On the Adaptable Chromatic Number of Graphs

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Abstract

The *adaptable chromatic number* of a graph G is the smallest integer k such that for any edge k-colouring of G there exists a vertex k-colouring of G in which the same colour never appears on an edge and both its endpoints. (Neither the edge nor the vertex colourings are necessarily proper in the usual sense.)

We give an efficient characterization of graphs with adaptable chromatic number at most two, and prove that it is NP-hard to decide if a given graph has adaptable chromatic number at most k, for any $k \geq 3$. The adaptable chromatic number cannot exceed the chromatic number; for complete graphs, the adaptable chromatic number seems to be near the square root of the chromatic number. On the other hand, there are graphs of arbitrarily high girth and chromatic number, in which the adaptable chromatic number coincides with the classical chromatic

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number. In analogy with well known properties of chromatic numbers, we also discuss the adaptable chromatic numbers of planar graphs, and of graphs with bounded degree, proving a Brooks-like result.

Keywords: chromatic number, girth, adaptable colouring, planar graphs, Brooks theorem, maximum degree, Lovasz local lemma, NP-hard problems, polynomial algorithms.

Mathematical Subject Classification: 05C15

1 Introduction

Recently a number of variants of the classical chromatic number have been introduced and investigated - the fractional chromatic number, the circular (or star) chromatic number, the vector chromatic number, the oriented chromatic number, the acyclic chromatic number, the local chromatic number, etc. [2, 3, 9, 10, 11, 12, 13, 14, 15, 16, 17]. The adaptable chromatic number is a novel variant, which is related to matrix partitions of graphs, trigraph homomorphisms, and full constraint satisfaction problems [4, 5, 6, 9].

Let G be a graph, and k a positive integer. We shall consider vertex and edge partitions of G into k classes. We find it convenient to call the parts (vertex or edge) *colours* $1, 2, \ldots, k$, and call the partitions (vertex or edge) *k-colourings*. Note that at this point no restrictions are made, and any partition of $V(G)$ or $E(G)$ is a vertex or edge colouring. As usual, the partitions are ordered, $V(G) = V_1 \cup V_2 \cup \ldots \cup V_k$, and $E(G) = E_1 \cup E_2 \cup \ldots \cup E_k$, and identified with the corresponding mappings $f : V(G) \to \{1, 2, \ldots, k\}$, and $F : E(G) \to \{1, 2, \ldots, k\}$. (Namely, $f(v) = i$ if and only if $v \in V_i$, and $F(uw) = j$ if and only if $uw \in E_j$.)

Let F be an edge k-colouring of G: we say that a vertex k-colouring f of G is *adapted to F*, if no edge uv is *monochromatic* in the sense that $f(u) = f(v) = F(uv)$.

We say that a graph G is *adaptably k-colourable* if for each edge k-colouring F of G, there is a vertex k-colouring f of G, adapted to F.

The *adaptable chromatic number* of G, denoted $\chi_a(G)$, is the smallest integer k such that G is adaptably k-colourable. (The same notation χ_a is sometimes used for the *acyclic* chromatic number [3]; however, we believe these two concepts are sufficiently different to accommodate this ambiguity.)

As usual, a vertex (or edge) colouring is called *proper* if adjacent vertices (respectively incident edges) have distinct colours. We note that a proper vertex k-colouring of G is adapted to *all* edge k-colourings; hence we obtain the following upper bound.

Proposition 1.1 *For any graph G, we have*

$$\chi_a(G) \leq \chi(G). \blacksquare$$

2 Low Adaptable Chromatic Numbers

In this section we characterize those graphs G which are adaptably 2-colourable.

Obviously, $\chi_a(G) = 1$ if and only if G has no edges (in which case we also have $\chi(G) = 1$). For $\chi_a(G) = 2$, we shall derive a polynomially checkable characterization. We shall use the following fact.

