Incidence coloring of k-degenerated graphs

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Abstract

We prove that the incidence coloring number of every k-degenerated graph G is at most $\Delta(G) + 2k - 1$. For K_4-minor free graphs ($k = 2$), we decrease this bound to $\Delta(G) + 2$, which is tight. For planar graphs ($k = 5$), we decrease this bound to $\Delta(G) + 7$.

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1 Introduction

All the graphs we consider are finite and simple. For a graph G, we respectively denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ its vertex set, edge set, minimum degree and maximum degree. For a vertex v in G we denote by $N_G(v)$ the set of its neighbors and by $d_G(v) = |N_G(v)|$ its degree. A vertex of degree k will be called a *k-vertex*.

An *incidence* in G is a pair (v, e) with $v \in V(G)$, $e \in E(G)$, such that v and e are incident. We denote by $I(G)$ the set of all incidences in G. For every vertex v, we denote by I_v the set of incidences of the form (v, vw) and by A_v the set of incidences of the form

(w, wv). Two incidences (v, e) and (w, f) are *adjacent* if one of the following holds: (i) $v = w$, (ii) $e = f$ or (iii) the edge vw equals e or f.

A k-*incidence coloring* of a graph G is a mapping σ of $I(G)$ to a set C of k colors such that adjacent incidences are assigned distinct colors. The *incidence chromatic number* $\chi_i(G)$ of G is the smallest k such that G admits a k-incidence coloring.

Incidence colorings have been introduced by Brualdi and Massey [3] in 1993. It is easy to see that for every graph G with at least one edge, $\chi_i(G) \geq \Delta(G) + 1$. Brualdi and Massey proved the following upper bound:

Theorem 1 [3] *For every graph G, $\chi_i(G) \leq 2\Delta(G)$.*

In [6], Guiduli observed that the concept of incidence coloring is a particular case of directed star arboricity, introduced by Algor and Alon [1]. Following an example from [1], Guiduli proved that there exist graphs G with $\chi_i(G) \geq \Delta(G) + \Omega(\log \Delta(G))$. He also proved the following upper bound:

Theorem 2 [6] *For every graph G, $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$.*

Concerning the incidence chromatic number of special classes of graphs, the following is known:

- For every $n \geq 2$, $\chi_i(K_n) = n = \Delta(K_n) + 1$ [3].
- For every $m \geq n \geq 2$, $\chi_i(K_{m,n}) = m + 2 = \Delta(K_{m,n}) + 2$ [3].
- For every tree T of order $n \geq 2$, $\chi_i(T) = \Delta(T) + 1$ [3].
- For every Halin graph G with $\Delta(G) \geq 5$, $\chi_i(G) = \Delta(G) + 1$ [5].
- For every outerplanar graph G with $\Delta(G) \geq 4$, $\chi_i(G) = \Delta(G) + 1$ [5].

In [4], Chen, Lam and Shiu proposed the following:

Conjecture 3 [4] *If G is a cubic graph then $\chi_i(G) \leq \Delta(G) + 2$.*

They proved that this conjecture is true for some classes of cubic graphs, for instance the class of Hamiltonian cubic graphs.

In view of these results we are interested in classes of graphs for which the incidence chromatic number is bounded by the maximum degree plus some fixed constant not depending on the maximum degree of the graph. We consider in particular the class of k-degenerated graphs (recall that a graph G is k-degenerated if $\delta(H) \leq k$ for every subgraph H of G), which includes for instance the classes of partial k-trees or of graphs embeddable on a surface of given genus. More precisely, we shall prove in this paper the following:

1. If G is a k-degenerated graph, then $\chi_i(G) \leq \Delta(G) + 2k - 1$.

2. If G is a K_4-minor free graph, then $\chi_i(G) \leq \Delta(G) + 2$, and this bound is tight.

3. If G is a planar graph, then $\chi_i(G) \leq \Delta(G) + 7$.

In fact we shall prove something stronger, namely that one can construct for these classes of graphs incidence colorings such that for every vertex v, the number of colors that are used on the incidences of the form (w, wv) is bounded by some fixed constant not depending on the maximum degree of the graph.

More precisely, we define a (k, ℓ)-*incidence coloring* of a graph G as a k-incidence coloring σ of G such that for every vertex $v \in V(G)$, $|\sigma(A_v)| \leq \ell$.

We end this section by introducing some notation that we shall use in the rest of the paper.

Let G be a graph. If v is a vertex in G and vw is an edge in G, we denote by $G \setminus v$ the graph obtained from G by deleting the vertex v and by $G \setminus vw$ the graph obtained from G by deleting the edge vw. If vx is not an edge in G, we denote by $G + vx$ the graph obtained from G by adding the edge vx.

Let G be a graph and σ' a *partial* incidence coloring of G, that is an incidence coloring only defined on some subset I of $I(G)$. For every uncolored incidence $(v, vw) \in I(G) \setminus I$, we denote by $F_G^{\sigma'}(v, vw)$ the set of *forbidden colors* of (v, vw), that is:

$$F_G^{\sigma'}(v, vw) \;=\; \sigma'(A_v) \cup \sigma'(I_v) \cup \sigma'(I_w).$$

We shall often say that we extend such a partial incidence coloring σ' to some incidence coloring σ of G. In that case, it should be understood that we set $\sigma(v, vw) = \sigma'(v, vw)$ for every incidence $(v, vw) \in I$.

Finally, we shall make extensive use of the fact that every (k, ℓ)-incidence coloring may be viewed as a (k', ℓ)-incidence coloring for any $k' > k$.

