ZERO PRODUCT PRESERVERS OF C*-ALGEBRAS

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Dedicated to Professor Bingren Li on the occasion of his 65th birthday (1941.10.7 –)

ABSTRACT. Let $\theta : \mathscr{A} \to \mathscr{B}$ be a zero-product preserving bounded linear map between C*-algebras. Here neither $\mathscr{A}$ nor $\mathscr{B}$ is necessarily unital. In this note, we investigate when θ gives rise to a Jordan homomorphism. In particular, we show that $\mathscr{A}$ and $\mathscr{B}$ are isomorphic as Jordan algebras if θ is bijective and sends zero products of self-adjoint elements to zero products. They are isomorphic as C*-algebras if θ is bijective and preserves the full zero product structure.

1. INTRODUCTION

Let $\mathscr{M}$ and $\mathscr{N}$ be algebras over a field $\mathbb{F}$ and $\theta : \mathscr{M} \to \mathscr{N}$ a linear map. We say that θ is a *zero-product preserving* map if $\theta(a)\theta(b) = 0$ in $\mathscr{N}$ whenever $ab = 0$ in $\mathscr{M}$. The canonical form of a linear zero product preserver, $\theta = h\varphi$, arises from an element h in the center of $\mathscr{N}$ and an algebra homomorphism $\varphi : \mathscr{M} \to \mathscr{N}$. In [6], we see that in many interesting cases zero-product preserving linear maps arise in this way.

We are now interested in the C^*-algebra case. There are 4 different versions of zero products: $ab = 0$, $ab^* = 0$, $a^*b = 0$ and $ab^* = a^*b = 0$. Surprisingly, the original version $ab = 0$ is the least, if any, geometrically meaningful, while the others mean a, b have orthogonal initial spaces, or orthogonal range spaces, or both. Using the orthogonality conditions, the author showed in [11] that a bounded linear map $\theta : \mathscr{A} \to \mathscr{B}$ between C*-algebras is a triple homomorphism if and only if θ preserves the fourth disjointness $ab^* = a^*b = 0$ and $\theta^{**}(1)$ is a partial isometry. Here, the triple product of a C*-algebra is defined by $\{a, b, c\} = (ab^*c + cb^*a)/2$, and $\theta^{**} : \mathscr{A}^{**} \to \mathscr{B}^{**}$ is the bidual map of θ. See also [3] for a similar result dealing with the case $ab = ba = 0$. We shall deal with the first and original case in this note. The other cases will be dealt with in a subsequent paper.

There is a common starting point of all these 4 versions. Namely, we can consider first the zero products $ab = 0$ of self-adjoint elements a, b in $\mathscr{A}_{sa}$. In [10] (see also [9]), Wolff shows that if $\theta : \mathscr{A} \to \mathscr{B}$ is a bounded linear map between unital C^*-algebras preserving the involution and zero products of self-adjoint elements in $\mathscr{A}$ then $\theta = \theta(1)J$ for a Jordan $*$-homomorphism J from $\mathscr{A}$ into $\mathscr{B}^{**}$. In [6], the involution preserving assumption is successfully removed. Modifying the arguments in [6], we will further relax the condition that the C*-algebras are unital in this

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note. In particular, we show that $\mathscr{A}$ and $\mathscr{B}$ are isomorphic as Jordan algebras if θ is bijective and sends self-adjoint elements with zero products in $\mathscr{A}$ to elements (not necessarily self-adjoint, though) with zero products in $\mathscr{B}$. They are isomorphic as C*-algebras if θ is bijective and preserves the full zero product structure.

2. Results

In the following, $\mathscr{A}, \mathscr{B}$ are always C*-algebras not necessarily with identities. $\mathscr{A}_{sa}$ denotes the (real) Jordan-Banach algebra consisting of all self-adjoint elements of $\mathscr{A}$.

Recall that a linear map J between two algebras is said to be a *Jordan homomorphism* if $J(xy+yx) = J(x)J(y) + J(y)J(x)$ for all x, y. If the underlying field has characteristic not 2, this condition is equivalent to that $J(x^2) = (Jx)^2$ for all x in the domain. We also have the identity $J(xyx) = J(x)J(y)J(x)$ for all x, y in this case.