Proposition 2.1 *Let E' be a subset of the edges of G. If $\chi(G - E') > |E'|$ then*

$$\chi_a(G) \leq \chi(G - E').$$

Proof. Let $\chi(G-E') = k$ and fix a proper k-colouring c of $G-E'$, with the corresponding vertex partition $V_1, V_2, \ldots, V_k$ of $V(G) = V(G-E')$. Consider now an arbitrary edge k-colouring F of G. We shall define a vertex colouring f adapted to F, by permuting the colours of the colouring c. Let E'' be the set of edges in E' that actually lie within the sets $V_1, V_2, \ldots, V_k$. There are fewer than k edges in E' by assumption, thus $|E''| \leq k - 1$. Each edge $e \in E''$ forbids the colour $F(e)$ for one of the sets V_i; let C_i be the set of permitted (not forbidden) colours for the set V_i. We now claim that the sets $C_1, C_2, \ldots, C_k$ admit a system of distinct representatives. Otherwise, by Hall's theorem, the union of some ℓ sets C_i has at most $\ell - 1$ elements, i.e., the same $k - \ell + 1$ colours are forbidden for ℓ distinct sets V_i, yielding a total of at least $\ell \cdot (k - \ell + 1) \geq k$ forbidden colours in all, contrary to $|E''| \leq k - 1$. (Note that $k \geq \ell$, whence the above inequality.) The distinct representatives define a colouring f adapted to F. $\blacksquare$

As a consequence of Lemma 2.1, the adaptable chromatic number of a graph G with at least two edges is at most the smallest chromatic number of a graph obtained from G by deleting one edge. Thus if the deletion of some edge of G leaves a bipartite graph, then the graph G is adaptably 2-colourable. This turns out to be also a necessary condition for a connected G to be adaptably 2-colourable.

In the proof of the equivalance we shall also introduce a structural characterization of these graphs. Figures 1 and 2 below depict two kinds of problem graphs. An *edge- bicycle* is a union of two cycles joined by a path (the path may have length zero, with $x_1 = x_2$); the cycles and the path are required to be edge-disjoint (but may intersect at vertices). An *edge-*K_4 is a union of four paths as shown in Figure 2, which are again required to be edge-disjoint (but may intersect at vertices). If the cycles C, C' are odd, we have an *odd* edge-bicycle, and if the cycles C_1, C_2, C_3, C_4 are all odd (C_4 is the boundary of the outer region), we have an *odd* edge-K_4. We note that similarly defined graphs with vertex-disjoint paths and cycles play a role in another concept of being close to a bipartite graph [8, 9].

