Coloring the Square of a K_4-minor Free Graph

Ko-Wei Lih
Institute of Mathematics
Academia Sinica
Nankang, Taipei 115, Taiwan
E-mail:makwlih@sinica.edu.tw

Wei-Fan Wang
Department of Mathematics
Zhejiang Normal University
Jinhua, Zhejiang 321004, China
E-mail:wwf@lnu.edu.cn

Xuding Zhu
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung 804, Taiwan
E-mail:zhu@ibm28.math.nsysu.edu.tw

Revised, December 24, 2002

Abstract

Let G be a K_4-minor free graph with maximum degree Δ. We prove that the chromatic number of the square of G is at most (i) $\Delta + 3$ if $2 \leq \Delta \leq 3$; or (ii) $\lfloor 3\Delta/2 \rfloor + 1$ if $\Delta \geq 4$. Examples are given to show the bounds can be attained.

Key words: Series-parallel graph, K_4-minor free graph, square, chromatic number

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, let $V(G)$, $E(G)$, $|G|$, $\Delta(G)$, and $\delta(G)$ denote, respectively, its vertex set, edge set, number of vertices, maximum degree, and minimum degree. For $x \in V(G)$, let $N_G(x)$ denote the set of neighbors of x in G and let $d_G(x)$ denote the degree, i.e., the number of neighbors, of x in G. A vertex of degree k is called a k-vertex. For two vertices $u, v \in V(G)$, let $\text{dist}_G(u, v)$ denote the distance between u and v, i.e., the length of a shortest path connecting them.

A k-coloring of a graph G is a mapping ϕ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every edge xy of G. We call G k-colorable if it has a k-coloring. The *chromatic number* $\chi(G)$ of G is the smallest integer k such that G is k-colorable. The *square* G^2 of a graph G is the graph defined on the vertex set $V(G)$ such that two vertices u and v are adjacent in G^2 if and only if $1 \leq \text{dist}_G(u, v) \leq 2$. A mapping ϕ is a k-coloring of G^2 if and only if $\phi(u) \neq \phi(v)$ whenever $1 \leq \text{dist}_G(u, v) \leq 2$.

Since G^2 contains a clique of order at least $\Delta(G)+1$, we have $\chi(G^2) \geq \Delta(G) + 1$ for any graph G. If G is a tree, then $\chi(G^2) = \Delta(G) + 1$. On the other hand, $\Delta(G^2) \leq \Delta^2(G)$. Hence $\chi(G^2) \leq \Delta^2(G) + 1$ for any graph G. The 5-cycle C_5 and the Petersen graph satisfy $\chi(G^2) = \Delta^2(G) + 1$.

Wegner [9] first investigated the chromatic number of the square of a planar graph. Wegner proved that $\chi(G^2) \leq 8$ for every planar graph G with $\Delta(G) = 3$ and conjectured that the upper bound could be reduced to 7, which has been confirmed by Thomassen [7]. For planar graphs of maximum degree $\Delta \geq 4$, Wegner [9] proposed the following conjecture. The bounds are sharp if the conjecture is true.

Conjecture 1 *Let G be a planar graph. Then*

$$\chi(G^2) \leq \begin{cases} \Delta(G) + 5 & \textit{if } 4 \leq \Delta(G) \leq 7; \\ \lfloor 3\Delta(G)/2 \rfloor + 1 & \textit{if } \Delta(G) \geq 8. \end{cases}$$

This conjecture remains open. The best upper bound so far is $5\Delta(G)/3 + 78$ established by Molloy and Salavatipour [5]. This improves other recently obtained upper bounds: $\lfloor 9\Delta/5 \rfloor + 2$ for $\Delta(G) \geq 749$ ([1]), $\lceil 9\Delta/5 \rceil + 1$ for $\Delta(G) \geq 47$ ([2]), and $2\Delta(G) + 25$ ([6]). For planar graphs of large girth, better upper bounds for $\chi(G^2)$ are known. Wang and Lih [8] proved that if G is a planar graph with girth $g(G)$, then $\chi(G^2) \leq \Delta(G) + 5$ when $g(G) \geq 7$, $\chi(G^2) \leq \Delta(G) + 10$ when $g(G) \geq 6$, and $\chi(G^2) \leq \Delta(G) + 16$ when $g(G) \geq 5$.