Lemma 2.1. *Let $J : \mathscr{A}_{sa} \longrightarrow \mathscr{B}$ be a bounded Jordan homomorphism. Then J sends zero products in $\mathscr{A}_{sa}$ to zero products in $\mathscr{B}$.*

Proof. Let a, b be self-adjoint elements in $\mathscr{A}$ and $ab = 0$. We want to prove that $J(a)J(b) = 0$. Without loss of generality, we can assume that $a \geq 0$. Let a' in A_{sa} satisfy that ${a'}^2 = a$. We have $a'b = 0$. By the identities $0 = J(a'ba') = J(a')J(b)J(a')$ and $0 = J(a'b + ba') = J(a')J(b) + J(b)J(a')$, we have $J(a)J(b) = J({a'}^2)J(b) = J(a')^2 J(b) = 0$. □

Recall that when we consider $\mathscr{A}^{**}$ as the enveloping W*-algebra of $\mathscr{A}$, the multiplier algebra $M(\mathscr{A})$ of $\mathscr{A}$ is the C*-subalgebra of $\mathscr{A}^{**}$,

$$M(\mathscr{A}) = \{x \in \mathscr{A}^{**} : x\mathscr{A} \subseteq \mathscr{A} \text{ and } \mathscr{A}x \subseteq \mathscr{A}\}.$$

Elements in $M(\mathscr{A})_{sa}$ can be approximated by both monotone increasing and decreasing bounded nets from $\tilde{\mathscr{A}_{sa}} = \mathscr{A}_{sa} \oplus \mathbb{R}1$ (see, e.g., [5]). In case $\mathscr{A}$ is unital, $M(\mathscr{A}) = \mathscr{A}$.

Lemma 2.2. *Let $\theta : \mathscr{A}_{sa} \to \mathscr{B}$ be a bounded linear map sending zero products in $\mathscr{A}_{sa}$ to zero products in $\mathscr{B}$. Then the restriction of θ^{**} induces a bounded linear map, denoted again by θ, from $M(\mathscr{A})_{sa}$ into $\mathscr{B}^{**}$, which sends zero products in $M(\mathscr{A})_{sa}$ to zero products in $\mathscr{B}^{**}$.*

Proof. First we consider the case $b \in \mathscr{A}_{sa}$, and p is an open projection in $\mathscr{A}^{**}$ such that $pb = 0$. For any self-adjoint element c in the hereditary C*-subalgebra $h(p) = p\mathscr{A}^{**}p \cap \mathscr{A}$ of $\mathscr{A}$, we have $cb = 0$ and thus $\theta(c)\theta(b) = 0$. By the weak* continuity of θ^{**}, we have $\theta^{**}(p\mathscr{A}_{sa}^{**}p)\theta(b) = 0$. In particular, $\theta^{**}(p)\theta(b) = 0$.

Let a, b be self-adjoint elements in $M(\mathscr{A})$ with $ab = 0$. We want to prove that $\theta(a)\theta(b) = 0$. Without loss of generality, we can assume both a, b are positive. Let $0 \leq a_\alpha + \lambda_\alpha \uparrow a$ be a monotone increasing net from $\tilde{\mathscr{A}_{sa}}$. Since $0 \leq b(a_\alpha + \lambda_\alpha)b \uparrow bab = 0$, we have $(a_\alpha + \lambda_\alpha)b = 0$ for all α. Similarly, there is a monotone increasing net $0 \leq b_\beta + s_\beta \uparrow b$ from $\tilde{\mathscr{A}_{sa}}$ such that $(a_\alpha + \lambda_\alpha)(b_\beta + s_\beta) = 0$ for all β. We can assume the real scalar $\lambda_\alpha \neq 0$. Then $s_\beta = 0$ for all β. In particular, we see that a_α commutes with all b_β. In the abelian C*-subalgebra of $M(\mathscr{A})$ generated by a_α, b_β and 1, we see that $a_\alpha + \lambda_\alpha$ can be approximated in norm by finite real linear combinations of open projections disjoint from b_β. By the first paragraph, we have $\theta(a_\alpha + \lambda_\alpha)\theta(b_\beta) = 0$.