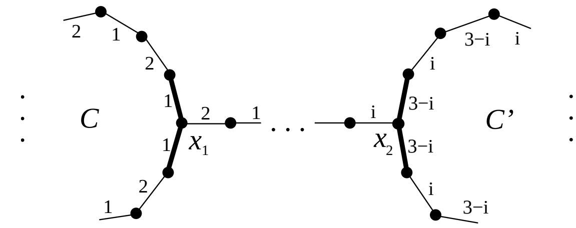

Figure 1: An odd edge-bicycle, with a marked edge 2-colouring

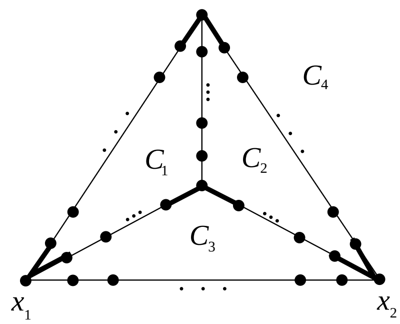

Figure 2: An odd edge-K_4, with a suggested edge 2-colouring

Theorem 2.1 *The following statements are equivalent for a connected graph G.*

1. *G is adaptably* 2*-colourable;*

2. *G does not have an odd edge-bicycle or an odd edge-*K_4*; and*

3. *there is an edge e such that* $G - e$ *is bipartite.*

Proof. We first show that 1 implies 2. Indeed, in the figures, we have suggested edge 2-colourings F of odd edge-bicycles and odd edge-K_4's which do not admit adapted vertex colouring. Suppose first that G contains an odd edge-bicycle. The edge 2-colouring F of G will colour by 1 the two edges of C incident to x_1, and colour the remaining edges of C alternately by 1 and 2; colour the edges of P alternately by 1 and 2, with the edge incident with x_1 coloured by 2. This determines the colours of all the edges of P; assume the edge of P incident with x_2 is coloured by i, and colour the two edges of C' incident with x_2 by $3 - i$. (In case $x_1 = x_2$, the two edges of C' incident to x_2 are coloured by 2.) Then colour the other edges of C' alternately by 1 and 2. The remaining edges of G are coloured arbitrarily. Note that because of the parity of the cycles, each odd cycle contains exactly one pair of consecutive edges which obtain the same colour; this is suggested by the heavy edges in the figure. In other words, the colouring is determined by the heavy edges, in the sense that such a pair of edges obtains the same colour, while all other consecutive pairs obtain distinct colours. In the case of the odd edge-K_4 in the figure, we only used this shorthand to describe an edge 2-colouring F.

There is no vertex colouring adapted to this edge 2-colouring F, since colouring x_1 by 1 forces a monochromatic edge on the odd cycle C, and colouring x_1 by 2 leads to x_2 being coloured so that it forces a monochromatic edge on the odd cycle C'.

We note that the same argument applies when C and C' have common edges in a ('consistent') way, i.e., so that the colours given to the common edges by F coincide. We can view the cycles C_1, C_2 of an odd edge-K_4 this way. In the edge 2-colouring suggested in Figure 2, these odd cycles can be viewed as joined by the path between x_1 and x_2, and intersecting in a consistent manner. In any event, it is easy to argue directly (as above) that the depicted edge 2-colouring F does not admit an adapted vertex colouring. This proves that a graph G containing an odd edge-K_4 is not adaptably 2-colourable either, so that 1 implies 2.

Since 3 implies 1 by Proposition 2.1, it remains to prove 2 implies 3. Assume that G does not have an odd edge-bicycle or an odd edge-K_4. Since

G is connected, the absence of odd edge-bicycles implies that it does not have two edge-disjoint odd cycles, i.e., that any two odd cycles intersect. Assume, for the contradiction, that there is no edge which lies in all odd cycles (else we are done). For two odd cycles C_1 and C_2 of G, a *segment* of $C_1 \cap C_2$ is a maximal subset of $C_1 \cap C_2$ which induces a path in both C_1 and C_2. (A cycle C is viewed as a set of edges). Choose two odd cycles C_1 and C_2 so that $C_1 \cap C_2$ has the minimum number of segments. We claim that $C_1 \cap C_2$ has exactly one segment. Otherwise let P_1 and P_2 be two consecutive segments of $C_1 \cap C_2$. Let the two end vertices of P_i be a_i and b_i. Let Q_i be the path of C_i connecting b_1 to a_2, and let R_i be the path of C_i connecting b_2 to a_1. If the lengths of Q_1 and Q_2 have different parity, then the lengths of R_1 and R_2 also have different parity. In this case $Q_1 \cup Q_2$ and $R_1 \cup R_2$ are two edge disjoint odd closed walks, and hence G has two edge disjoint odd cycles, contrary to our assumption. Thus Q_1 and Q_2 have the same parity. Let C_2' be obtained from C_2 by replacing Q_2 with Q_1. Then $C_1 \cap C_2'$ has fewer segments than $C_1 \cap C_2$, contrary to our choice of C_1 and C_2. For future reference we also note that since R_1 and R_2 also have the same parity, a similar argument applies to replacing R_2 with R_1.