2 k-degenerated graphs

The aim of this section is to prove the following:

Theorem 4 *Every k-degenerated graph G admits a $(\Delta(G)+2k-1, k)$-incidence coloring.*

Proof. Suppose to the contrary that the theorem is false and let G be a minimal counter-example. We can assume without loss of generality that G is connected. Let v be a t-vertex in G, $t \leq k$, with $N_G(v) = \{x_1, \ldots, x_t\}$ and let $G' = G \setminus v$. Due to the minimality of G, there exists a $(\Delta(G) + 2k - 1, k)$-incidence coloring σ' of G'. We shall extend σ' to a $(\Delta(G) + 2k - 1, k)$-incidence coloring σ of G. We start by proving the following:

CLAIM. *For every i, $1 \leq i \leq t$, there exists a color a_i such that $a_i \notin F_G^{\sigma'}(v, vx_i) \cup \{a_1, \ldots, a_{i-1}\}$ and $|\sigma'(A_{x_i}) \cup \{a_i\}| \leq k$.*

Consider first $i = 1$. If $\sigma'(A_{x_1}) \neq \emptyset$ then a_1 can be any color in $\sigma'(A_{x_1})$, otherwise (that is if $d_G(x_1) = 1$) a_1 can be any color. Suppose now that we have obtained $i-1$ colors $a_1, \ldots, a_{i-1}$, $i - 1 < t$, satisfying the claim. If $|\sigma'(A_{x_i})| = k$, we take any $a_i \in \sigma'(A_{x_i}) \setminus \{a_1, \ldots, a_{i-1}\}$ (recall that $i \leq k$). Otherwise, $|F_G^{\sigma'}(v, vx_i) \cup \{a_1, \ldots, a_{i-1}\}| \leq d_G(x_i) - 1 + i - 1 \leq \Delta(G) + i - 2 \leq \Delta(G) + k - 2$. Therefore, one can choose some color $a_i \notin F_G^{\sigma'}(v, vx_i) \cup \{a_1, \ldots, a_{i-1}\}$ and the claim is proved.

Thanks to the above claim, we can set $\sigma(v, vx_i) = a_i$ for every i, $1 \leq i \leq t$.

Now, since for every i, $1 \leq i \leq t$, we have $|\sigma'(A_{x_i}) \cup \{a_i\}| \leq k$, we get that the number of forbidden colors for the incidence (x_i, x_iv) satisfies $|\sigma'(I_{x_i}) \cup \sigma'(A_{x_i}) \cup \{a_1, \ldots, a_t\}| \leq \Delta(G) - 1 + k + t - 1 \leq \Delta(G) + 2k - 2$. Hence for every i, $1 \leq i \leq t$, there exists one free color $b_i \notin F_G^{\sigma'}(x_i, x_iv) \cup \{a_1, \ldots, a_t\}$ and we can set $\sigma(x_i, x_iv) = b_i$.

The so-obtained coloring σ is clearly a $(\Delta(G) + 2k - 1, k)$-incidence coloring of G. We thus get a contradiction and the theorem is proved. ∎

Since K_4-minor free graphs are 2-degenerated, we get in particular that every K_4-minor free graph G admits a $(\Delta(G) + 3, 2)$-incidence coloring. This result will be improved in Section 3.

Similarly, since planar graphs are 5-degenerated, we get that every planar graph G admits a $(\Delta(G) + 9, 5)$-incidence coloring. This result will be improved in Section 4.

3 K_4-minor free graphs

We shall make use of the following structural lemma due to Lih, Wang and Zhu [7]. For a graph G and a vertex $v \in V(G)$, we denote by $D_G(v)$ the cardinality of the set

$$\{u \in V(G) \mid [d_G(u) \geq 3 \text{ and } uv \in E(G)] \text{ or } [\exists w \in V(G),\ d_G(w) = 2,\ uw, wv \in E(G)]\}.$$

Then we have:

Lemma 5 [7] *Let G be a K_4-minor free graph. Then one of the following holds:*
(1) $\delta(G) \leq 1$;
(2) There exist two adjacent 2-vertices;
(3) There exists a vertex u with $d_G(u) \geq 3$ such that $D_G(u) \leq 2$.

We can now prove the main result of this section:

Theorem 6 *Every K_4-minor free graph G admits a $(\Delta(G) + 2, 2)$-incidence coloring.*

Proof. Suppose that the theorem is false and let G be a minimal counter-example. We can assume without loss of generality that G is connected. According to lemma 5, we have three cases to consider.

1. G contains a 1-vertex v.

Let w denote the unique neighbor of v in G. Due to the minimality of G, there exists a $(\Delta(G)+2,2)$-incidence coloring σ' of $G' = G \setminus v$. Since $|F_G^{\sigma'}(w,wv)| = |\sigma'(I_w) \cup \sigma'(A_w)| \leq d_G(w) - 1 + 2 \leq \Delta(G)+1$, there exists a color a such that $a \notin F_G^{\sigma'}(w,wv)$. We can then extend σ' to a $(\Delta(G)+2,2)$-incidence coloring σ of G by setting $\sigma(w,wv) = a$ and $\sigma(v,vw) = b$ for some $b \in \sigma'(A_w)$ (if G' has no edge we simply take $b \neq a$).

2. $\delta(G) > 1$ and G contains two adjacent 2-vertices v and w.

   If $\Delta(G) = 2$ then $\chi_i(G) \leq 4$ by Theorem 1. Moreover, every 4-incidence coloring of a cycle is clearly a (4,2)-incidence coloring.

   Therefore, $\Delta(G) \geq 3$. Let $G' = G \setminus vw$. Denote by v' the unique neighbor of v and by w' the unique neighbor of w in G'. Due to the minimality of G, there exists a $(\Delta(G)+2,2)$-incidence coloring σ' of G'. Let $a = \sigma'(w',w'w)$, $b = \sigma'(w,ww')$, $c = \sigma'(v',v'v)$ and $d = \sigma'(v,vv')$.

   Suppose first that $|\{a,b,c,d\}| = 4$. We can extend σ' to a $(\Delta(G)+2,2)$-incidence coloring σ of G by setting $\sigma(v,vw) = a$ and $\sigma(w,wv) = c$.

   Now, if $|\{a,b,c,d\}| = 3$, we can extend σ' to a $(\Delta(G)+2,2)$-incidence coloring σ of G by setting $\sigma(v,vw) = e$ and $\sigma(w,wv) = f$ for any $e,f \notin \{a,b,c,d\}$, since $\Delta(G)+2 \geq 5$.

