Minimal circular-imperfect graphs of large clique number and large independence number

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Abstract

This paper constructs some classes of minimal circular-imperfect graphs. In particular, it is proved that there are minimal circular-imperfect graphs whose independence number and clique number are both arbitrarily large.

Keywords: circular chromatic number, circular clique number, circular-perfect graph, minimal circular-imperfect graph

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1 Introduction

Suppose G and H are graphs. A *homomorphism* of G to H is a mapping $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. A homomorphism of G to H is also called an *H-colouring* of G. We say a graph G is *H-colourable*, and write $G \leq H$, if there exists a homomorphism of G to H. A graph G is called *uniquely H-colourable* if there is an onto homomorphism f of G to H, and for any other homomorphism h of G to H, $h = \phi \circ f$ for an automorphism of ϕ of H. Two graphs G and H are *homomorphically equivalent*, written as $G \approx H$, if $G \leq H$ and $H \leq G$. Obviously, $\approx$ is an equivalence relation. Let $\mathcal{G}$ be the set of equivalence classes graphs under the equivalence relation "$\approx$". Then $\leq$ is a partial order on $\mathcal{G}$. A homomorphism f of a graph G to a subgraph H of G is called a *retraction* if the restriction of f to H is identity. A graph G is called a *core* if G does not admit a homomorphism to any of its proper subgraphs.

An *n-colouring* of a graph G is a mapping $f : V(G) \to \{1, 2, \cdots, n\}$ such that for every edge xy of G, $f(x) \neq f(y)$. The *chromatic number* $\chi(G)$ of G is the least integer n for which G has an n-colouring. It is easy to see that a homomorphism of G to K_n is equivalent to an n-colouring of G. Thus the chromatic number of G can be defined as

$$\chi(G) = \min\{n : G \leq K_n\}.$$

An *n-clique* in a graph G is a subgraph of G isomorphic to K_n. The *clique number* of G, denoted by $\omega(G)$, is the largest n for which G has an n-clique. So the clique number of G is

$$\omega(G) = \max\{n : K_n \leq G\}.$$

Thus the set $Z_{\mathcal{G}} = \{K_n : n = 1, 2, \cdots\}$ of complete graphs, which form an increasing chain in the partial order $(\mathcal{G}, \leq)$, is used as a scale that measures the chromatic number and the clique number of graphs [9]. The circular chromatic number and the circular clique number of graphs are, respectively, the refinements of the chromatic number and the clique number, defined through a finer scale.

Suppose $p \geq 2q$ are positive integers. The *circular clique* $K_{p/q}$ has vertex set $\{0, 1, \cdots, p-1\}$ in which ij is an edge if $q \leq |i - j| \leq p - q$. Let $Q_{\mathcal{G}} = \{K_1\} \cup \{K_{p/q} : p/q \geq 2 \text{ is a rational number}\}$. It is known [3, 6] that for any rationals p/q and p'/q', $K_{p/q} \leq K_{p'/q'}$ if and only if $p/q \leq p'/q'$. So the set $Q_{\mathcal{G}}$ also form a chain in the partial order $(\mathcal{G}, \leq)$. As $Z_{\mathcal{G}} \subseteq Q_{\mathcal{G}}$, the chain $Q_{\mathcal{G}}$ provides a finer scale. Given a graph G, the *circular chromatic number* of G is defined as

$$\chi_c(G) = \inf\{p/q : G \leq K_{p/q}\}$$

and the *circular clique number* of G is defined as

$$\omega_c(G) = \sup\{p/q : K_{p/q} \leq G\}.$$

It is known [6, 3, 11] that the infimum and supremum in the definitions above are attained and can be replaced by the minimum and the maximum, respectively. It follows from the definition that for any graph G,

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G). \tag{1}$$

Moreover, $\omega_c(G) < \omega(G) + 1$ and $\chi(G) < \chi_c(G) + 1$. Observe that if $p/q = p'/q'$ then $K_{p/q}$ and $K_{p'/q'}$ are homomorphically equivalent, although they are non-isomorphic if $p \neq p'$. In case $(p, q) = 1$, then we call the circular clique $K_{p/q}$ a *prime circular clique*. A homomorphism f of G to $K_{p/q}$ is also called a (p, q)-colouring of G. A uniquely $K_{p/q}$-colourable graph is also called uniquely (p, q)-*colourable*. It is obvious that a graph H is uniquely H-colourable if and only if H is a core.