Choose odd cycles C_1 and C_2 so that $C_1 \cap C_2$ has only one segment, and among all pairs of odd cycles whose intersection has only one segment, $|C_1 \cap C_2|$ is minimum. Let $P = C_1 \cap C_2$. Then $(C_1 \cup C_2) \setminus P$ is an even cycle. Let e be an edge of P. By our assumption, $G - e$ has an odd cycle C_3, and moreover C_3 intersects both C_1 and C_2. As above, if $C_1 \cap C_3$ has more than one segment, then we can replace a subpath Q_3 of C_3 with a subpath Q_1 of C_1, to obtain another odd cycle C_3' such that the intersection $C_1 \cap C_3'$ has fewer segments than $C_1 \cap C_3$. Moreover, we may assume that $e \notin Q_1$ (or else we could use the R_i's instead of the Q_i's). Thus we assume that $C_1 \cap C_3$ has only one segment P', and, similarly, $C_2 \cap C_3$ has only one segment P''. If $P \cap P' \neq \emptyset$, then since $P \cap P' \subseteq C_2$, and $C_3 \cap C_2$ has only one segment, it follows that either $C_1 \cap C_3 \subseteq P - e$ or $C_2 \cap C_3 \subseteq P - e$, contrary to the choice of C_1 and C_2. Thus we assume that $P \cap P' = \emptyset$, and similarly $P \cap P'' = \emptyset$. Now $C_3 \setminus C_1 \setminus C_2$ consists of two paths, say B_1 and B_2 connecting C_1 and C_2 (either of which might be a single vertex belonging to P). As C_3 is an odd cycle, either $((C_1 \cup C_2) \setminus P) \cup B_1$ contains an odd cycle, or $((C_1 \cup C_2) \setminus P) \cup B_2$ contains an odd cycle. Hence either $C_1 \cup C_2 \cup B_1$ or $C_1 \cup C_2 \cup B_2$ is an odd edge-K_4, contrary to our assumption. ∎

Here is our main conclusion of this section.

Corollary 2.1 *A graph G is adaptably 2-colourable if and only if each con-*

nected component of G can be made bipartite by the deletion of one edge.

The corollary represents a polynomial time algorithm to test whether or not $\chi_a(G) \leq 2$.

We also note in passing that a graph G in which any (not necessarily distinct) three odd cycles have a common edge cannot have an odd edge-bicycle nor an odd edge-K_4. In such a case, Theorem 2.1 guarantees that *all* odd cycles of G have a common edge – namely an edge e for which $G - e$ is bipartite. In other words, *if any three (not necessarily distinct) odd cycles of G have a common edge, then all odd cycles of G have a common edge.*

3 High Adaptable Chromatic Numbers

In this section we prove that recognizing graphs of higher adaptable chromatic number is NP-hard. We first present an auxiliary construction of graphs in which the adaptable chromatic number and the ordinary chromatic number are the same, and arbitrarily high.

Theorem 3.1 *For any positive integer k, there is a graph G with $\chi(G) = \chi_a(G) = k$.*

Proof. We shall construct, for each $k \geq 1$, a graph G_k such that $\chi_a(G_k) = \chi(G_k) = k$. For $k = 1, 2$, let $G_k = K_k$; in these cases clearly $\chi_a(K_k) = \chi(K_k) = k$. For $k \geq 2$, we construct G_{k+1} by taking a disjoint union of k copies of G_k, say $G_k^1, G_k^2, \cdots, G_k^k$, and adding a vertex u adjacent to all vertices of all the copies of G_k. Figure 3 shows the graphs G_3 and G_4.

It is clear from this definition that $\chi(G_{k+1}) = \chi(G_k) + 1 = k + 1$. To prove that $\chi_a(G_{k+1}) = k + 1$, we shall recursively define an edge k-colouring F_k of G_{k+1} which admits no adapted vertex colouring. The trivial edge 1-colouring F_1 of $G_2 = K_2$ obviously has this property. Thus suppose G_k admits an edge $(k-1)$-colouring F_{k-1} for which there is no adapted vertex colouring. We define F_k as follows (see the illustrations in Figure 3). The edges from the central vertex u to all vertices of G_k^i receive colour i. The edges within the copy G_k^i are coloured according to the colouring F_{k-1}, except all edges of colour i in F_{k-1} are coloured by k instead. This is done for all $i = 1, 2, \ldots, k$; note that G_k^k is coloured by F_{k-1} without any changes.