3. None of the two previous cases occurs. In that case, G contains a vertex v with $d_G(v) \geq 3$ and $D_G(v) \leq 2$.

   Suppose first that $D_G(v) = 1$ and denote by $x_1,\ldots,x_t$ the 2-neighbors of v. We clearly have $t \geq 2$ and all these 2-vertices are linked to a k-vertex w, $k \geq 3$ (recall that G has no 1-vertex and no pair of adjacent 2-vertices). Moreover, if $t = 2$ then G necessarily contains the edge vw since $d_G(v) \geq 3$. We consider the following two subcases.

   (a) $vw \in E(G)$.
   Let $G' = G \backslash vx_1$ and let σ' be a $(\Delta(G)+2,2)$-incidence coloring of G'. Moreover, let $a = \sigma'(v,vw)$ and $b = \sigma'(w,wv)$. Note that a is a legal color for the incidence (x_1,x_1w). Thus by a recoloring if necessary, we may assume that $\sigma'(x_1,x_1w) = a$. We can extend σ' to a $(\Delta(G)+2,2)$-incidence coloring σ of G as follows. We first set $\sigma(x_1,x_1v) = b$. Now, since $|F_G^{\sigma}(v,vx_1)| = |\sigma'(I_v \setminus (v,vx_1)) \cup \sigma'(A_v)| \leq d_G(v) - 1 + 2 \leq \Delta(G)+1$, there exists a color c such that $c \notin F_G^{\sigma'}(v,vx_1)$ and we set $\sigma(v,vx_1) = c$.

   (b) $vw \notin E(G)$.
   In that case, we have $t \geq 3$. Let $G' = G \setminus v$ and let σ' be a $(\Delta(G)+2,2)$-incidence coloring of G'. Observe that the color $\sigma'(x_1,x_1w)$ can be used for coloring all the incidences (x_i,x_iw), $2 \leq i \leq t$. Therefore, we can choose σ' in such a way that $\sigma'(x_i,x_iw) = a$ for every i, $1 \leq i \leq t$. We can extend σ' to a $(\Delta(G)+2,2)$-incidence coloring σ of G as follows. Since for every i, $1 \leq i \leq t$, $F_G^{\sigma'}(x_i,x_iv) = \sigma'(I_w) \cup \{a\}$ and $d_G(w) \leq \Delta(G)$, there exists one free color, say b, for coloring the incidences (x_i,x_iv). We then set $\sigma(x_i,x_iv) = b$, $1 \leq i \leq t$. Finally, only the two colors a and b are forbidden for coloring the t incidences of the form (v,vx_i) by distinct colors. Since $t \leq \Delta(G)$ this can be done.

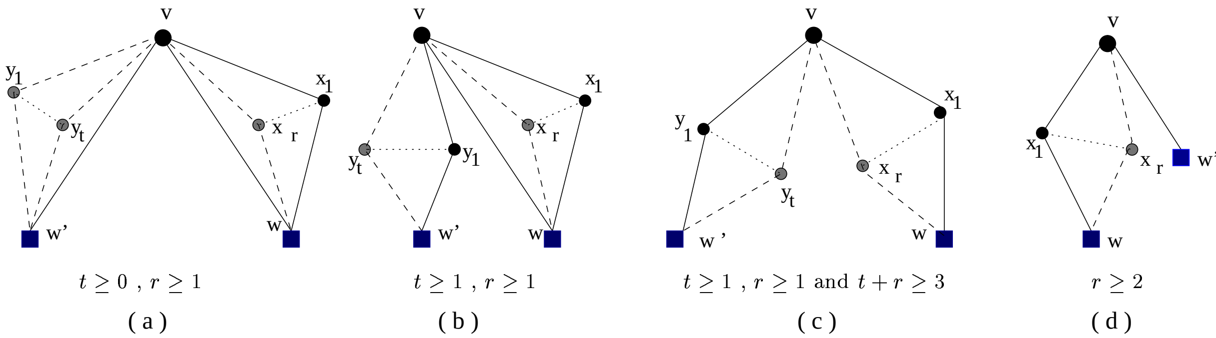

Figure 1: Configurations for the proof of Theorem 6

Suppose now that $D_G(v) = 2$. We have four subcases to consider, according to the four configurations depicted in Figure 1.

(a) Let $G' = G \setminus vx_1$ and let σ' be a $(\Delta(G) + 2, 2)$-incidence coloring of G'. As above, we can choose σ' in such a way that $\sigma'(y_i, y_iw') = a$ and $\sigma'(y_i, y_iv) = \sigma'(w', w'v) = b$ for every i, $1 \leq i \leq t$. Moreover, let $c = \sigma'(v, vw)$, $d = \sigma'(w, wv)$, $e = \sigma'(x_1, x_1w)$ and $f = \sigma'(w, wx_1)$.

We can extend σ' to a $(\Delta(G) + 2, 2)$-incidence coloring σ of G as follows. We first set $\sigma(x_1, x_1w) = c$ and $\sigma(x_1, x_1v) = d$. Now, since $|F^{\sigma}_G(v, vx_1)| = |\sigma'(I_v \setminus (v, vx_1)) \cup \sigma'(A_v)| \leq d_G(v) - 1 + 2 \leq \Delta(G) + 1$, there exists a color g such that $g \notin F^{\sigma'}_G(v, vx_1)$ and we set $\sigma(v, vx_1) = g$.

(b) This case is solved as in the previous case, except that we do not need to consider the incidence (v, vw').

(c) Let $G' = G \setminus v$ and let σ' be a $(\Delta(G) + 2, 2)$-incidence coloring of G'. Since we have at least two possibilities for choosing $\sigma'(x_i, x_iw)$, $1 \leq i \leq r$, and $\sigma'(y_j, y_jw')$, $1 \leq j \leq t$, σ' can be chosen in such a way that $\sigma'(x_i, x_iw) = a$ for every i, $1 \leq i \leq r$, $\sigma'(y_j, y_jw') = b$ for every j, $1 \leq j \leq t$, and $a \neq b$.

