FIXED POINT THEOREMS FOR GENERAL CONTRACTIVE MAPPINGS WITH W-DISTANCES IN METRIC SPACES

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Abstract. In this paper, using the concept of w-distances on a metric space, we first prove a generalized fixed point theorem for mappings without continuity in a complete metric space. Using this result, we obtain new and well-known fixed point theorems in a complete metric space.

1. Introduction

Let X be a metric space with metric d. A function $p : X \times X \to [0, \infty)$ is said to be a *w-distance* [8] on X if the following are satisfied:

(i) $p(x, z) \le p(x, y) + p(y, z)$ for all $x, y, z \in X$;
(ii) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
(iii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \le \delta$ and $p(z, y) \le \delta$ imply $d(x, y) \le \varepsilon$.

Using the concept of w-distances, Kada, Suzuki and Takahashi [8] improved Caristi's fixed point theorem [1], Ekeland's variational principle [4] and the nonconvex minimization theorem according to Takahashi [15].

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *contractive* if there exists $r \in [0, 1)$ such that $d(Tx, Ty) \le r\ d(x, y)$ for all $x, y \in X$. Such a mapping is also called *r-contractive.* A mapping $T : X \to X$ is said to be *Kannan* [9] if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le \alpha\{d(x, Tx) + d(y, Ty)\}$$

for all $x, y \in X$. A mapping $T : X \to X$ is said to be *contractively nonspreading* [2, 19, 7] if there exists $\beta \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le \beta\{d(x, Ty) + d(y, Tx)\}$$

for all $x, y \in X$. A mapping $T : X \to X$ is called *contractively hybrid* [6] if there exists $\gamma \in [0, \frac{1}{3})$ such that

$$d(Tx, Ty) \le r\{d(Tx, y) + d(Ty, x) + d(x, y)\}$$

for all $x, y \in X$. Recently, motivated by generalized hybrid mappings [10] in a Hilbert space, Hasegawa, Komiya and Takahashi [6] introduced the concept of contractively generalized hybrid mappings on metric spaces, and studied fixed point theorems for such mappings on complete metric spaces. Let (X, d) be a metric

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space. A mapping $T : X \to X$ is called *contractively generalized hybrid* [6] if there exist $\alpha, \beta \in \mathbb{R}$ and $r \in [0, 1)$ such that

$$\begin{aligned}\alpha d(Tx, Ty) + &(1 - \alpha)d(x, Ty) \\ &\leq r\{\beta d(Tx, y) + (1 - \beta)d(x, y)\}\end{aligned} \tag{1.1}$$

for all $x, y \in X$. Such a mapping T is also called *contractively* (α, β, r)*-generalized hybrid*; see also [3, 13]. For example, a ontractively $(\alpha,\ \beta, r)$-generalized hybrid mapping is r-contractive for $\alpha = 1$ and $\beta = 0$. It is contractively nonspreading for $\alpha = 1 + r$ and $\beta = 1$. Furthermore, it is ontractively hybrid for $\alpha = 1 + \frac{r}{2}$ and $\beta = \frac{1}{2}$; see Hasegawa, Komiya and Takahashi [6].

In this paper, motivated by w-distances and Hasegawa, Komiya and Takahashi [6], we first prove a generalized fixed point theorem for mappings without continuity in a complete metric space. Using this result, we obtain new and well-known fixed point theorems in a complete metric space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let X be a metric space with metric d. Then we denote by $W(X)$ the set of all w-distances on X. A w-distance p on X is called *symmetric* if $p(x, y) = p(y, x)$ for all $x, y \in X$. We denote by $W_0(X)$ the set of all symmetric w-distances on X. Note that the metric d is an element of $W_0(X)$. We also know that there are many important examples of w-distances on X; see [8, 16]. We denote by $WC_1(X)$ the set of all mappings T of X into itself such that there exist $p \in W(X)$ and $r \in [0, 1)$ satisfying

$$p(Tx, Ty) \leq rp(x, y)$$

for all $x, y \in X$. Such a mapping T is called *p-contractive*. Shioji, Suzuki and Takahashi [14] also introduced the sets $WC_2(X)$, $WC_0(X)$, $WK_1(X)$, $WK_2(X)$ and $WK_0(X)$ of mappings of X into itself as follows:

$T \in WC_2(X)$ if and only if there exist $p \in W(X)$ and $r \in [0, 1)$ such that

$$p(Tx, Ty) \leq rp(y, x) \text{ for all } x, y \in X;$$

$T \in WC_0(X)$ if and only if there exist $p \in W_0(X)$ and $r \in [0, 1)$ such that

$$p(Tx, Ty) \leq rp(x, y) \text{ for all } x, y \in X;$$

$T \in WK_1(X)$ if and only if there exist $p \in W(X)$ and $\alpha \in [0, 1/2)$ such that

$$p(Tx, Ty) \leq \alpha\{p(Tx, x) + p(Ty, y)\} \text{ for all } x, y \in X;$$

$T \in WK_2(X)$ if and only if there exist $p \in W(X)$ and $\alpha \in [0, 1/2)$ such that

$$p(Tx, Ty) \leq \alpha\{p(Tx, x) + p(y, Ty)\} \text{ for all } x, y \in X;$$

$T \in WK_0(X)$ if and only if there exist $p \in W_0(X)$ and $\alpha \in [0, 1/2)$ such that

$$p(Tx, Ty) \leq \alpha\{p(Tx, x) + p(Ty, y)\} \text{ for all } x, y \in X.$$

In particular, a mapping $T \in WK_1(X)$ is called *p-Kannan*. The following lemma was proved in [14].

Lemma 2.1 ([14]). *Let X be a metric space with metric d, let p be a w-distance on X, let T be a mapping of X into itself and let u be a point in X such that*

$$\lim_{m,n\to\infty} p(T^m u, T^n u) = 0.$$

Then for every $x \in X$, $\lim_{k\to\infty} p(T^k u, x)$ and $\lim_{k\to\infty} p(x, T^k u)$ exist. Moreover, let β and γ be functions from X to $[0,\infty)$ defined by

$$\beta(x) = \lim_{k\to\infty} p(T^k u, x) \text{ and } \gamma(x) = \lim_{k\to\infty} p(x, T^k u).$$

Then the following hold:

(i) *β is lower semicontinuous on X;*
(ii) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\beta(x) \leq \delta$ and $\beta(y) \leq \delta$ imply $d(x,y) \leq \varepsilon$. In particular, the set $\{x \in X : \beta(x) = 0\}$ consists of at most one point;*
(iii) *the functions q_0 and q_1 from $X \times X$ to $[0,\infty)$ defined by*

$$q_0(x,y) = \beta(x) + \beta(y) \text{ and } q_1(x,y) = \gamma(x) + \beta(y)$$

are w-distances on X.

Shioji, Suzuki and Takahashi [14] proved the following theorem from Lemma 2.1.

Theorem 2.2 ([14]). *Let (X,d) be a metric space. Then*

$$WC_1(X) = WC_0(X) = WK_1(X) = WK_0(X) \subset WC_2(X) = WK_2(X).$$

Iemoto, Takahashi and Yingtaweesittikul [7] also introduced the following class of mappings of X into itself. Let p be a w-distance on X. A mapping $T : X \to X$ is called *p-contractively nonspreading* if there exists $\alpha \in [0, 1/2)$ such that

$$p(Tx, Ty) \leq \alpha\{p(Tx, y) + p(x, Ty)\} \quad \forall x, y \in X.$$

In [7], they proved the following result from Lemma 2.1.

Theorem 2.3 ([7]). *Let (X,d) be a metric space and let p be a w-distance on X such that $p(x,x) = 0$ for all $x \in X$. Let T be a p-contractively nonspreading mapping of X into itself. Then T is in $WC_0(X)$.*

Let ℓ^∞ be the Banach space of bounded sequences with the supremum norm. A linear functional μ on ℓ^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1,1,1,....)$. For $x = (x_1, x_2, x_3, ...)$, the value $\mu(x)$ is also denoted by $\mu_n(x_n)$. A mean μ on ℓ^∞ is called a *Banach limit* if it satisfies $\mu_n(x_n) = \mu_n(x_{n+1})$. If μ is a Banach limit on ℓ^∞, then for $x = (x_1, x_2, x_3, ...) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $x = (x_1, x_2, x_3, ...) \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(x) = \mu_n(x_n) = a$. For details, we can refer [16].