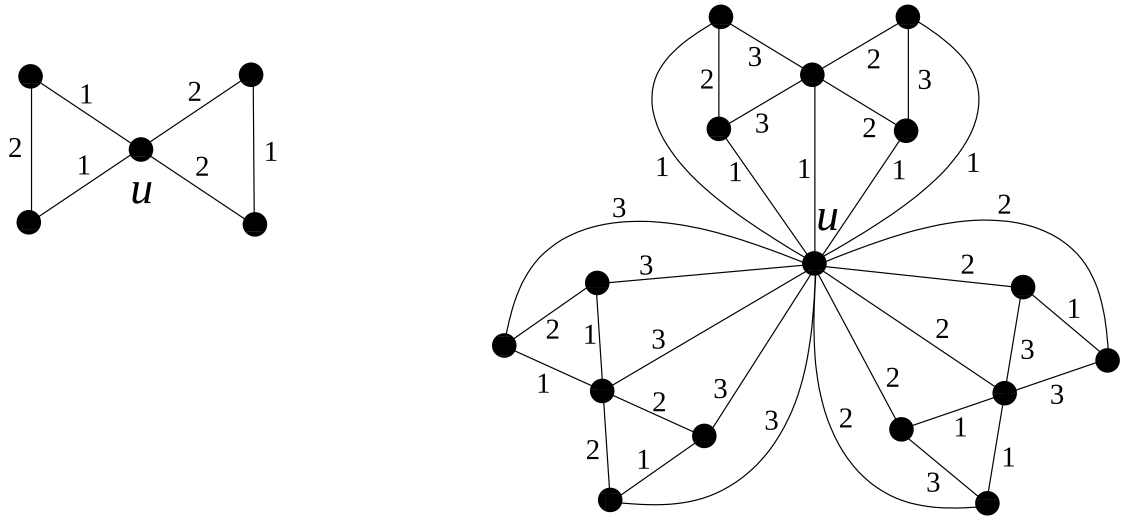

Figure 3: Graphs G_3 and G_4, with edge colourings F_2 and F_3

It is clear that there is no vertex colouring of G_{k+1} adapted to F_k: if u is coloured i, then G_k^i requires a vertex colouring adapted to F_{k-1}, which does not exist by assumption. ∎

Let G_k be the graph constructed above, with $k \geq 3$. Observe that u is the unique universal vertex of G_k. Let G'_k be obtained from G_k by splitting u into two vertices u_a and u_b. The edges incident to u in G_k are distributed to u_a and u_b as follows. If $k \geq 4$, then u_a is adjacent to the unique universal vertex of each copy G_{k-1}^i, $i = 1, 2, \ldots, k-1$, and u_b is adjacent to all the other neighbours of u. For $k = 3$, each of u_a, u_b is adjacent to one vertex of each $G_2^i = K_2$ in G_3. Figure 4 below shows the graphs G'_3 and G'_4, as well as suggests how we may naturally lift the edge-colouring F_k of G_{k+1} to an edge-colouring F'_k of G'_{k+1}.

Lemma 3.1 *Assume $k \geq 2$. The graph G'_{k+1} satisfies the following property. For any pair of distinct colours $i, j \in \{1, 2, \cdots, k\}$, there exists a vertex colouring f of G'_{k+1} with $f(u_a) = i, f(u_b) = j$ which is adapted to F'_k. On the other hand, no vertex colouring adapted to F'_k assigns u_a and u_b the same colour.*

Proof. It is easy to verify that $\chi(G'_{k+1}) \leq k$, and hence G'_{k+1} is adaptably k-colourable; since it contains G_k as a subgraph, we have $\chi_a(G'_{k+1}) = k$.