A graph G is called *perfect* if for every induced subgraph H of G, $\omega(H) = \chi(H)$. Perfect graphs have been studied extensively since the concept was introduced by Berge in 1961 [2]. Berge noted that odd cycles of length at least 5 and their complements are minimal imperfect graphs, i.e., these graphs are not perfect but their proper induced subgraphs are perfect. Berge conjectured that these are the only minimal imperfect graphs. This conjecture, called the Strong Perfect Graph Conjecture, after resisting attacks for more than 40 years, has been turned into a theorem by Chudnovsky, Robertson, Seymour and Thomas [4] in 2002.

The concept of perfect graphs is naturally extended to circular colourings. A graph G is called *circular-perfect* if for every induced subgraph H of G, $\chi_c(H) = \omega_c(H)$. Circular-perfect graphs was introduced in [14], and has been studied in a few papers [1, 5, 7, 8, 10, 12, 13]. By (1), $\chi(H) = \omega(H)$ implies that $\chi_c(H) = \omega_c(H)$. Thus every perfect graph is circular-perfect. On the other hand, it is proved in [14] that circular cliques are circular perfect. In particular, odd cycles, which are $K_{(2k+1)/k}$ and the complement of odd cycles, which are $K_{(2k+1)/2}$ are circular-perfect. So the family of perfect graphs is a proper subfamily of the family of circular-perfect graphs. Although some sufficient conditions for a graph to be circular-perfect and some necessary conditions for a graph to be circular-perfect are obtained in the literature [12, 13], not much is known about the structure of circular-perfect graphs. Inspired by the Strong Perfect Graph Theorem, one might expect a simple forbidden induced subgraph characterization of circular-perfect graphs. We call a graph *circular-imperfect* if it is not circular-perfect. A graph G is called *minimal circular-imperfect* if G is circular-imperfect, but every proper induced subgraph H of G is circular-perfect. The question is which graphs are minimal circular-imperfect.

For a graph G, the *independence number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G. A common feature of the minimal imperfect graphs G is that they all satisfies

$$\min\{\alpha(G), \omega(G)\} = 2.$$

Minimal circular-imperfect graphs have been studied in [5, 7, 8]. Not many minimal circular-imperfect graphs are known. The currently known minimal circular-imperfect graphs include (1): The square of the cycle C_{3k+1} (the complement of $K_{(3k+1)/3}$) and some of its spanning subgraphs, and (2): the graph obtained from an odd cycle of length at least 5 or its complement by adding a universal vertex, and (3): the graph obtained from a cycle $C_{(2p+1)(2q+1)}$ with vertices $v_1, v_2, \cdots, v_{(2p+1)(2q+1)}$ ($p \geq 1$ and $q \geq 1$) by adding a vertex u adjacent to $v_{i(2q+1)}$ for $i = 1, 2, \cdots, 2p+1$. These graphs G satisfy

$$\min\{\alpha(G), \omega(G)\} \leq 3.$$

The question whether $\min\{\alpha(G), \omega(G)\} \leq 3$ for all minimal circular-imperfect graphs is asked in [5]. A weaker question is this: is there a constant c such that every minimal circular-imperfect graphs G have $\min\{\alpha(G), \omega(G)\} \leq c$? This paper answers this question in the negative. We prove that for each even integer k, there is a minimal circular-imperfect graph G with $\min\{\alpha(G), \omega(G)\} = k$. We also construct a new infinite family of minimal circular-imperfect graphs with $\min\{\alpha(G), \omega(G)\} = 4$, and a new infinite family of minimal circular-imperfect graphs G with $\min\{\alpha(G), \omega(G)\} = 2$.