We can extend σ' to a $(\Delta(G)+2, 2)$-incidence coloring σ of G as follows. We first set $\sigma(v, vx_1) = b$ and $\sigma(v, vy_1) = a$. Since $r < d_G(v) \leq \Delta(G)$, there exists a color $c \notin \{a, b\} \cup \{\sigma'(w, wx_i) : 1 \leq i \leq r\}$. Similarly, since $t < d_G(v) \leq \Delta(G)$, there exists a color $d \notin \{a, b\} \cup \{\sigma'(w', w'y_j) : 1 \leq j \leq t\}$. We then set $\sigma(x_i, x_iv) = c$, $1 \leq i \leq r$, and $\sigma(y_j, y_jv) = d$, $1 \leq j \leq t$. Finally, we need $r + t - 2$ colors distinct from a, b, c and d for coloring the incidences (v, vx_i), $2 \leq i \leq r$, and (v, vy_j), $2 \leq j \leq t$. This can be done since $r+t = d_G(v) \leq \Delta(G)$.

(d) We consider two subcases, according to the degree of v in G.

   i. $d_G(v) < \Delta(G)$.
   Let $G' = G \setminus \{vx_i : 1 \leq i \leq r\}$ and let σ' be a $(\Delta(G) + 2, 2)$-incidence coloring of G'. As in case (a), σ' can be chosen in such a way that $\sigma'(x_i, x_iw) = a$, $1 \leq i \leq r$. Moreover, let $b = \sigma'(w', w'v)$ and $c = \sigma'(v, vw')$. We can extend σ' to a $(\Delta(G) + 2, 2)$-incidence coloring σ of G as follows. Let $F = \bigcup_{i=1}^{r} F^{\sigma'}_G(x_i, x_iv) = \{a\} \cup \{\sigma'(w, wx_i) : 1 \leq i \leq r\}$; since $r < d_G(v) < \Delta(G)$, we can set $\sigma(x_i, x_iv) = d$ for every i, $1 \leq i \leq r$, with $d \notin F$. Finally, we need r colors distinct from a, b, c and d for coloring

the incidences (v, vx_i), $1 \leq i \leq r$. This can be done since $r \leq d_G(v) - 1 \leq \Delta(G) - 2$.

ii. $d_G(v) = \Delta(G)$.
Let $G' = G \backslash \{x_i,\ 1 \leq i \leq r\}$ and let σ' be a $(\Delta(G)+2, 2)$-incidence coloring of G'. Let $a = \sigma'(w', w'v)$ and $b = \sigma'(v, vw')$, $b \neq a$. If $d_G(w) = \Delta(G) - 1$, then $D_G(w) = 1$, which is a case covered already. Thus we may assume that $d_G(w) = \Delta(G)$.
Let v' be the unique neighbor of w in G'. Let $c = \sigma'(v', v'w)$ and $d = \sigma'(w, wv')$, $d \neq c$. We set $\sigma(x_i, x_i w) = c$, $1 \leq i \leq r$. Now, if $a \neq c$, we set $\sigma(x_i, x_i v) = a$ for every i, $1 \leq i \leq r$, and, if $a = c$, we set $\sigma(x_i, x_i v) = z$, $1 \leq i \leq r$, for some $z \notin \{a, b, d\}$. It is easy to verify that in both cases $|\bigcup_{k=1}^{r} F_G^{\sigma}(v, vx_k)| = |\bigcup_{k=1}^{r} F_G^{\sigma}(w, wx_k)| = 3$. Since $r = d_G(w) - 1 = \Delta(G) - 1$, we can color the incidences (v, vx_i) and the incidences (w, wx_i), $1 \leq i \leq r$.

Therefore, we get in each case a $(\Delta(G)+2, 2)$-incidence coloring of G, which contradicts our assumption, and the theorem is proved. ∎

The following proposition shows that the bound given in Theorem 6 is tight:

Proposition 7 *For every $k \geq 1$, there exist infinitely many K_4-minor free graphs with maximum degree k and incidence chromatic number $k + 2$.*

Proof. Let G_k be the K_4-minor free graph obtained by linking two vertices u and v by k distinct paths of length 2 whose inner 2-vertices are denoted respectively by $w_1, w_2, \ldots, w_k$. We clearly have $\Delta(G_k) = k$. Suppose to the contrary that G_k admits a $(k + 1)$-incidence coloring. Since we need k distinct colors for coloring I_u, all the incidences of the form $(w_i, w_i u)$ are assigned the same color, say a. Similarly, since we need k distinct colors for coloring I_v, all the incidences of the form $(w_i, w_i v)$ are assigned the same color, say b. But the color b has to be distinct from a and from the k colors assigned to I_u, a contradiction.

Finally, every K_4-minor free graph with maximum degree k and containing G_k as a subgraph has incidence chromatic number $k+2$. (Such graphs can be obtained for instance by linking distinct paths of arbitrary length to intermediate vertices $w_1, w_2, \ldots, w_k$). ∎

4 Planar graphs

We shall use the following structural lemma which follows from Euler's formula [2]:

Lemma 8 *Let G be a planar graph. Then one of the following holds:*
(1) $\delta(G) \leq 2$;
(2) There exists an edge vw in G with $d_G(v) = 3$ and $d_G(w) \leq 10$;
(3) There exists an edge vw in G with $d_G(v) = 4$ and $d_G(w) \leq 8$;
(4) There exists an edge vw in G with $d_G(v) = 5$ and $d_G(w) \leq 6$.

We can now prove the main result of this section:

Theorem 9 *Every planar graph G admits a $(\Delta(G)+7,7)$-incidence coloring.*

Proof. Suppose that the theorem is false and let G be a minimal counter-example. We can assume without loss of generality that G is connected. Observe first that we necessarily have $\Delta(G) \geq 8$ since otherwise we get by Theorem 1 that $\chi_i(G) \leq 2\Delta(G) \leq \Delta(G)+7$ and every $(\Delta(G)+7)$-incidence coloring of G is obviously a $(\Delta(G)+7,7)$-incidence coloring.

We consider five cases, according to Lemma 8 (for each case, we assume that none of the previous cases occur).

1. G contains a 1-vertex v.

   Let w denote the unique neighbor of v in G. Due to the minimality of G, the graph $G' = G \setminus v$ admits a $(\Delta(G)+7,7)$-incidence coloring σ'. We extend σ' to a $(\Delta(G)+7,7)$-incidence coloring σ of G as follows. Since $|F_G^{\sigma'}(w,wv)| = |\sigma'(I_w) \cup \sigma'(A_w)| \leq \Delta(G)-1+7 = \Delta(G)+6$, there is a free color, say a, that can be assigned to the incidence (w,wv) and we thus set $\sigma(w,wv)=a$. Now, it suffices to set $\sigma(v,vw)=b$ for any color b in $\sigma'(A_w)$.