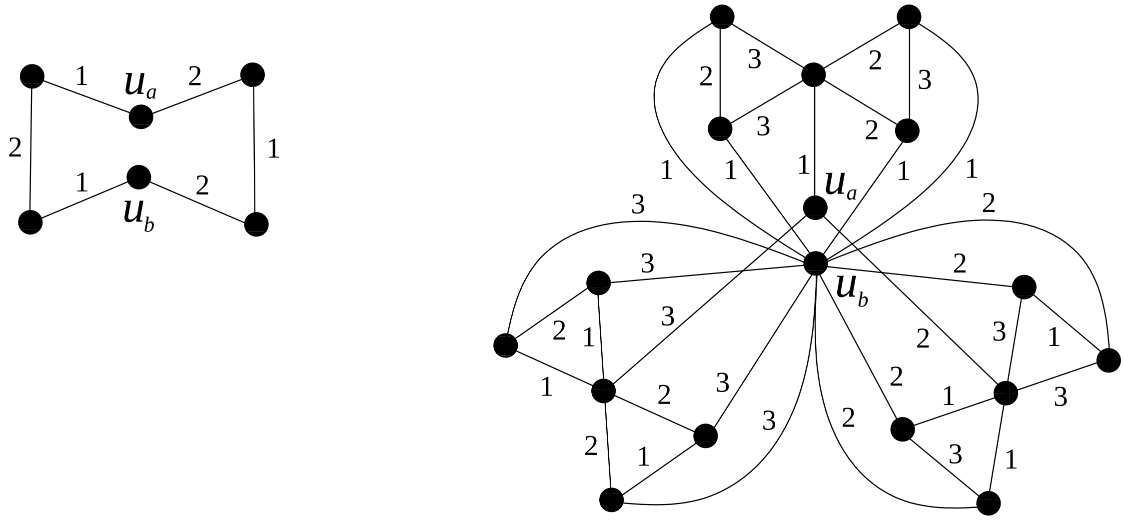

Figure 4: Graphs G'_3 and G'_4, with colourings F'_2 and F'_3

However, if f is a colouring adapted to F'_k which colours u_a and u_b by the same colour, then by letting $f(u) = f(u_a)$, we obtain a colouring of G_{k+1} adapted to F_k, which is contrary to Theorem 3.1. Note that the symmetry of G'_k ensures that the vertices u_a, u_b can be coloured by any pair of distinct colours. So a mapping $f : \{u_a, u_b\} \to \{1, 2, \cdots, k\}$ can be extended to a colouring of G'_{k+1} adapted to F'_k if and only if $f(u_a) \neq f(u_b)$. ∎

Suppose G is a graph and $e = xy$ is an edge of G. To *replace* e by a copy of G'_k means to delete e, add a copy of G'_k, and identify x with u_a and y with u_b. Given an arbitrary graph G, we denote by $G \cdot G_k$ the graph obtained from G by replacing each edge of G with a (distinct) copy of G'_k.

Theorem 3.2 *We have $\chi(G) \leq k - 1$ if and only if $\chi_a(G \cdot G_k) \leq k - 1$.*

Proof. If G is $(k-1)$-colourable, we may lift these colours to the corresponding vertices of $G \cdot G_k$, and extend this colouring to the copies of G_k by Lemma 3.1. Thus $\chi_a(G \cdot G_k) \leq \chi((G \cdot G_k) \leq k - 1$.

On the other hand, if G is not $(k-1)$-colourable, then let F be the edge $(k-1)$-colouring of $G \cdot G_k$ which is the union of the edge $(k-1)$-colourings F_{k-1} of the copies of G'_k. Then in any vertex $(k-1)$-colouring of $G \cdot G_k$, a copy of G_k will have the same colour i on both u_a and u_b. Hence the restriction of the colouring to the vertices of that copy of G_k is not adapted to F_{k-1}, and the colouring is not adapted to F. ∎

Corollary 3.1 *If $k \geq 3$, it is NP-hard to decide whether or not $\chi_a(G) \leq k$.* ∎

We observe that the complexity questions are less resolved if we seek to decide the existence of a vertex k-colouring adapted to one given edge k-colouring F of G. For $k \geq 4$, the problem is NP-hard, even for complete graphs. However, the complexity of this problem is not known when $k = 3$ even when G is restricted to be a complete graph; there is some evidence that this problem is not NP-complete, but no polynomial-time algorithm is known [4, 6].

4 High Girth

The k-chromatic graphs G with $\chi_a(G) = k$ constructed in the previous section all contain triangles. In this section we prove the existence of graphs with high girth and high adaptable chromatic number. In fact, we can arrange to have these graphs also satisfy $\chi_a(G) = \chi(G)$.