2 Glueing of circular cliques

Suppose H and H' are graphs and K and K' are n-cliques ($n \geq 1$) of H and H', respectively. Let G be a graph obtained from the disjoint union of H and H' by identifying K and K' into a single n-clique. It is well-known and easy to see that if H and H' are perfect graphs, then G is also perfect. However, this is in general not true for circular-perfect graphs.

Lemma 1 *Suppose H and H' are graphs and K and K' are n-cliques ($n \geq 1$) of H and H', respectively. Let G be a graph obtained from the disjoint union of H and H' by identifying K and K' into a single n-clique. Then $\omega_c(G) = \max\{\omega_c(H), \omega_c(H')\}$.*

Proof. Since G contains H, H' as subgraphs, we have $\omega_c(G) \geq \max\{\omega_c(H), \omega_c(H')\}$. Assume to the contrary that $\omega_c(G) = p/q > \max\{\omega_c(H), \omega_c(H')\}$, where $(p, q) = 1$. Then G contains an induced subgraph Q isomorphic to $K_{p/q}$. Assume the vertices of Q are $v_0, v_1, \cdots, v_{p-1}$, and v_i is adjacent to v_j if and only if $q \leq |i-j| \leq p-q$.. As $p/q > \max\{\omega_c(H), \omega_c(H')\}$, Q contains vertices of $H-K$ and vertices of $H'-K'$. In particular, Q is not a clique (as vertices of $H-K$ and vertices of $H'-K'$ are not adjacent), and $q \geq 2$. Without loss of generality, we may assume that $v_0 \in V(H-K)$ and $v_1 \in V(H'-K')$. Then $N_Q(v_0) \cap N_Q(v_1) = \{v_{q+1}, v_{q+2}, \cdots, v_{p-q}\}$ is a subset of the identified n-clique. As v_{q+1} is not adjacent to v_{q+2}, we conclude that $N_Q(v_0) \cap N_Q(v_1)$ is 1-clique, i.e., $p-q = q+1$. Thus Q is an odd cycle. But an odd cycle containing vertices of both $H-K$ and $H'-K'$ cannot be an induced subgraph of G. This proves the lemma. ∎

If G is obtained from the disjoint union of H and H' by identifying a vertex x of H with a vertex x' of H', then it is obvious that $\chi_c(G) = \max\{\chi_c(H), \chi_c(H')\}$. Hence we have the following lemma:

Lemma 2 *Suppose H and H' are circular-perfect graphs and x and x' are vertices of H and H', respectively. Let G be a graph obtained from the disjoint union of H and H' by identifying x and x' into a single vertex. Then G is circular-perfect.*

But identifying a clique of size greater than 1 of two circular-perfect graphs may result in a circular-imperfect graph. This is due to the fact that for a (p, q)-colourable graph H and a clique K of H, it is possible that some (p, q)-colourings of K cannot be extended to a (p, q)-colouring of H. In particular, the following special case will be used in our construction of minimal circular-imperfect graphs.

Lemma 3 *Suppose $(p, q) = 1$ and H and H' are two copies of $K_{p/q}$ with $V(H) = \{v_0, v_1, \cdots, v_{p-1}\}$ and $V(H') = \{v'_0, v'_1, \cdots, v'_{p-1}\}$, in which v_i is adjacent to v_j (respectively, v'_i is adjacent to v'_j) if and only if $q \leq |i-j| \leq p-q$. Let G be obtained from the disjoint union of H and H' by identifying v_0 and v'_0 into a single vertex u, and identifying v_i and v'_j into a single vertex u'. If $q \leq i < j \leq p/2$, then $\chi_c(G) > p/q$. In particular, G is circular-imperfect.*