2. G contains a 2-vertex v.

   Let w and w' denote the two neighbors of v in G. Suppose first that ww' is an edge in G. Due to the minimality of G, the graph $G' = G \setminus v$ admits a $(\Delta(G)+7,7)$-incidence coloring σ'. Let $a = \sigma'(w,ww')$ and $b = \sigma'(w',w'w)$. We extend σ' to a $(\Delta(G)+7,7)$-incidence coloring σ of G as follows. We first set $\sigma(v,vw)=b$ and $\sigma(v,vw')=a$. Since $|F_G^{\sigma}(w,wv)| = d_G(w)-1+|\sigma(A_w)| \leq \Delta(G)-1+7 = \Delta(G)+6$, there exists a color $c \notin F_G^{\sigma}(w,wv)$. Similarly, since $|F_G^{\sigma}(w',w'v)| \leq \Delta(G)+6$, there exists a color $d \notin F_G^{\sigma}(w',w'v)$. By setting $\sigma(w,wv)=c$ and $\sigma(w',w'v)=d$, we clearly obtained a $(\Delta(G)+7,7)$-incidence coloring of G.

   Now, if ww' is not an edge in G, we consider the graph $G' = (G \setminus v)+ww'$. Due to the minimality of G, the graph G' admits a $(\Delta(G)+7,7)$-incidence coloring σ'. Again, let $a = \sigma'(w,ww')$ and $b = \sigma'(w',w'w)$. We extend σ' to a $(\Delta(G)+7,7)$-incidence coloring σ of G as follows. We first set $\sigma(w,wv)=a$ and $\sigma(w',w'v)=b$.

   If $|\sigma'(A_w)| \geq 2$ then there exists a color $c \in \sigma'(A_w)$ such that $c \neq b$ and we set $\sigma(v,vw)=c$. Otherwise, that is $\sigma'(A_w)=\{b\}$, we have $|F_G^{\sigma}(v,vw)| = |\sigma'(I_w) \cup \sigma'(A_w)| \leq \Delta(G)+1$; therefore, there exists a color $c \notin F_G^{\sigma}(v,vw)$ and we set $\sigma(v,vw)=c$.

   Now, if $|\sigma'(A_{w'})| \geq 3$ then there exists a color $d \in \sigma'(A_{w'})$ such that $d \notin \{a,c\}$ and we set $\sigma(v,vw')=d$. Otherwise, we have $|F_G^{\sigma}(v,vw')| = |\sigma'(I_{w'}) \cup \sigma'(A_{w'}) \cup \{c\}| \leq \Delta(G)+2+1 = \Delta(G)+3$; therefore, there exists a color $d \notin F_G^{\sigma}(v,vw')$ and we set $\sigma(v,vw')=d$.

3. G contains an edge vw such that $d_G(v)=3$ and $3 \leq d_G(w) \leq 10$.

   Due to the minimality of G, the graph $G' = G \setminus vw$ admits a $(\Delta(G)+7,7)$-incidence coloring σ'. We extend σ' to a $(\Delta(G)+7,7)$-incidence coloring σ of G as follows.

   For coloring the incidence (w,wv) we consider the five following subcases.

(a) $d_G(w) \leq 7$.
Since $|F_G^{\sigma'}(w, wv)| = |\sigma'(I_w) \cup \sigma'(A_w) \cup \sigma'(I_v)| \leq 6+6+2 = 14$ and $\Delta(G)+7 \geq 8+7 = 15$, there exists a color $a \notin F_G^{\sigma'}(w, wv)$ and we set $\sigma(w, wv) = a$.

(b) $d_G(w) = 8$ and ($\Delta(G) \geq 10$ or $|\sigma'(A_w)| \leq 5$).
Since $|\sigma'(I_w)| = d_G(w)-1$, we have $|F_G^{\sigma'}(w, wv)| = d_G(w)-1+|\sigma'(A_w) \cup \sigma'(I_v)|$. If $\Delta(G) \geq 10$, we get $|F_G^{\sigma'}(w, wv)| \leq 7+7+2 = 16 \leq \Delta(G)+6$. Similarly, if $|\sigma'(A_w)| \leq 5$ we get $|F_G^{\sigma'}(w, wv)| \leq \Delta(G)-1+5+2 = \Delta(G)+6$. In both cases, there exists a color a such that $a \notin F_G^{\sigma'}(w, wv)$ and we set $\sigma(w, wv) = a$.

(c) $d_G(w) = 8$, $\Delta(G) \leq 9$ and $|\sigma'(A_w)| \geq 6$.
Let $N_G(v) = \{w, x_1, x_2\}$. We first claim that we can recolor the two incidences (v, vx_1) and (v, vx_2) by using two colors c_1 and c_2 such that $c_1, c_2 \in \sigma'(A_w) \cup \sigma'(I_w)$.
To see that, observe that if for some $i \in \{1, 2\}$, $|\sigma'(A_{x_i})| = 7$, then $|\sigma'(A_{x_i}) \setminus \sigma'(A_v)| \geq 7-2+1 = 6$ and $|\sigma'(A_w) \cup \sigma'(I_w)| \geq 6+7 = 13$. Since the total number of colors is at most 16, $|(\sigma'(A_{x_i}) \setminus \sigma'(A_v)) \cap (\sigma'(A_w) \cup \sigma'(I_w))| \geq 3$. So we have at least three possible choices for c_i. On the other hand, if $|\sigma'(A_{x_i})| \leq 6$ then we have at least three possible choices for c_i since $|\sigma'(A_w) \cup \sigma'(I_w)| \geq 13$ and $|\sigma'(I_{x_i}) \cup \sigma'(A_v)| \leq 9+2-1 = 10$.
By setting $\sigma(v, vx_1) = c_1$ and $\sigma(v, vx_2) = c_2$, we get that $|F_G^{\sigma'}(w, wv)| = |\sigma'(A_w) \cup \sigma'(I_w) \cup \sigma'(I_v)| = |\sigma'(A_w) \cup \sigma'(I_w)| \leq 7+d_G(w)-1 = d_G(w)+6 \leq \Delta(G)+6$. Therefore, there exists a color a such that $a \notin F_G^{\sigma'}(w, wv)$ and we can set $\sigma(w, wv) = a$.