A graph G has a graph H as a *minor* if H can be obtained from a subgraph of G by contracting edges, and G is called H*-minor free* if G does not have H as a minor. A graph G is called a *series-parallel* graph if G can be obtained from K_2 by applying a sequence of operations, where each operation is either to duplicate an edge (i.e., replace an edge with two parallel edges) or to subdivide an edge (i.e., replace an edge with a path of length 2). It is well-known [3] that a graph G is an outerplanar graph if and only if G is K_4-minor free and $K_{2,3}$-minor free. A graph G is K_4-minor free if and only if each block of G is a series-parallel graph. Thus the class of K_4-minor free graphs is a class of planar graphs that contains both outerplanar graphs and series-parallel graphs. We will establish the following in this paper.

Theorem 1 *Let G be a K_4-minor free graph. Then*

$$\chi(G^2) \leq \begin{cases} \Delta(G) + 3 & \text{if } 2 \leq \Delta(G) \leq 3; \\ \lfloor 3\Delta(G)/2 \rfloor + 1 & \text{if } \Delta(G) \geq 4. \end{cases}$$

Before proving Theorem 1, we show that the result is best possible. For $\Delta = 2$, we have $\Delta(C_5) = 2$ and $\chi(C_5^2) = 5$. For $\Delta = 3$, let G be the graph consisting of three internally disjoint paths joining two vertices x and y, where two of the paths are of length 2, and the third one is of length 3. Then $\Delta(G) = 3$ and $\chi(G^2) = \chi(K_6) = 6$. For $\Delta = 2k \geq 4$, let G_{2k} be the graph consisting of k internally disjoint paths joining x and y, k internally disjoint paths joining x and z, and k internally disjoint paths joining y and z. All these paths are of length 2, except one path joining x and y is of length 1, and one path joining x and z is of length 1. Then $\Delta(G_{2k}) = 2k$ and $\chi(G_{2k}^2) = \chi(K_{3k+1}) = 3k+1$. For $\Delta = 2k+1 \geq 5$, let G_{2k+1} be obtained from G_{2k} by adding a new path of length 2 joining y and z. Then $\Delta(G_{2k+1}) = 2k+1$ and $\chi(G_{2k+1}^2) = \chi(K_{3k+2}) = 3k+2$.

2 Proof of Theorem 1

Let $u \sim v$ denote that u and v are adjacent in G, that is, $uv \in E(G)$. Define $S_G(u) = \{x \mid d_G(x) \geq 3$ such that $u \sim x$ or there exists a 2-vertex z satisfying $u \sim z$ and $z \sim x\}$. Let $D_G(u) = |S_G(u)|$. It is well-known [4] that every K_4-minor free graph contains a vertex of degree at most two.

Lemma 2 *Let G be a K_4-minor free graph. Then one of the following holds.*

(i) $\delta(G) \leq 1$*;*

(ii) *There exist two adjacent 2-vertices;*

(iii) *There exists a vertex u with $d_G(u) \geq 3$ such that $D_G(u) \leq 2$.*

Proof. Suppose that the lemma is false. Let G be a counterexample, i.e., G is a K_4-minor free graph satisfying the following properties:

(a) $\delta(G) = 2$;

(b) Every 2-vertex is adjacent to two vertices of degree ≥ 3;

(c) Every vertex u with $d_G(u) \geq 3$ has $D_G(u) \geq 3$.

Let H be the graph obtained from G by removing all 2-vertices and joining any two nonadjacent vertices that had a 2-vertex as a common neighbor. Then H has minimum degree at least 3 because $N_H(v) = S_G(v)$ for each vertex v with $d_G(v) \geq 3$. However, H is a K_4-minor free graph by its very construction. A contradiction is henceforth obtained. □

Proof of Theorem 1. The case for $\Delta(G) = 2$ follows directly from the well-known Brooks' Theorem since $\Delta(G) = 2$ implies $\Delta(G^2) \leq 4$. In the following we assume that $\Delta(G) \geq 3$.

Our proof proceeds by induction on the number of vertices of G. To make notation simpler, define $K(\Delta) = 6$ if $\Delta = 3$ and $K(\Delta) = \lfloor 3\Delta/2 \rfloor + 1$ if $\Delta \geq 4$.

The conclusion follows immediately if $|G| \leq 6$. Now assume that $|G| \geq 7$ and Theorem 1 holds for K_4-minor free graphs H with $|H| < |G|$.