Proof. Assume f is a (p, q)-coloring of G. Without loss of generality, assume that $f(u) = 0$. As $K_{p/q}$ is uniquely (p, q)-colourable, the restriction of f to each of H and H' can be viewed as an automorphism of $K_{p/q}$. Thus by considering the restriction of f to H, we should have $f(u') = i$ or $p-i$. By considering the restriction of f to H' we should have $f(u') = j$ or $p-j$. As $q \leq i < j \leq p/2$, such a homomorphism f does not exist. Therefore $\chi_c(H) > p/q$. As $\omega_c(H) = p/q$ by Lemma 1, it follows that G is circular-imperfect. ∎

Now we construct an infinite family of minimal circular-imperfect graphs G with $\min\{\alpha(G), \omega(G)\} = 4$.

Theorem 1 *Let H, H' be two disjoint copies of $K_{(2k+1)/2}$, with $V(H) = \{0, 1, \cdots, 2k\}$ and $V(H') = \{0', 1', \cdots, (2k)'\}$. Let G be obtained from the disjoint union of H and H' by identifying 0 and $0'$ into a single vertex v, and identifying i and j' into a single vertex v'. If $2 \leq i < j \leq k$, then G is minimal circular-imperfect.*

Proof. It follows from Lemma 3 that G is circular-imperfect. To show that G is minimal circular-imperfect, we need to show that for any proper induced subgraph Q of G, $\omega_c(Q) = \chi_c(Q)$. Let Q be obtained from G by deleting some vertices. If v or v' is deleted, then by Lemma 2, Q is circular-perfect. Assume none of v and v' is deleted. Let Q_1 be the restriction of Q to H (including the edge vv'), and let Q_2 be the intersection of Q with H' (including the edge vv'). If each of H, H' contains a deleted vertex, then Q_1, Q_2 are perfect graphs, as $K_{(2k+1)/2}$ is a minimal imperfect graph. Therefore Q is perfect. Assume only H contains deleted vertices. Then $\omega_c(Q) = (2k+1)/2$. Since Q_1 is a proper induced subgraph of $K_{(2k+1)/2}$, Q_1 admits a homomorphism to K_k. Note that any edge of $K_{(2k+1)/2}$ is contained in a copy of K_k. Thus there is a homomorphism h of Q_1 to $K_{(2k+1)/2}$ such that $h(v) = 0$ and $h(v') = j$. Now h can be extended to a homomorphism of Q to $K_{(2k+1)/2}$ by letting $f(s') = s$ for each $s \in Q_2 = H'$. So $\chi_c(Q) = \omega_c(Q)$. ∎

When $k \geq 4$, the graph G constructed above has $\min\{\omega(G), \alpha(G)\} = 4$. Thus we have constructed an infinite family of minimal circular-imperfect graphs G with $\min\{\omega(G), \alpha(G)\} = 4$. Minimal circular-imperfect graphs G with $\min\{\omega(G), \alpha(G)\} = 2$ are known. Indeed, a graph obtained from the complement of an odd cycle of length at least 5 by adding a universal vertex is such a minimal circular-imperfect graph. The following is a new family of minimal circular-imperfect graphs G with $\min\{\omega(G), \alpha(G)\} = 2$.

Theorem 2 *Let H, H' be two disjoint copies of $K_{4k/(2k-1)}$, with $V(H) = \{0, 1, \cdots, 4k-1\}$ and $V(H') = \{0', 1', \cdots, (4k-1)'\}$. Let G be obtained from the disjoint union of H and H' by identifying 0 and $0'$ into a single vertex v and identifying $k-1$ and k' into a single vertex v'. Then G is minimal circular-imperfect.*

