that the fractional chromatic number of an infinite graph may be strictly greater than the supremum of the fractional chromatic numbers of its finite subgraphs.

References

[1] B. Bauslaugh, Cores and compactness of infinite directed graphs, *J. Comb. Th. Ser. B*, 68 (1996) 255-276.

[2] J. A. Bondy and P. Hell, A note on the star chromatic number, *J. Graph Theory* **14** 1990, 479-482.

[3] I. Leader, The fractional chromatic number of infinite graphs, *J. Graph Theory* **20** (1995) 411–417.

[4] A. Vince, Star chromatic number, *J. Graph Theory* **12** (1988) 551-559.

[5] X. Zhu, Star chromatic number and product of graphs, *J. Graph Theory* **16** (1992) 557-569.