If G has a vertex x of degree 1, then let $H = G - x$. If G has two adjacent 2-vertices x and y, then let H be the graph obtained from G by deleting x and adding an edge joining the two neighbors of x. In each of the above cases, H is a K_4-minor free graph with $\Delta(H) \leq \Delta(G)$. Moreover, $G^2 - x$ is a subgraph of H^2. By the induction hypothesis, H^2 is $K(\Delta)$-colorable. So $G^2 - x$ is $K(\Delta)$-colorable. Since x has degree $< K(\Delta)$ in G^2, any $K(\Delta)$-coloring of $G^2 - x$ can be extended to a $K(\Delta)$-coloring of G^2.

Thus we may assume that $\delta(G) = 2$ and any two 2-vertices of G are not adjacent. By Lemma 2, G has a vertex u such that $d_G(u) \geq 3$ and $D_G(u) \leq 2$. For $t \in S_G(u)$, let $M(u,t)$ denote the set of all 2-vertices in G that are adjacent to both u and t and let $m(t) = |M(u,t)|$.

Obviously $D_G(u) \geq 1$. Assume that $D_G(u) = 1$ and $S_G(u) = \{z\}$. Then all the neighbors of u are either z or some neighbors of z. Since $d_G(u) \geq 3$, we see that $m(z) \geq 2$. Let $w \in M(u,z)$ and $H = G - w$. Then H is a K_4-minor free graph with $\Delta(H) \leq \Delta(G)$. Moreover, $G^2 - w = H^2$. By the induction hypothesis, H^2 has a $K(\Delta)$-coloring ϕ. Since $d_{G^2}(w) \leq \Delta(G) + 1 < K(\Delta)$, ϕ can be extended to a $K(\Delta)$-coloring of G^2.

Assume that $D_G(u) = 2$. Let $S_G(u) = \{x, y\}$. Thus all the neighbors of u are either x, y, or some neighbors of x or y. Without loss of generality, we suppose $m(x) \geq m(y)$. Since $d_G(u) \geq 3$, we have $m(x) \geq 1$. Let $w \in M(u,x)$. The proof is divided into the following three cases.

Case 1. $x \sim u$.

Let $H = G - w$. Then H is a K_4-minor free graph with $\Delta(H) \leq \Delta(G)$ and $G^2 - w = H^2$. By the induction hypothesis, H^2 has a $K(\Delta)$-coloring ϕ. In order to extend ϕ to a $K(\Delta)$-coloring of G^2, it suffices to show that w has degree at most $K(\Delta) - 1$ in G^2. Since x is adjacent to u, $d_{G^2}(w) \leq d_G(u) + d_G(x) - m(x) - 1$. ¿From $m(x) + m(y) \geq d_G(u) - 2$ and $m(x) \geq m(y)$, we conclude that $m(x) \geq \lceil (d_G(u) - 2)/2 \rceil = \lceil d_G(u)/2 \rceil - 1$. Hence

$$\begin{aligned} d_{G^2}(w) &\leq d_G(u) + d_G(x) - m(x) - 1 \\ &\leq d_G(u) + d_G(x) - \lceil d_G(u)/2 \rceil \\ &\leq \lfloor 3\Delta(G)/2 \rfloor \\ &= K(\Delta) - 1. \end{aligned}$$

Case 2. $x \not\sim u$ and $y \not\sim u$.

Similarly as above, let $H = G - w$. Then H is a K_4-minor free graph with $\Delta(H) \leq \Delta(G)$. Note that x and u have distance at most 2 in H since $d_G(u) \geq 3$ and $m(x) \geq m(y)$. Hence $G^2 - w = H^2$. It is easy to see that $m(x) + m(y) = d_G(u)$ and $d_{G^2}(w) = d_G(u) + d_G(x) - m(x) + 1$. From $m(x) \geq m(y)$ we conclude that $m(x) \geq \lceil d_G(u)/2 \rceil$. Hence

$$\begin{aligned} d_{G^2}(w) &\leq d_G(u) + d_G(x) - m(x) + 1 \\ &\leq d_G(u) + d_G(x) - \lceil d_G(u)/2 \rceil + 1 \\ &\leq \lfloor 3\Delta(G)/2 \rfloor + 1 \\ &= K(\Delta). \end{aligned}$$

By the induction hypothesis, H^2 has a $K(\Delta)$-coloring ϕ, which acts as a partial coloring of G^2 such that all the vertices except w have been properly colored. Since $d_{G^2}(u) \leq \Delta(G) + 2 \leq K(\Delta) - 1$, u has at most $K(\Delta) - 2$ colored neighbors in G^2 in this partial coloring. Hence there are at least two choices for properly coloring u before coloring w. One of these two colorings of u will imply that the neighbors of w have at most $K(\Delta)-1$ colors. So the coloring ϕ can be extended to a $K(\Delta)$-coloring of G^2.