Proof. By Lemma 3, G is circular-imperfect. Now we show that for any proper induced subgraph Q of G, $\omega_c(Q) = \chi_c(Q)$. Let Q be obtained from G by deleting some vertices. If v or v' is deleted, then by Lemma 2, Q is circular perfect. Assume none of v, v' is deleted. Let Q_1 be the restriction of Q to H (including the edge vv'), and let Q_2 be the intersection of Q with H' (including the edge vv'). Without loss of generality, we assume that Q_1 is proper induced subgraph of H. As Q_1 and Q_2 are circular-perfect, each of Q_1, Q_2 retracts to its maximum circular cliques. It is easy to verify that each proper induced subgraph of $K_{4k/(2k-1)}$ has minimum degree at most 2. Thus the circular clique contained in Q_1 is either an edge or a cycle. Thus Q_1 retracts to an edge, or an odd cycle. If Q_2 is also a proper induced subgraph of H', then Q_2 also retracts to an edge, or an odd cycle. As Q is obtained from Q_1 and Q_2 by identifying an edge, Q also retracts to a vertex, or an edge, or an odd cycle. Assume $Q_2 = H'$. Then $\omega_c(Q) = 4k/(2k-1)$. If Q_1 retracts to an edge, then by retracting Q_1 to the edge vv', we conclude that Q retracts to $Q_2 = K_{4k/(2k-1)}$. If Q_1 retracts to an odd cycle of length $2t+1$, then it is easy to verify that each edge of $K_{4k/(2k-1)}$ is contained in a cycle of length $2t+1$. Hence we can retract Q_1 to an odd cycle of Q_2 containing edge vv', and hence Q retracts to Q_2, and $\chi_c(Q) = \omega_c(Q)$. ■

3 Graphs of large independence number and large clique number

In this section, we construct, for each odd integer $k \geq 3$, a minimal circular-imperfect graph G_k with $\min\{\omega(G_k), \alpha(G_k)\} = k+1$. Suppose $k \geq 3$ is an odd integer. Let G_k be the graph constructed as follows:

- Take a copy of $K_{3k/2}$ whose vertex set is $V = \{v_0, v_1, \dots, v_{3k-1}\}$ (in which $v_i \sim v_j$ if and only if $2 \leq |i-j| \leq 3k-2$).
- Add vertices $u_0, u_1, \dots, u_{k-1}$. Join u_0 to $v_2, v_5, \dots, v_{3k-4}, v_{3k-2}$, and for $i = 1, 2, \dots, k-1$, join u_i to $v_{3i+2}, v_{3i+5}, \dots, v_{3k-1}, v_1, v_4, \dots, v_{3i-2}$.
- Add vertex x and join x to $u_0, u_1, \dots, u_{k-1}$.

Figure 1 below is the graph G_5. The subset $V = \{v_0, v_1, \cdots, v_{14}\}$ of $V(G_5)$ induces a copy of $K_{15/2}$, i.e., v_iv_j is an edge if $2 \leq |i-j| \leq 13$. For simplicity, these edges are not shown in the figure.

Let $U = \{u_0, u_1, \dots, u_{k-1}\}$. It is obvious that the graph G_k has clique number $\lfloor 3k/2 \rfloor$ and has independence number $k+1$ (the set $U \cup \{v_0\}$ is an independent

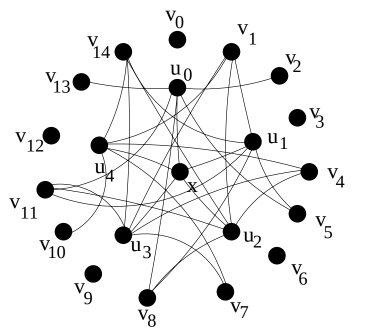

Figure 1: The graph G_5, with edges between the v_i's missing.

set of G_k). Thus $\min\{\omega(G_k), \alpha(G_k)\} = k+1$. In the following we shall show that G_k is minimal circular-imperfect. First we prove that G_k is circular-imperfect.