3. Generalized fixed point theorem

In this section, we prove a fixed point theorem for mappings with w-distances in complete metric spaces. Before proving it, we need the following lemma proved by Kada, Suzuki and Takahashi [8]; see also [16].

Lemma 3.1 ([8]). *Let (X,d) be a complete metric space and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, and let $x,y,z \in X$. Then the following hold:*

(i) *If $p(x_n,y) \le \alpha_n$ and $p(x_n,z) \le \beta_n$ for all $n \in \mathbb{N}$, then $y=z$. In particular, if $p(x,y)=0$ and $p(x,z)=0$, then $y=z$;*
(ii) *if $p(x_n,y_n) \le \alpha_n$ and $p(x_n,z) \le \beta_n$ for all $n \in \mathbb{N}$, then the sequence $\{y_n\}$ converges to z;*
(iii) *if $p(x_n,x_m) \le \alpha_n$ for all $n,m \in \mathbb{N}$ with $m>n$, then the sequence $\{x_n\}$ is a Cauchy sequence;*
(iv) *if $p(y,x_n) \le \alpha_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Theorem 3.2. *Let (X,d) be a complete metric space, let $p \in W_0(X)$ and let $\{x_n\}$ be a sequence in X such that $\{p(x_n,x)\}$ is bounded for some $x \in X$. Let T be a mapping of X into itself. Suppose that there exist a real number $r \in [0,1)$ and a mean μ on ℓ^∞ such that*

$$\mu_n p(x_n,Ty) \le r\mu_n p(x_n,y), \quad \forall y \in X.$$

Then, the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u.*

Proof. Since $\{p(x_n,x)\}$ is bounded for some $x \in X$, we have that, for any $y \in X$, $\{p(x_n,y)\}$ is bounded. In fact, we have that, for any $n \in \mathbb{N}$,

$$p(x_n,y) \le p(x_n,x) + p(x,y) \le \sup_{m\in\mathbb{N}} p(x_m,x) + p(x,y) < \infty.$$

Using a mean μ on ℓ^∞, we can define a function $g: X \to \mathbb{R}$ as follows:

$$g(y) = \mu_n p(x_n,y), \quad \forall y \in X.$$

For any $z \in X$, consider a sequence $\{T^n z\}$ in X. We have that, for any $m,n \in \mathbb{N}$,

$$p(T^m z, T^{m+1}z) \le p(T^m z, x_n) + p(x_n, T^{m+1}z).$$

Since μ is a mean on ℓ^∞ and p is symmetric, we have that, for any $m \in \mathbb{N}$,

$$\begin{aligned} p(T^m z, T^{m+1}z) &\le \mu_n p(T^m z, x_n) + \mu_n p(x_n, T^{m+1}z) \\ &= \mu_n p(x_n, T^m z) + \mu_n p(x_n, T^{m+1}z) \\ &\le r\mu_n p(x_n, T^{m-1}z) + r\mu_n p(x_n, T^m z) \\ &\le \dots \\ &\le r^m \mu_n p(x_n,z) + r^m \mu_n p(x_n,Tz) \\ &\le r^m \mu_n p(x_n,z) + r^{m+1}\mu_n p(x_n,z) \\ &= r^m(1+r)\mu_n p(x_n,z) \\ &= r^m(1+r)g(z). \end{aligned} \tag{3.1}$$

We have from (3.1) that, for any $l, m \in \mathbb{N}$ with $m > l$,

$$\begin{aligned} p(T^l z, T^m z) &\le p(T^l z, T^{l+1} z) + p(T^{l+1} z, T^{l+2} z) + \cdots + p(T^{m-1} z, T^m z) \\ &\le r^l (1+r) g(z) + r^{l+1}(1+r) g(z) + \cdots + r^{m-1}(1+r) g(z) \\ &\le r^l (1+r) g(z) + r^{l+1}(1+r) g(z) + \cdots + r^{m-1}(1+r) g(z) + \dots \\ &= r^l (1+r) g(z)(1 + r + r^2 + r^3 + \dots) \\ &= r^l (1+r) g(z) \frac{1}{1-r} \end{aligned} \tag{3.2}$$

and $r^l(1+r)g(z)\frac{1}{1-r} \to 0$ as $l \to \infty$. We have from Lemma 3.1 that $\{T^m z\}$ is a Cauchy sequence in X. Since X is complete, we have that $\{T^m z\}$ converges. Let $T^m z \to u$. We know from the definition of p that, for any $n \in \mathbb{N}$, $y \mapsto p(x_n, y)$ is lower semicontinuous. Using this, we have that, for any $n \in \mathbb{N}$,

$$p(x_n, u) \le \liminf_{m\to\infty} p(x_n, T^m z)$$

and hence

$$g(u) = \mu_n p(x_n, u) \le \mu_n \left(\liminf_{m\to\infty} p(x_n, T^m z) \right). \tag{3.3}$$