Theorem 4.1 *For any integers $k, g \geq 3$, there is a graph G^* of girth at least g such that $\chi_a(G^*) = \chi(G^*) = k$.*

Proof. Let $K_{n,n,\cdots,n}$ be the complete k-partite graph with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$, where for each i, the set V_i is $\{v_{i,1}, v_{i,2}, \cdots, v_{i,n}\}$. Let G be the random subgraph of $K_{n,n,\cdots,n}$, in which each edge of $K_{n,n,\cdots,n}$ belong to G with probability $p = 1/n^{1-\epsilon}$. Then colour the edges of G independently and uniformly at random by colours $1, 2, \cdots, k-1$.

Assume n is sufficiently large. Let A denote the following event: there exist $i, j \in \{1, 2, \cdots, k\}$, $i \neq j$, a colour $t \in \{1, 2, \cdots, k-1\}$, and sets $U \subset V_i, W \subset V_j$ with $|U| = |W| = \lceil \frac{n}{2k^2} \rceil$, such that there is no edge of colour t between U and W.

Then $Pr[A] \leq k^3 n^{\frac{n}{k^2}} (1 - p/k)^{\frac{n^2}{4k^4}} = O(e^{-n^{\epsilon}})$. So with probability more than 1/2, A does not happen, provided that n is large enough.

The number of expected cycles of length smaller than g is at most $\sum_{l=3}^{g-1}(kn)^l 2l! p^l = O(n^{g\epsilon})$. This implies that with probability more than 1/2, the random graph has fewer than $n/2$ cycles of length less than g (provided that n is large enough). Therefore with positive probability, the random graph G has fewer than $n/2$ cycles of length less than g and for which A does not happen. In particular, there is a subgraph G' of $K_{n,n,\cdots,n}$ such that the following is true:

(1) G' has at most $n/2$ cycles of length less than g, and

(2) for any $i, j \in \{1, 2, \cdots, n\}$, $i \neq j$, for any colour $t \in \{1, 2, \cdots, k-1\}$, and for any sets $U \subseteq V_i$, $W \subseteq V_j$ with $|U| = |W| = \lceil \frac{n}{2k^2} \rceil$, there is an edge of colour t between U and W.

Let G^* be obtained from G' by deleting $n/2$ vertices from each V_i so that G^* contains no cycle of length less than g. We consider the edge colouring F of G^* induced by G', and claim that G^* has no colouring adapted to F.

Assume to the contrary that G^* has a colouring f adapted to the edge $(k-1)$-colouring F. Then one of the colour classes, say $f^{-1}(t)$, has size $|f^{-1}(t)| \geq kn/2(k-1)$. This implies that there are two distinct indices $i, j \in \{1, 2, \cdots, k\}$ such that $|f^{-1}(t) \cap V_i| \geq \frac{n}{2k^2}$ and $|f^{-1}(t) \cap V_j| \geq \frac{n}{2k^2}$. But by (2) above, there is an edge $e \in E_t$ between two vertices of $f^{-1}(t)$, contrary to the assumption that f is adapted to F. This proves that $\chi_a(G^*) \geq k$. Since G^* is obviously k-colourable, we obtain $\chi_a(G^*) = \chi(G^*) = k$. ∎

5 Degree-Bounded and Planar Graphs

Like the classical chromatic number, the adaptable chromatic number is bounded by a function of the maximum degree Δ. However, while the Brooks bound for the chromatic number is Δ (+1), the adaptable chromatic number is bounded by roughly $3\sqrt{\Delta}$.