Lemma 4 *Suppose $k = 2d+1 \geq 3$ is an odd integer. Then G_k is circular-imperfect.*

Proof. Since G_k contains $K_{3k/2}$ as a subgraph and does not contain K_{3d+2}, we have $3d+2 > \omega_c(G_k) \geq 3k/2$. If $\omega_c(G_k) = p/q$ and $(p,q) = 1$, then G_k contains $K_{p/q}$ as an induced subgraph. In particular, $p \leq |V(G)| \leq 4k+1$. As there is no fraction p/q such that $3k/2 = (6d+3)/2 < p/q < 3d+2$ with $p \leq 4k+1 = 8d+5$, we conclude that $\omega_c(G_k) = 3k/2$.

To prove that G_k is circular-imperfect, it suffices to show that G_k is not $(3k,2)$-colourable. Assume to the contrary that c is a $(3k,2)$-colouring of G_k. The restriction of c to the copy of $K_{3k/2}$ induced by V can be viewed as an automorphism of $K_{3k/2}$. So without loss of generality, we may assume that $c(v_i) = i$ for $i = 0, 1, \ldots, 3k-1$. As u_0 is adjacent to $v_2, v_5, \ldots, v_{3k-4}, v_{3k-2}$, this forces $c(u_0) = 0$. For $i = 1, 2, \cdots, k-1$, as u_i is adjacent to $v_{3i+2}, v_{3i+5}, \ldots, v_{3k-1}, v_1, v_4, \ldots, v_{3i-2}$, this forces $c(u_i) = 3i$. Then there is no legal colour for x. So G_k is not $(3k,2)$-colourable and hence G_k is circular-imperfect. ∎

In the remainder of this section, we shall prove that G_k is minimal circular-imperfect.

For $i = 0, 1, \ldots, k-1$, let $V_i = N_{G_k}(v_i) \cap U$. By the construction of G_k, we have

- $V_{3i} = \emptyset$ for $i = 0, 1, \ldots, k-1$.
- $V_{3k-1} = U - \{u_0\}$.

- $V_{3k-2} = \{u_0\}$.
- $V_{3i+1} = \{u_{i+1}, u_{i+2}, \ldots, u_{k-1}\}$ for $i = 0, 1, \ldots, k-2$.
- $V_{3i+2} = \{u_0, u_1, \ldots, u_i\}$ for $i = 0, 1, \ldots, k-2$.

Thus we have the following lemma.

Lemma 5 *For the sets V_i defined above, we have*

1 $V_{3i+1} \cap V_{3i+2} = \emptyset$ *and* $V_{3i+1} \cup V_{3i+2} = U$ *for* $i = 0, 1, \ldots, k-1$.

2 $V_{3k-1} \supseteq V_1 \supseteq V_4 \supseteq \ldots \supseteq V_{3(k-2)+1}$.

3 $V_{3k-4} \supseteq V_{3k-7} \supseteq \ldots \supseteq V_5 \supseteq V_2 \supseteq V_{3k-2}$.

Lemma 6 *Assume H is an induced subgraph of G_k. If $\omega_c(H) \neq \omega(H)$ then $\omega_c(H) = 5/2$ or $7/3$ or $3k/2$. Moreover, $\omega_c(H) = 3k/2$ if and only if $V \subseteq V(H)$, and if $\omega_c(H) = 5/2$ or $7/3$, then any odd cycle of H contains x. In particular, every induced odd cycle of length at least 5 of G_k is either a C_5 or a C_7 and contains x.*

Proof. Let K be a maximum circular clique of H. Assume first that $V(K) \cap U = \emptyset$. Then $V(K) \subseteq V$. If $V(K) = V$, then $\omega_c(H) = 3k/2$. Otherwise K is a proper induced subgraph of $K_{3k/2}$, and hence K is a perfect graph, which implies that $\omega(K) = \omega_c(K) = \chi_c(K) = \chi(K)$. Hence $\omega_c(H)$ is an integer.