(d) $9 \leq d_G(w) \leq 10$ and ($\Delta(G) \geq 12$ or $|\sigma'(A_w)| \leq 5$).
Again, since $|\sigma'(I_w)| = d_G(w)-1$, we have $|F_G^{\sigma'}(w, wv)| = d_G(w)-1+|\sigma'(A_w) \cup \sigma'(I_v)|$. If $\Delta(G) \geq 12$, we get $|F_G^{\sigma'}(w, wv)| \leq 9+7+2 = 18 \leq \Delta(G)+6$. Similarly, if $|\sigma'(A_w)| \leq 5$ we get $|F_G^{\sigma'}(w, wv)| \leq \Delta(G)-1+5+2 = \Delta(G)+6$. In both cases, there exists a color a such that $a \notin F_G^{\sigma'}(w, wv)$ and we set $\sigma(w, wv) = a$.

(e) $9 \leq d_G(w) \leq 10$, $\Delta(G) \leq 11$ and $|\sigma'(A_w)| \geq 6$.
Let $N_G(v) = \{w, x_1, x_2\}$. We first claim that we can recolor the two incidences (v, vx_1) and (v, vx_2) by using two colors c_1 and c_2 such that $c_1, c_2 \in \sigma'(A_w) \cup \sigma'(I_w)$.
To see that, observe that if for some $i \in \{1, 2\}$, $|\sigma'(A_{x_i})| = 7$, then $|\sigma'(A_{x_i}) \setminus \sigma'(A_v)| \geq 7-2+1 = 6$ and $|\sigma'(A_w) \cup \sigma'(I_w)| \geq 6+8 = 14$. Since the total number of colors is at most 18, $|(\sigma'(A_{x_i}) \setminus \sigma'(A_v)) \cap (\sigma'(A_w) \cup \sigma'(I_w))| \geq 2$. So we have at least two possible choices for c_i. On the other hand, if $|\sigma'(A_{x_i})| \leq 6$ then we have at least two possible choices for c_i since $|\sigma'(A_w) \cup \sigma'(I_w)| \geq 6+d_G(w)-1 \geq 14$ and $|\sigma'(I_{x_i}) \cup \sigma'(A_v)| \leq 11+2-1 = 12$.
By setting $\sigma(v, vx_1) = c_1$ and $\sigma(v, vx_2) = c_2$, we get that $|F_G^{\sigma'}(w, wv)| = |\sigma'(A_w) \cup \sigma'(I_w) \cup \sigma'(I_v)| = |\sigma'(A_w) \cup \sigma'(I_w)| \leq 7+d_G(w)-1 = d_G(w)+6 \leq \Delta(G)+6$. Therefore, there exists a color a such that $a \notin F_G^{\sigma'}(w, wv)$ and we can set $\sigma(w, wv) = a$.

It remains to color the incidence (v, vw). If $|\sigma(A_w)| \leq 6$ then there exists a color d such that $d \notin F_G^{\sigma}(v, vw)$ since in that case $|F_G^{\sigma}(v, vw)| = |\sigma(I_v) \cup \sigma(A_v) \cup \sigma(I_w)| \leq$

$2+3+d_G(w)-1 = d_G(w)+4 \leq \Delta(G)+4$. We then set $\sigma(v,vw)=d$. Otherwise, if $|\sigma(A_w)|=7$, there exists a color e such that $e \in \sigma(A_w)\setminus(\sigma(A_v)\cup\sigma(I_v))$ since $|\sigma(A_v)\cup\sigma(I_v)|\leq 5$. We then set $\sigma(v,vw)=e$.

4. G contains an edge vw such that $d_G(v)=4$ and $4\leq d_G(w)\leq 8$.

   Recall that we have $\Delta(G)\geq 8$ and thus at least 15 colors. Due to the minimality of G, the graph $G' = G\setminus vw$ admits a $(\Delta(G)+7,7)$-incidence coloring σ'. We extend σ' to a $(\Delta(G)+7,7)$-incidence coloring σ of G as follows.

   If $d_G(w)\leq 6$, we have $|F_G^{\sigma'}(w,wv)| = |\sigma'(I_w)\cup\sigma'(A_w)\cup\sigma'(I_v)| \leq 5+5+3=13$ and $|F_G^{\sigma'}(v,vw)| = |\sigma'(I_v)\cup\sigma'(A_v)\cup\sigma'(I_w)|\leq 3+3+5=11$. Since we have at least 15 colors, one can choose to colors a and b such that $a\notin F_G^{\sigma'}(w,wv)$, $b\notin F_G^{\sigma'}(v,vw)$ and $a\neq b$. We then set $\sigma(w,wv)=a$ and $\sigma(v,vw)=b$.

   Assume from now on that $7\leq d_G(w)\leq 8$. We shall first color the incidence (w,wv). Let $N_G(v)=\{w,x_1,x_2,x_3\}$. We consider seven subcases, according to $d_G(w)$, $\Delta(G)$ and $|\sigma'(A_w)|$.

   (a) $\Delta(G)\geq 11$.
   We have $|F_G^{\sigma'}(w,wv)| = |\sigma'(I_w)\cup\sigma'(A_w)\cup\sigma'(I_v)|\leq 7+7+3=17$. Since we have at least 18 colors, there exists a color a such that $a\notin F_G^{\sigma'}(w,wv)$ and we set $\sigma(w,wv)=a$.

   (b) $|\sigma'(A_w)|\leq 6$ and $|\sigma'(A_w)|+4\leq\Delta(G)\leq 10$.
   Since $|F_G^{\sigma'}(w,wv)| = |\sigma'(I_w)\cup\sigma'(A_w)\cup\sigma'(I_v)|\leq 7+|\sigma'(A_w)|+3 = |\sigma'(A_w)|+10\leq\Delta(G)+6$, there exists a color a such that $a\notin F_G^{\sigma'}(w,wv)$ and we set $\sigma(w,wv)=a$.