Theorem 5.1 *For any graph G with maximum degree Δ we have*

$$\chi_a(G) \leq \lceil \sqrt{e(2\Delta - 1)} \rceil.$$

Proof. Let $k = \lceil \sqrt{e(2\Delta - 1)} \rceil$. Let G be a graph, and F an edge k-colouring of G. We shall use the Lovasz Local Lemma [1]. Consider the set of all vertex k-colourings of G; let A_e denote the event that the edge e is monochromatic. It is easy to see that the probability of A_e is $\frac{1}{k^2}$. Moreover, if edges e and e' are not incident, the events A_e and $A_{e'}$ are independent; thus each A_e is dependent on at most $2\Delta - 2$ other events. Note that our choice of k ensures that

$$e\frac{1}{k^2}(2\Delta - 1) \leq 1.$$

This is just the condition under which the local lemma guarantees a positive probability that none of the events A_e occur [1]. Thus there exists a vertex colouring f in which none of the events A_e occur, i.e., such that f is adapted to F. ∎

Next we make the following observation about the adaptable chromatic number of planar graphs.

Theorem 5.2 *For every planar graph G we have $\chi_a(G) \leq 4$, and there exist planar graphs G with $\chi_a(G) = 4$.*

Proof. The first statement is a consequence of the four colour theorem. The second statement follows from Theorem 3.1, since for $k = 4$ the graph G_k is planar, as shown in figure 3. ∎

6 Complete Graphs

The adaptable chromatic number of complete graphs is an interesting parameter. It turns out that for $n > 2$ we have $\chi_a(K_n) < \chi(K_n)$. In fact, Theorem 5.1 implies that $\chi_a(K_n) \leq c\sqrt{n}$ for a constant $c < 3$.

Corollary 6.1 *For any n,*

$$\chi_a(K_n) \leq \lceil \sqrt{e(2n-3)} \rceil.$$

We also offer the following constructive lower bound.

Theorem 6.1 $\chi_a(K_{2^n}) > n$.

Proof. For $n \geq 1$, let K_{2^n} be the complete graph whose vertices are $0, 1$-sequences of length n, i.e., $V(G) = \{x = (x_1 x_2 \cdots x_n) : x_i \in \{0, 1\}\}$. Now we define an edge n-colouring F_n of K_{2^n} by colouring an edge xy by colour t, where $x_i = y_i$ for $i = 1, 2, \cdots, t-1$ and $x_t \neq y_t\}$. We shall prove by induction on n that K_{2^n} has no colouring adapted to F_n. If $n = 1$, this is obviously true. Assume $n \geq 2$, and assume to the contrary that there is a colouring f_n adapted to F_n. For each $0,1$-sequence w of length $n-1$, at least one of the vertices $w0, w1$ is not coloured by colour n. Let $i_w = 0$ if $w0$ is not coloured by n, and $i_w = 1$ otherwise. Then the set $X = \{wi_w : w$ is a $0,1$-sequence of length $n-1$ induces a copy of $K_{2^{n-1}}$ with the edge $(n-1)$-colouring F_{n-1}. The colouring f_n then induces an $(n-1)$-colouring of $K_{2^{n-1}}$ adapted to F_{n-1}, a contradiction. ∎

Since the parameter is monotone, i.e., if H is a subgraph of G then $\chi_a(G) \geq \chi_a(H)$, we have the following lower bound.

Corollary 6.2 *For any integer n,*

$$\chi_a(K_n) \geq \lfloor \log_2 n \rfloor + 1.$$

For small values of n, the above lower bound is achieved: we have verified that if $n \leq 10$, then $\chi_a(K_n) = \lfloor \log_2 n \rfloor + 1$. However, for large values of n, it can be shown that using a random colouring of the edges one obtains $\chi_a(K_n) \geq \sqrt{n/\log n}$. (We are grateful to Robert Šámal and Alexander Kostochka, who independently made this observation.) Thus the upper bound from Corollary 6.1 is not far from the truth, and

$$\sqrt{n/\log n} \leq \chi_a(K_n) \leq c\,\sqrt{n)}$$

.

We conclude the paper, with the following problems.

1. Tighten the bounds for $\chi_a(K_n)$.
2. Let $f(n) = \min\{\chi_a(G) : \chi(G) = n\}$.
   - Is it true that $f(n) = \chi_a(K_n)$?
   - Is it true that $\lim_{n\to\infty} f(n) = \infty$?
3. Prove that every planar graph G has $\chi_a(G) \leq 4$, without using the four colour theorem.

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