On the other hand, we have from (3.2) that, for any $l, m, n \in \mathbb{N}$ with $m > l$,

$$\begin{aligned} p(x_n, T^m z) &\le p(x_n, T^l z) + p(T^l z, T^m z) \\ &\le p(x_n, T^l z) + r^l (1+r) g(z) \frac{1}{1-r} \end{aligned}$$

and hence

$$\limsup_{m\to\infty} p(x_n, T^m z) \le p(x_n, T^l z) + r^l (1+r) g(z) \frac{1}{1-r}.$$

Applying μ to both sides of the inequality, we have that

$$\mu_n \left(\limsup_{m\to\infty} p(x_n, T^m z) \right) \le \mu_n p(x_n, T^l z) + r^l (1+r) g(z) \frac{1}{1-r}.$$

Letting $l \to \infty$, we get that

$$\mu_n \left(\limsup_{m\to\infty} p(x_n, T^m z) \right) \le \liminf_{l\to\infty} \mu_n p(x_n, T^l z). \tag{3.4}$$

We have from (3.3) and (3.4) that

$$\begin{aligned} g(u) = \mu_n p(x_n, u) &\le \mu_n \left(\liminf_{m\to\infty} p(x_n, T^m z) \right) \\ &\le \mu_n \left(\limsup_{m\to\infty} p(x_n, T^m z) \right) \\ &\le \liminf_{m\to\infty} \mu_n p(x_n, T^m z) \\ &= \liminf_{m\to\infty} g(T^m z) \\ &\le \limsup_{m\to\infty} g(T^m z). \end{aligned} \tag{3.5}$$

Furthermore, from

$$g(T^m z) = \mu_n p(x_n, T^m z) \le r \mu_n p(x_n, T^{m-1} z) \le \cdots \le r^m \mu_n p(x_n, z) = r^m g(z),$$

we have that

$$\limsup_{m\to\infty} g(T^m z) \le 0. \tag{3.6}$$

Therefore, we obtain from (3.5) and (3.6) that $g(u) \le 0$. This implies that

$$g(u) = \mu_n p(x_n, u) = 0.$$

We show that u is a fixed point of T. Since

$$p(Tu, u) \le p(Tu, x_n) + p(x_n, u)$$

for all $n \in \mathbb{N}$, we have

$$\begin{aligned} p(Tu, u) &\le \mu_n p(x_n, Tu) + \mu_n p(x_n, u) \\ &\le r\mu_n p(x_n, u) + \mu_n p(x_n, u) \\ &= r0 + 0 = 0 \end{aligned}$$

and hence $p(Tu, u) = 0$. We also have that

$$p(Tu, Tu) \le p(Tu, x_n) + p(x_n, Tu)$$

for all $n \in \mathbb{N}$. From this, we have that

$$\begin{aligned} p(Tu, Tu) &\le \mu_n p(x_n, Tu) + \mu_n p(x_n, Tu) \\ &\le r\mu_n p(x_n, u) + r\mu_n p(x_n, u) \\ &= r0 + r0 = 0 \end{aligned}$$

and hence $p(Tu, Tu) = 0$. We have from Lemma 3.1 that $Tu = u$. We show that such a fixed point u is unique. Let $Tu = u$ and $Tv = v$. Since $0 \le r < 1$ and

$$\mu_n p(x_n, u) = \mu_n p(x_n, Tu) \le r\mu_n p(x_n, u),$$

we obtain $\mu_n p(x_n, u) = 0$. Similarly, we have $\mu_n p(x_n, v) = 0$. Since

$$p(u, v) \le p(u, x_n) + p(x_n, v)$$

for all $n \in \mathbb{N}$, we have

$$\begin{aligned} p(u, v) &\le \mu_n p(x_n, u) + \mu_n p(x_n, v) \\ &= 0 + 0 = 0 \end{aligned}$$

and hence $p(u, v) = 0$. Furthermore, since

$$p(u, u) \le p(u, x_n) + p(x_n, u)$$

for all $n \in \mathbb{N}$, we have

$$\begin{aligned} p(u, u) &\le \mu_n p(x_n, u) + \mu_n p(x_n, u) \\ &= 0 + 0 = 0 \end{aligned}$$

and $p(u, u) = 0$. We have from Lemma 3.1 that $u = v$. This completes the proof. □