Assume next that K contains u_i for some i. If K does not contain x, then K is a clique, because $N_{G_k - x}(u_i)$ induces a clique. (Observe that if $q \geq 2$, then for each vertex v of $K_{p/q}$, $N_{K_{p/q}}(v)$ does not induce a clique.) Hence $\omega_c(H)$ is an integer. If K contains x, then K contains no triangle, because x is not contained in any triangle. (Observe that circular cliques are vertex-transitive. So if K contains a triangle, then every vertex is contained in a triangle.) This implies that u_i has at most one neighbour in V (again because $N_{G_k - x}(u_i)$ induces a clique). If u_i has no neighbour in $K_{3k/2}$, then $K = K_2$. If u_i has one neighbour in $K_{3k/2}$, then K is an odd cycle. It is easy to see that the only induced odd cycles of G_k have length 5 and 7. Hence $\omega_c(H) = 5/2$ or $7/3$. Moreover, each odd cycle of H contains x. This implies that every induced odd cycle of length at least 5 of G_k is either a C_5 or a C_7 and contains x. ∎

Theorem 3 *For any odd integer $k \geq 3$, the graph G_k is minimal circular-imperfect.*

Proof. Assume to the contrary that G_k is not minimal circular imperfect. Let H be a minimal circular-imperfect graph contained in G_k. Then $\chi_c(H) > \omega_c(H)$ and $\chi_c(H') = \omega_c(H')$ for any proper induced subgraph H' of H. Thus H is a core. Let $X = V(H) \cap V$, $Y = V(H) \cap U$ and let $Q = H[X]$. Observe that every vertex of Y is adjacent to some vertices of X, for otherwise H is not a core.

We divide the proof into three cases.

Case 1 $\omega_c(H) = 3k/2$.

By Lemma 6, $X = V$. It follows from the proof of Lemma 4 that $\chi_c(G_k - x) \leq 3k/2$. Since $\chi_c(H) > \omega_c(H) = 3k/2$, we have $x \in V(H)$. Since H is a proper induced subgraph of G_k, there is an index j such that $u_j \notin Y$. Let $c(v_i) = i$, $c(u_i) = 3i$ for $u_i \in Y$ and letting $v(x) = 3j$. Then c is a $(3k, 2)$-colouring of H, in contrary to our assumption that $\chi_c(H) > \omega_c(H) = 3k/2$.

Case 2 $\omega_c(H) \in \{5/2, 7/2\}$.

By Lemma 6, $H - x$ is bipartite, say with partite sets A and B. We observed before, each vertex $u \in Y$ is adjacent to some vertices of X. Since any two neighbours of u in V are adjacent, and since H contains no triangle, each vertex $u \in Y$ has exactly one neighbour in X.

Consider the set $S = \{v_{3k-1}, v_1, v_4, \cdots, v_{3(k-2)+1}\}$ in this order; and the set $T = \{v_{3k-4}, v_{3k-7}, \cdots, v_5, v_2, v_{3k-2}\}$ in this order. Let v be the smallest vertex of S adjacent to some vertices of Y (if it exists), and let v' be the smallest vertex of T adjacent to some vertices of Y (if ti exists). It follows from Lemma 5 that any vertex of Y is adjacent to exactly one of v and v'. The vertices v and v' exist and belong to different parts of the bipartite graph $H - x$, for otherwise H itself is a bipartite graph. Assume $v \in A$ and $v' \in B$. This implies that that $N_H(A \cap Y) \cap X$ is an independent set, and $N_H(B \cap Y) \cap X$ is an independent set (for otherwise $H - x$ is not bipartite). Let $u \in N_H(v) \cap Y$ and $u' \in N_H(v') \cap Y$. If $v \sim v'$, then let f be defined as follows:

$$f(a) = \begin{cases} a & \text{if } a \in \{v, v', x, u, u'\}, \\ u & \text{if } a \in B \cap Y, \\ u' & \text{if } a \in A \cap Y, \\ v & \text{if } a \in A \cap X, \\ v' & \text{if } a \in B \cap X. \end{cases}$$

Then f is a homomorphism from H to a subgraph of H, which is a copy of C_5, in contrary to the assumption that H is a core and circular-imperfect.