By the weak* continuity of θ^{**} again, we see that $\theta(a_\alpha + \lambda_\alpha)\theta(b) = \lim_\beta \theta(a_\alpha + \lambda_\alpha)\theta(b_\beta) = 0$ for each α, and then $\theta(a)\theta(b) = \lim_\alpha \theta(a_\alpha + \lambda_\alpha)\theta(b) = 0$. □

With Lemma 2.2, results in [6] concerning zero product preservers of unital C*-algebras can be extended easily to the non-unital case. We restate [6, Lemmas 4.4 and 4.5] below, but now here $\mathscr{A}$ does not necessarily have an identity.

Lemma 2.3. *Let $\theta : \mathscr{A} \to \mathscr{B}$ be a bounded linear map sending zero products in $\mathscr{A}_{sa}$ to zero products in $\mathscr{B}$. For any a in $M(\mathscr{A})$, we have*
(i) $\theta(1)\theta(a) = \theta(a)\theta(1)$,
(ii) $\theta(1)\theta(a^2) = (\theta(a))^2$.
If $\theta(1)$ is invertible then $\theta = \theta(1)J$ for a bounded Jordan homomorphism J from $\mathscr{A}$ into $\mathscr{B}$.

Theorem 2.4. *Two C*-algebras $\mathscr{A}$ and $\mathscr{B}$ are isomorphic as Jordan algebras if and only if there is a bounded bijective linear map θ between them sending zero products in $\mathscr{A}_{sa}$ to zero products in $\mathscr{B}$. If θ is just surjective, then $\mathscr{B}$ is isomorphic to the C*-algebra $\mathscr{A}/\ker\theta$ as Jordan algebras.*

Proof. One way follows from Lemma 2.1. Conversely, suppose $\theta(\mathscr{A}) = \mathscr{B}$. Since $\theta(1)\theta(a^2) = \theta(a)^2$ for all a in $\mathscr{A}$ and $\mathscr{B} = \mathscr{B}^2$, we have $\theta(1)\mathscr{B} = \mathscr{B}$. Thus, the central element $\theta(1)$ is invertible. Lemma 2.3 applies, by noting that closed Jordan ideals of C*-algebras are two-sided ideals [7]. □

In case θ preserves all zero products in $\mathscr{A}$, we have the following non-unital version of [6, Theorem 4.11].

Theorem 2.5. *Let θ be a surjective bounded linear map from a C*-algebra $\mathscr{A}$ onto a C*-algebra $\mathscr{B}$. Suppose that $\theta(a)\theta(b) = 0$ for all $a, b \in \mathscr{A}$ with $ab = 0$. Then $\theta(1)$ is a central element and invertible in $M(\mathscr{B})$. Moreover, $\theta = \theta(1)\varphi$ for a surjective algebra homomorphism φ from $\mathscr{A}$ onto $\mathscr{B}$.*

Proof. First, we have already seen in the proof of Theorem 2.4 that $\theta(1)$ is a central element and invertible in $M(\mathscr{B})$. Second, we observe that to utilize the results [6, Theorems 4.12 and 4.13] of Brešar [4], and [6, Lemma 4.14] of Akemann and Pedersen [2], one does not need to assume $\mathscr{A}$ or $\mathscr{B}$ is unital. Together with our new Theorem 2.4, which is a non-unital version of [6, Theorem 4.6], we can now make use of the same proof of [6, Theorem 4.11] to establish the assertion. □

Motivated by the theory of Banach lattices (see, e.g., [1]), we call two C*-algebras being *d-isomorphic* if there is a bounded bijective linear map between them sending zero-products to zero-products. We end this note with the following

Corollary 2.6. *Two C*-algebras are d-isomorphic if and only if they are $*$-isomorphic.*

Proof. The conclusion follows from Theorem 2.5 and a result of Sakai [8, Theorem 4.1.20] stating that two algebraic isomorphic C*-algebras are indeed $*$-isomorphic. □

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