As a direct consequence of Theorem 3.2, we obtain the following theorem proved by Hasegawa, Komiya and Takahashi [6].

Theorem 3.3 ([6]). *Let (X, d) be a complete metric space and let T be a mapping of X into itself. Suppose that there exist a real number r with $0 \le r < 1$ and an element $x \in X$ such that $\{T^n x\}$ is bounded and*

$$\mu_n d(T^n x, Ty) \le r \mu_n d(T^n x, y), \quad \forall y \in X$$

for some mean μ on l^∞. Then, the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u.*

Proof. We know that the metric d is one of symmetric w-distances on X; see [8, 16]. We also have that $\{d(T^n x, x)\}$ is bounded because $\{T^n x\}$ is bounded. Thus we have the desired result from Theorem 3.2. □

4. Applications

In this section, using Theorem 3.2, we prove new and well-known fixed point theorems in a complete metric space. We first prove a fixed point theorem for generalized hybrid mappings with w-distances in a metric space. Let (X, d) be a metric space and let p be a w-distance on X. A mapping $T : X \to X$ is called *p-contractively generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ and $r \in [0, 1)$ such that

$$\alpha p(Tx, Ty) + (1 - \alpha)p(x, Ty) \le r\{\beta p(Tx, y) + (1 - \beta)p(x, y)\} \tag{4.1}$$

for all $x, y \in X$. We call such a mapping T a *p-contractively (α,β,r)-generalized hybrid* mapping. We know that the class of the mappings above covers well-known mappings in a metric space. For example, a p-contractively (α,β,r)-generalized hybrid mapping T is p-contractive for $\alpha = 1$ and $\beta = 0$, i.e., there exists $r \in [0, 1)$ such that

$$p(Tx, Ty) \le rp(x, y), \quad \forall x, y \in X.$$

Theorem 4.1. *Let (X, d) be a complete metric space and let p be a symmetric w-distance on X. Let $T : X \to X$ be a p-contractively generalized hybrid mapping. Then T has a fixed point in X if and only if $\{p(T^n x, x)\}$ is bounded for some $x \in X$. In this case, the following hold:*

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u.*

Proof. Since $T : X \to X$ is a p-contractively generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ and $r \in [0, 1)$ such that

$$\alpha p(Tx, Ty) + (1 - \alpha)p(x, Ty) \le r\{\beta p(Tx, y) + (1 - \beta)p(x, y)\} \tag{4.2}$$

for all $x, y \in X$. If $F(T) \ne \emptyset$, then $\{T^n u\} = \{u\}$ for $u \in F(T)$. So, $\{p(T^n u, u)\} = \{p(u, u)\}$ is bounded. We show the reverse. Take $x \in X$ such that $\{p(T^n x, x)\}$ is bounded. Then we have that, for any $y \in X$ and $n \in \mathbb{N}$, $\{p(T^n x, y)\}$ is bounded. In fact, we have that

$$p(T^n x, y) \le p(T^n x, x) + p(x, y) \le \sup_{m \in \mathbb{N}} p(T^m x, x) + p(x, y) < \infty. \tag{4.3}$$

We also have from (4.2) that, for any $y \in X$,

$$\begin{aligned} \alpha p(T^{n+1} x, Ty) + &(1 - \alpha)p(T^n x, Ty) \\ &\le r\{\beta p(T^{n+1} x, y) + (1 - \beta)p(T^n x, y)\}. \end{aligned}$$

Applying a Banach limit μ to both sides of the inequality, we have

$$\mu_n(\alpha p(T^{n+1}x, Ty) + (1-\alpha)p(T^n x, Ty)) \\ \leq \mu_n(r\{\beta p(T^{n+1}x, y) + (1-\beta)p(T^n x, y)\}).$$