If $v \not\sim v'$, then without loss of generality, we may assume that $v = v_{3i+1}$ and $v' = v_{3i+2}$ for some i. Since X contains no vertex which is adjacent to both v_{3i+1} and v_{3i+2} and since $H-x$ is connected, we conclude that $X = \{v_{3i}, v_{3i+1}, v_{3i+2}, v_{3i+3}\}$. Let f be defined as follows:

$$f(a) = \begin{cases} a & \text{if } a \in X \cup \{x, u, u'\}, \\ u & \text{if } a \in B \cap Y, \\ u' & \text{if } a \in A \cap Y. \end{cases}$$

Then f is a homomorphism from H to a subgraph of H, which is either a copy of C_7, again in contrary to the assumption that H is a core and circular-imperfect.

Case 3 $\omega_c(H) = \omega(H) = n$.

Certainly we have $n \geq 3$, for otherwise H is bipartite and hence perfect. We shall derive a contradiction by showing that H is n-colourable.

It is obvious that either $\omega(Q) = n$ or $n-1$. If $\omega(Q) = n-1$, then H is n-colourable: Colour Q by colours $\{0, 1, \ldots, n-2\}$, colour all the vertices of Y by colour $n-1$, and colour x by colour 0. Thus we assume $\omega(Q) = n$. As $H-x$ is perfect, $H-x$ has an n-colouring c.

Assume there is an index i such that both vertices v_{3i+1}, v_{3i+2} belong to X. By Lemma 5, $V_{3i+1} \cup V_{3i+2} = U$, and each vertex of Y is adjacent to either v_{3i+1} or v_{3i+2}. If $c(v_{3i+1}) = c(v_{3i+2})$, then for any $u \in Y$, $c(u) \neq c(v_{3i+1})$. Hence c can be extended to an n-colouring of H by letting $c(x) = c(v_{3i+1})$. If $c(v_{3i+1}) \neq c(v_{3i+2})$, then it follows from Lemma 5 that if $u \in V_{3i+1}$, then $(N_{H-x}(u) \setminus \{v_{3i+1}\}) \subseteq N_H(v_{3i+2})$; if $u \in V_{3i+2}$, then $(N_{H-x}(u) \setminus \{v_{3i+2}\}) \subseteq N_H(v_{3i+1})$. Thus we can re-colour the vertices of Y by letting $c(u) = c(v_{3i+2})$ for $u \in V_{3i+1}$ and $c(u) = c(v_{3i+1})$ for $u \in V_{3i+2}$. This can be extended to an n-colouring of H by colouring x with a colour different from $c(v_{3i+1})$ and $c(v_{3i+2})$.

Thus for any index i, at most one of v_{3i+1}, v_{3i+2} is contained in $V(H)$.

Next we show that $v_{3i} \notin X$ for any i. Assume to the contrary that $v_{3i} \in X$. If $\{v_{3i+1}, v_{3i-1}\} \cap X = \emptyset$, then for any $u \in U$, $N_{H-x}(u) \subseteq N_H(v_{3i})$. Hence we can re-colour all the vertices of Y by colour $c(v_{3i})$, which then can be extended to an n-colouring of H by colouring x with a colour different from $c(v_{3i})$. If $v_{3i+1} \in X$, then since $v_{3i+2} \notin X$, we conclude that $N_H(v_{3i}) \subseteq N_H(v_{3i+1})$, in contrary to the assumption that H is a core. The same contradiction arise if $v_{3i-1} \in X$. This proves that $v_{3i} \notin X$ for any index i.

Thus Q is a complete graph. By Lemma 5, X has two vertices v_i, v_j such that $V_i \subseteq V_j$. If $u \in Y$ is colored by $c(v_j)$, then u is not adjacent to v_j and hence is not adjacent to v_i. We re-colour the vertex u by colour $c(v_i)$. Extend this new n-colouring of $H-x$ to an n-colouring of H by colouring x with colour $c(v_j)$. ∎

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