   (c) $\Delta(G)=8$ and $|\sigma'(A_w)|=5$.
   In that case we have $11\leq|\sigma'(A_w)\cup\sigma'(I_w)|\leq 12$ and 15 possible colors. We prove first that we can recolor the three incidences (v,vx_1), (v,vx_2) and (v,vx_3) by using three colors c_1,c_2,c_3 with $c_1\in[\sigma'(A_w)\cup\sigma'(I_w)]$.
   If $|\sigma'(A_{x_1})|=7$ then $[\sigma'(A_{x_1})\setminus\sigma'(A_v)]\cap[\sigma'(I_w)\cup\sigma'(A_w)]\neq\emptyset$, since $|\sigma'(A_{x_1})\setminus\sigma'(A_v)|\geq 7-3+1=5$. Therefore, there exists a color $c_1\in[\sigma'(A_{x_1})\setminus\sigma'(A_v)]\cap[\sigma'(I_w)\cup\sigma'(A_w)]$. On the other hand, if $|\sigma'(A_{x_1})|\leq 6$ then $|\sigma'(I_{x_1})\cup\sigma'(A_v)|\leq 8+3-1=10$; therefore, there exists a color $c_1\in[\sigma'(A_w)\cup\sigma'(I_w)]\setminus[\sigma'(I_{x_1})\cup\sigma'(A_v)]$.
   We still have to find two distinct colors c_2 and c_3, both distinct from c_1, that can be respectively assigned to the incidences (v,vx_2) and (v,vx_3). This can be done since for every i, $i\in\{2,3\}$, if $|\sigma'(A_{x_i})|=7$ then the number of possible choices is $|\sigma'(A_{x_i})\setminus(\sigma'(A_v)\cup\{c_1\}|\geq 7-3+1-1=4$ while if $|\sigma'(A_{x_i})|\leq 6$ the number of forbidden choices is $|\sigma'(I_{x_i})\cup\sigma'(A_v)\cup\{c_1\}|\leq 8+3-1+1=11$.
   Therefore, we can set $\sigma(v,vx_1)=c_1$, $\sigma(v,vx_2)=c_2$ and $\sigma(v,vx_3)=c_3$. Now, since we have $|F_G^{\sigma}(w,wv)| = |\sigma(A_w)\cup\sigma(I_w)\cup\sigma(I_v)|\leq 12+2=14=\Delta(G)+6$, there exists a color a such that $a\notin F_G^{\sigma}(w,wv)$ and we set $\sigma(w,wv)=a$.

   (d) $d_G(w)=7$, $\Delta(G)=8$ and $|\sigma'(A_w)|=6$.
   In that case we have $|\sigma'(A_w)\cup\sigma'(I_w)|=12$ and 15 possible colors. Using the same argument as in the previous case, we can recolor the three incidences

(v, vx_1), (v, vx_2) and (v, vx_3) by using three colors c_1, c_2, c_3 with $c_1 \in [\sigma'(A_w) \cup \sigma'(I_w)]$. Therefore, there exists a color a such that $a \notin F_G^\sigma(w, wv)$ and we set $\sigma(w, wv) = a$.

(e) $d_G(w) = 8$, $\Delta(G) = 8$ and $|\sigma'(A_w)| \geq 6$.

In that case we have $13 \leq |\sigma'(A_w) \cup \sigma'(I_w)| \leq 14$ and 15 possible colors. For every i, $1 \leq i \leq 3$, if $|\sigma'(A_{x_i})| = 7$ then $|[\sigma'(A_{x_i}) \setminus \sigma'(A_v)] \cap [\sigma'(A_w) \cup \sigma'(I_w)]| \geq 3$ since $|\sigma'(A_{x_i}) \setminus \sigma'(A_v)| \geq 7-3+1 = 5$. On the other hand, if $|\sigma'(A_{x_i})| \leq 6$ then $|[\sigma'(A_w) \cup \sigma'(I_w)] \setminus [\sigma'(I_{x_i}) \cup \sigma'(A_v)]| \geq 13-8-3+1 = 3$. Therefore, we can find three distinct colors $c_1, c_2, c_3 \in \sigma'(A_w) \cup \sigma'(I_w)$ such that $c_1 \notin [\sigma'(I_{x_1}) \cup \sigma'(A_v)]$, $|\sigma'(A_{x_1}) \cup \{c_1\}| \leq 7$, $c_2 \notin [\sigma'(I_{x_2}) \cup \sigma'(A_v)]$, $|\sigma'(A_{x_2}) \cup \{c_2\}| \leq 7$, $c_3 \notin [\sigma'(I_{x_3}) \cup \sigma'(A_v)]$ and $|\sigma'(A_{x_3}) \cup \{c_3\}| \leq 7$. We then set $\sigma(v, vx_1) = c_1$, $\sigma(v, vx_2) = c_2$ and $\sigma(v, vx_3) = c_3$. Now, we have $|F_G^\sigma(w, wv)| \leq 14 = \Delta(G) + 6$; therefore, there exists a color a such that $a \notin F_G^\sigma(w, wv)$ and we set $\sigma(w, wv) = a$.

(f) $\Delta(G) = 9$ and $|\sigma'(A_w)| \geq 6$.

Consider first the case $d_G(w) = 7$. We then have $|F_G^{\sigma'}(w, wv)| = |\sigma'(I_w) \cup \sigma'(A_w) \cup \sigma'(I_v)| \leq 6+6+3 = 15 \leq \Delta(G) + 6$; therefore, there exists a color a such that $a \notin F_G^\sigma(w, wv)$ and we set $\sigma(w, wv) = a$.