Then, we obtain

$$\alpha\mu_n p(T^{n+1}x, Ty) + (1-\alpha)\mu_n p(T^n x, Ty) \\ \leq r\beta\mu_n p(T^{n+1}x, y) + r(1-\beta)\mu_n p(T^n x, y)$$

and hence

$$\alpha\mu_n p(T^n x, Ty) + (1-\alpha)\mu_n p(T^n x, Ty) \\ \leq r\beta\mu_n p(T^n x, y) + r(1-\beta)\mu_n p(T^n x, y).$$

This implies that

$$\mu_n p(T^n x, Ty) \leq r\mu_n p(T^n x, y)$$

for all $y \in X$. By Theorem 3.2, T has a unique fixed point u in X. Furthermore, for any $z \in X$, the sequence $\{T^n z\}$ converges to u. □

Using Theorem 4.1, we prove a fixed point theorem for p-contractive mappings in a complete metric space.

Theorem 4.2. *Let (X, d) be a complete metric space and let p be a w-distance on X. Let $T : X \to X$ be a p-contractive mapping, i.e., there exists a real number r with $0 \leq r < 1$ such that*

$$p(Tx, Ty) \leq rp(x, y)$$

for all $x, y \in X$. Then, the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u.*

Proof. Putting $\alpha = 1$ and $\beta = 0$ in (4.2), we have that

$$p(Tx, Ty) \leq rp(x, y)$$

for all $x, y \in X$. From Theorem 2.2, we have $WC_1(X) = WC_0(X)$. Then there exist a symmetric $q \in W_0(X)$ and a real number $\lambda \in [0, 1)$ such that

$$q(Tx, Ty) \leq \lambda q(x, y), \quad \forall x, y \in X. \tag{4.4}$$

Take $x \in X$ and $n \in \mathbb{N}$. Replacing x by $T^{n-1}x$ and y by $T^n x$ in (4.4), we have that

$$q(T^n x, T^{n+1}x) \leq \lambda q(T^{n-1}x, T^n x). \tag{4.5}$$

Thus we have that, for any $n \in \mathbb{N}$,

$$\begin{aligned} q(x, T^n x) &\leq q(x, Tx) + q(Tx, T^2 x) + \cdots + q(T^{n-1}x, T^n x) \\ &\leq q(x, Tx) + \lambda q(x, Tx) + \cdots + \lambda^{n-1} q(x, Tx) \\ &\leq q(x, Tx) + \lambda q(x, Tx) + \cdots + \lambda^{n-1} q(x, Tx) + \dots \\ &= q(x, Tx)(1 + \lambda + \cdots + \lambda^{n-1} + \dots) \\ &= q(x, Tx)\frac{1}{1-\lambda} \end{aligned}$$

and hence $\{q(x, T^n x)\} = \{q(T^n x, x)\}$ is bounded. We also have that

$$\mu_n q(T^n x, Ty) = \mu_n q(T^{n+1}x, Ty) \leq \lambda\mu_n q(T^n x, y), \quad \forall y \in X. \tag{4.6}$$

Therefore, we have the desired result from Theorem 4.1. □

The following is a fixed point theorem for p-Kannan mappings in a complete metric space.

Theorem 4.3. *Let (X, d) be a complete metric space and let p be a w-distance on X. Let $T \in WK_1(X)$, i.e., there exists $\alpha \in [0, 1/2)$ such that*

$$p(Tx, Ty) \leq \alpha\{p(Tx, x) + p(Ty, y)\} \text{ for all } x, y \in X.$$

Then, the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u.*

Proof. From Theorem 2.2, we have $WK_1(X) = WC_0(X)$. From Theorem 4.2, we have the desired result. □

Using Theorems 4.2 and 2.3, we also have the following fixed point theorem.

Theorem 4.4. *Let (X, d) be a complete metric space and let p be a w-distance on X such that $p(x, x) = 0$ for all $x \in X$. Let $T : X \to X$ be p-contractively nonspreading, i.e., there exists a real number γ with $0 \leq \gamma < \frac{1}{2}$ such that*

$$p(Tx, Ty) \leq \gamma\{p(Tx, y) + p(x, Ty)\}$$

for all $x, y \in X$. Then, the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u.*

Proof. We know from Theorem 2.3 that the mapping T is in $WC_0(X)$. So, we have the desired result from Theorem 4.2. □

Concerning that $\{p(T^n x, x)\}$ is bounded for some $x \in X$ in Theorem 4.1, we have the following lemma.