Case 3. $x \not\sim u$ and $y \sim u$.

If $m(x) = m(y)$, we may interchange x and y, and it falls under Case 1. So assume that $m(x) > m(y)$.

First suppose $d_G(u)$ is odd. Then $m(x) + m(y) = d_G(u) - 1$ is even.

If $m(x) \geq m(y) + 4$, then $m(x) \geq (d_G(u) + 3)/2$. This implies that

$$\begin{aligned} d_{G^2}(w) &\leq d_G(u) + d_G(x) - m(x) + 1 \\ &\leq \lfloor 3\Delta(G)/2 \rfloor \\ &= K(\Delta) - 1. \end{aligned}$$

Let $H = G - w$. As before, $H^2 = G^2 - w$ and every $K(\Delta)$-coloring of H^2 can be extended to a $K(\Delta)$-coloring of G^2.

If $m(x) = m(y) + 2$, we have $m(x) = (d_G(u) + 1)/2$. If $d_G(u) < \Delta(G)$, then $d_{G^2}(w) \leq \lfloor 3\Delta(G)/2 \rfloor = K(\Delta) - 1$, and the same argument applies to this case. Thus we assume that $d_G(u) = \Delta(G)$.

If $x \sim y$, then $d_{G^2}(w) \leq d_G(u) + d_G(x) - 1 - (m(x) - 1) \leq \lfloor 3\Delta(G)/2 \rfloor$. Any $K(\Delta)$-coloring of $G^2 - w$ can be extended to a $K(\Delta)$-coloring of G^2. Assume now that $x \not\sim y$. In this case, the calculation shows that $d_{G^2}(w) \leq K(\Delta)$. We now show how to choose a $K(\Delta)$-coloring of $G^2 - w$ so that two neighbors of w in G^2 have the same color. Such a $K(\Delta)$-coloring then can be extended to w.

Since $d_G(u) = \Delta(G) \geq 5$, we may choose some $w' \in M(u, y) \neq \emptyset$. Let $H = G - w - w' + xy$. It is easy to see that H is also a K_4-minor free graph with $\Delta(H) \leq \Delta(G)$. Moreover, $G^2 - w - w'$ is a subgraph of H^2. By the induction hypothesis, H^2 has a $K(\Delta)$-coloring ϕ. Note that x has a color different from those of y and all neighbors of y in H. Going back to the graph G, we color w' with the color $\phi(x)$. This coloring can be extended to w because x and w' are neighbors of w in G^2 with the same color.

Now we consider the case that $d_G(u)$ is even. Then $m(x) + m(y) = d_G(u) - 1$ is odd. If $m(x) \geq m(y) + 3$, we have $m(x) \geq d_G(u)/2 + 1$. If $d_G(u) < \Delta(G)$, then

$$\begin{aligned} d_{G^2}(w) &\leq d_G(u) + d_G(x) - m(x) + 1 \\ &\leq d_G(x) + d_G(u)/2 \\ &\leq \Delta(G) + (\Delta(G) - 1)/2 \\ &\leq \lfloor 3\Delta(G)/2 \rfloor \\ &= K(\Delta) - 1. \end{aligned}$$

If $d_G(u) = \Delta(G)$, the same calculation shows that $d_{G^2}(w) \leq d_G(x) + d_G(u)/2 \leq \lfloor 3\Delta(G)/2 \rfloor$. So any $K(\Delta)$-coloring of $G^2 - w$ can be extended to G^2.