Suppose now $d_G(w) = 8$. In that case we have $13 \leq |\sigma'(A_w) \cup \sigma'(I_w)| \leq 14$ and 16 possible colors. We prove first that we can recolor the three incidences (v, vx_1), (v, vx_2) and (v, vx_3) by using three colors c_1, c_2, c_3 with $c_1, c_2 \in [\sigma'(A_w) \cup \sigma'(I_w)]$. For $i \in \{1, 2\}$, if $|\sigma'(A_{x_i})| = 7$ then the number of possible choices for c_i is $|[\sigma'(A_{x_i}) \setminus \sigma'(A_v)] \cap [\sigma'(A_w) \cup \sigma'(I_w)]| \geq 2$ since $|\sigma'(A_{x_i}) \setminus \sigma'(A_v)| \geq 7-3+1 = 5$, while if $|\sigma'(A_{x_i})| \leq 6$ then the number of possible choices for c_i is $|[\sigma'(A_w) \cup \sigma'(I_w)] \setminus [\sigma'(I_{x_i}) \cup \sigma'(A_v)]| \geq 13-9-3+1 = 2$. Now, if $|\sigma'(A_{x_3})| = 7$ then the number of possible choices for c_3 is $|\sigma'(A_{x_3}) \setminus (\sigma'(A_v) \cup \{c_1, c_2\})| \geq 7-3+1-2 = 3$, while if $|\sigma'(A_{x_3})| \leq 6$ then the number of forbidden choices for c_3 is $|\sigma'(I_{x_3}) \cup \sigma'(A_v) \cup \{c_1, c_2\}| \leq 9+3-1+2 = 13$. Therefore, we can find the three required colors c_1, c_2 and c_3 and we set $\sigma(v, vx_1) = c_1$, $\sigma(v, vx_2) = c_2$ and $\sigma(v, vx_3) = c_3$.

Now, since we have $|F_G^\sigma(w, wv)| \leq 14+1 = 15 \leq \Delta(G)+6$, there exists a color a such that $a \notin F_G^\sigma(w, wv)$ and we set $\sigma(w, wv) = a$.

(g) $\Delta(G) = 10$ and $|\sigma'(A_w)| = 7$.

(This case is similar to case (c), we give it for completeness.)

In that case we have $13 \leq |\sigma'(A_w) \cup \sigma'(I_w)| \leq 14$ and 17 possible colors. We prove first that we can recolor the three incidences (v, vx_1), (v, vx_2) and (v, vx_3) by using three colors c_1, c_2, c_3 with $c_1 \in [\sigma'(A_w) \cup \sigma'(I_w)]$.

If $|\sigma'(A_{x_1})| = 7$ then $[\sigma'(A_{x_1}) \setminus \sigma'(A_v)] \cap [\sigma'(I_w) \cup \sigma'(A_w)] \neq \emptyset$, since $|\sigma'(A_{x_1}) \setminus \sigma'(A_v)| \geq 7-3+1 = 5$. Therefore, there exists a color $c_1 \in [\sigma'(A_{x_1}) \setminus \sigma'(A_v)] \cap [\sigma'(I_w) \cup \sigma'(A_w)]$. On the other hand, if $|\sigma'(A_{x_1})| \leq 6$ then $|\sigma'(I_{x_1}) \cup \sigma'(A_v)| \leq 10+3-1 = 12$; therefore, there exists a color $c_1 \in [\sigma'(A_w) \cup \sigma'(I_w)] \setminus [\sigma'(I_{x_1}) \cup \sigma'(A_v)]$.

We still have to find two distinct colors c_2 and c_3, both distinct from c_1, that can be respectively assigned to the incidences (v, vx_2) and (v, vx_3). This can be done since for every i, $i \in \{2, 3\}$, if $|\sigma'(A_{x_i})| = 7$ then the number of possible choices is $|\sigma'(A_{x_i}) \setminus (\sigma'(A_v) \cup \{c_1\}| \geq 7-3+1-1 = 4$ while if $|\sigma'(A_{x_i})| \leq 6$ the number of forbidden choices is $|\sigma'(I_{x_i}) \cup \sigma'(A_v) \cup \{c_1\}| \leq 10+3-1+1 = 13$. Therefore, we can set $\sigma(v, vx_1) = c_1$, $\sigma(v, vx_2) = c_2$ and $\sigma(v, vx_3) = c_3$. Now, since we have $|F_G^\sigma(w, wv)| \leq 14+2 = 16 = \Delta(G)+6$, there exists a color a such that $a \notin F_G^\sigma(w, wv)$ and we set $\sigma(w, wv) = a$.

It remains now to color the incidence (v, vw). If $|\sigma(A_w)| = 7$ then there exists a color b such that $b \in \sigma(A_w) \setminus [\sigma(A_v) \cup \sigma(I_v))]$ since $|\sigma(A_v) \cup \sigma(I_v)| \leq 6$ and we set $\sigma(v, vw) = b$. On the other hand, if $|\sigma(A_w)| \leq 6$, there exists a color b such that $b \notin F_G^\sigma(v, vw)$ since $|F_G^\sigma(v, vw)| = |\sigma(A_v) \cup \sigma(I_v) \cup \sigma(I_w)| \leq 3+3+8 = 14 \leq \Delta(G)+6$. We can thus set $\sigma(v, vw) = b$.

5. G contains an edge vw such that $d_G(v) = 5$ and $5 \leq d_G(w) \leq 6$.

   Recall first that $\Delta(G) \geq 8$. Due to the minimality of G, the graph $G' = G \setminus vw$ admits a $(\Delta(G)+7, 7)$-incidence coloring σ'. We extend σ' to a $(\Delta(G)+7, 7)$-incidence coloring σ of G as follows.

   Since $|F_G^{\sigma'}(w, wv)| = |\sigma'(A_w) \cup \sigma'(I_w) \cup \sigma'(I_v)| \leq 5+5+4 = 14 \leq \Delta(G)+6$, there exists a color a such that $a \notin F_G^{\sigma'}(w, wv)$ and we set $\sigma(w, wv) = a$.

   Now, since $|F_G^\sigma(v, vw)| = |\sigma'(A_v) \cup \sigma'(I_v) \cup \sigma'(I_w) \cup \{a\}| \leq 4+4+5+1 = 14 \leq \Delta(G)+6$, there exists a color b such that $b \notin F_G^\sigma(v, vw)$ and we set $\sigma(v, vw) = b$.

It is easy to check that in all cases, we have obtained a $(\Delta(G)+7, 7)$-incidence coloring of G, which contradicts our assumption, and the theorem is proved. ■

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