Lemma 4.5. *Let (X, d) be a complete metric space and let p be a w-distance on X such that $p(x, x) = 0$ for all $x \in X$. Let $T : X \to X$ be a p-contractively (α,β,r)-generalized hybrid mapping, i.e., there exist $\alpha, \beta \in \mathbb{R}$ and $r \in [0, 1)$ such that*

$$\alpha p(Tx, Ty) + (1 - \alpha)p(x, Ty) \leq r\{\beta p(Tx, y) + (1 - \beta)p(x, y)\} \tag{4.7}$$

for all $x, y \in X$. Furthermore, α, β and r satisfy

$$\beta \geq 0, \ \alpha - r\beta > 0 \text{ and } r < \frac{\alpha}{1 + \beta}.$$

Then, $\{p(T^n x, x)\}$ is bounded for all $x \in X$.

Proof. Take $x \in X$ and $n \in \mathbb{N}$. Replacing x by $T^n x$ and y by $T^{n-1}x$ in (4.7), we have

$$\begin{aligned} \alpha p(T^{n+1}x, T^n x) &+ (1 - \alpha)p(T^n x, T^n x) \\ &\leq r\{\beta p(T^{n+1}x, T^{n-1}x) + (1 - \beta)p(T^n x, T^{n-1}x)\}. \end{aligned} \tag{4.8}$$

From $\beta \geq 0$ and (4.8), we have

$$\begin{aligned} \alpha p(T^{n+1}x, T^n x) \leq r\{&\beta(p(T^{n+1}x, T^n x) \\ &+ p(T^n x, T^{n-1}x)) + (1 - \beta)p(T^n x, T^{n-1}x)\} \end{aligned} \tag{4.9}$$

and hence

$$(\alpha - r\beta)p(T^{n+1}x, T^n x) \leq rp(T^n x, T^{n-1}x). \tag{4.10}$$

From $\alpha - r\beta > 0$ we have

$$p(T^{n+1}x, T^n x) \leq \frac{r}{\alpha - r\beta} p(T^n x, T^{n-1}x). \tag{4.11}$$

From $r < \frac{\alpha}{1+\beta}$, we have $r < \alpha - r\beta$ and

$$0 \leq \frac{r}{\alpha - r\beta} < 1.$$

Putting $\lambda = \frac{r}{\alpha - r\beta}$, we have that for any $n \in \mathbb{N}$,

$$\begin{aligned} p(T^n x, x) &\leq p(T^n x, T^{n-1}x) + p(T^{n-1}x, T^{n-2}x) + \cdots + p(T^2 x, Tx) + p(Tx, x) \\ &\leq \lambda^{n-1} p(Tx, x) + \lambda^{n-2} p(Tx, x) + \cdots + \lambda p(Tx, x) + p(Tx, x) \\ &\leq p(Tx, x)(1 + \lambda + \cdots + \lambda^{n-1} + \dots) \\ &= p(Tx, x)\frac{1}{1-\lambda}. \end{aligned}$$

Then the sequence $\{p(T^n x, x)\}$ is bounded. □

Using Theorem 4.1 and Lemma 4.5, we prove the following fixed point theorem proved by Hasegawa, Komiya and Takahashi [6].

Theorem 4.6 ([6])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be an (α,β,r)-contractively generalized hybrid mapping, i.e., there exist $\alpha, \beta \in \mathbb{R}$ and $r \in [0, 1)$ such that*

$$\alpha d(Tx, Ty) + (1 - \alpha)d(x, Ty) \leq r\{\beta d(Tx, y) + (1 - \beta)d(x, y)\}$$

for all $x, y \in X$. Furthermore, α, β and r satisfy

$$\beta \geq 0,\ \ \alpha - r\beta > 0 \text{ and } r < \frac{\alpha}{1+\beta}.$$

Then, the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u.*

Proof. Since $d(x, y) = d(y, x)$ and $d(x, x) = 0$ for all $x, y \in X$, we have the desired result from Theorem 4.1 and Lemma 4.5. □

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