Assume that $m(x) = m(y) + 1$. Since $d_G(u) \geq 4$, $m(y) = d_G(u)/2 - 1 \geq 1$. Choose $w' \in M(u, y)$. Then

$$\begin{aligned} d_{G^2}(w') &\leq d_G(u) + d_G(y) - m(y) - 1 \\ &\leq d_G(u) + d_G(y) - d_G(u)/2 \\ &\leq \lfloor 3\Delta(G)/2 \rfloor \\ &= K(\Delta) - 1. \end{aligned}$$

Let $H = G - w'$. Then $H^2 = G^2 - w'$, and any $K(\Delta)$-coloring of H^2 can be extended to G^2. □

3 Remarks

A graph G is called *k-degenerate* if every subgraph H of G has $\delta(H) \leq k$. The *degeneracy* of G is the minimum k such that G is k-degenerate. The *coloring number* $\mathrm{col}(G)$ of a graph G is one plus the degeneracy of G.

In most cases of the proof of Theorem 1, we found a vertex w of G such that $d_{G^2}(w) \leq K(\Delta) - 1$ and $G^2 - w$ was a subgraph of the square of a K_4-minor free graph H with $|H| = |G| - 1$. Yet there were two cases where $d_{G^2}(w) = K(\Delta)$, in which case $K(\Delta) = \lfloor 3\Delta(G)/2 \rfloor$, so the proof actually establishes the following result.

Theorem 3 *Let G be a K_4-minor free graph. Then*

$$\mathrm{col}(G^2) \leq \begin{cases} \Delta(G) + 3 & \text{if } 2 \leq \Delta(G) \leq 3; \\ \lfloor 3\Delta(G)/2 \rfloor + 2 & \text{if } \Delta(G) \geq 4. \end{cases}$$

For $\Delta(G) = 2$ and 3, the above upper bound for $\mathrm{col}(G^2)$ is the same as the upper bound for $\chi(G^2)$ in Theorem 1, hence it is sharp. If $\Delta(G)$ is an even number $2k \geq 4$, then let G'_{2k} be the graph consisting of vertices x_1, x_2, x_3, x_4 and k internally disjoint paths of length 2 joining x_i and x_{i+1} (addition modulo 4). It is easy to verify that G'_{2k} is a K_4-minor free graph, $\Delta(G'_{2k}) = 2k$, and $\delta((G'_{2k})^2) = 3k + 1$. So the coloring number of $(G'_{2k})^2$ is at least $3k + 2$. If $\Delta(G)$ is an odd number $2k + 1 \geq 5$, then let G'_{2k+1} be obtained from G'_{2k} as follows. Replace a length 2 path between x_1 and x_2 by an edge and a length 2 path between x_3 and x_4 by an edge. Then add one length 2 path connecting x_1 and x_4 and one length 2 path connecting x_2 and x_3. Then $\Delta(G'_{2k+1}) = 2k + 1$ and $\delta((G'_{2k+1})^2) = 3k + 2$. So the coloring number of $(G'_{2k+1})^2$ is at least $3k + 3$. In conclusion, Theorem 3 gives sharp bounds.

References

[1] G. Agnarsson and M. M. Halldórsson, Coloring powers of planar graphs, submitted to SIAM J. Discrete Math, 2000.

[2] O. V. Borodin, H. J. Broersma, A. Glebov, and J. van den Heuvel, Stars and bunches in planar graphs. part II: general planar graphs and colourings, CDAM Research Report, London School of Economics and Political Science, 2002, a translated and adapted version of a paper that appeared in Diskretn. Anal. Issled. Oper. Ser. 1, 8(2001) pp. 9-33 (in Russian).

[3] G. Chartrand and F. Harary, Planar permutation graphs, Ann. Inst. H. Poincaré Sect. B (N.S.) 3 (1967) 433-438.

[4] R. J. Duffin, Topology of series-parallel networks, J. Math. Anal. Appl. 10 (1965) 303-318.

[5] M. Molloy and M. R. Salavatipour, A bound on the chromatic number of the square of a graph, preprint, 2002.

[6] J. van den Heuvel and S. McGuinness, Coloring the square of a planar graph, J. Graph Theory 42(2003) 110-124.

[7] C. Thomassen, Applications of Tutte cycles, Technical Report, Technical University of Denmark, 2001.

[8] Wei-Fan Wang and Ko-Wei Lih, Labeling planar graphs with conditions on girth and distance two, to appear in SIAM J. Discrete Math.

[9] G. Wegner, Graphs with given diameter and a coloring problem, Technical Report, University of Dortmund, 1977.