BANACH-STONE THEOREMS FOR VECTOR VALUED FUNCTIONS ON COMPLETELY REGULAR SPACES

LEI LI AND NGAI-CHING WONG

Abstract. We obtain several Banach-Stone type theorems for vector-valued functions in this paper. Let X, Y be realcompact or metric spaces, E, F locally convex spaces, and ϕ a bijective linear map from $C(X, E)$ onto $C(Y, F)$. If ϕ preserves zero set containments, i.e.,

$$z(f) \subseteq z(g) \Longleftrightarrow z(\phi(f)) \subseteq z(\phi(g)), \quad \forall\, f, g \in C(X, E),$$

then X is homeomorphic to Y, and ϕ is a weighted composition operator. The above conclusion also holds if we assume a seemingly weaker condition that ϕ preserves nonvanishing functions, i.e.,

$$z(f) = \emptyset \Longleftrightarrow z(\phi f) = \emptyset, \quad \forall\, f \in C(X, E).$$

These two results are special cases of the theorems in a very general setting in this paper, covering bounded continuous vector-valued functions on general completely regular spaces, and uniformly continuous vector-valued functions on metric spaces. Our results extend and generalize many recent ones, while our arguments are not usually seen in the literature.

1. Introduction

The classical Banach-Stone theorem states that the geometric structure of the Banach space $C(X)$ of continuous scalar-valued functions on a compact (Hausdorff) space X determines X. In the cases a Banach space E or its Banach dual E^* is strictly convex, Jerison [20] and Lau [22], respectively, showed that the vector-valued function space $C(X, E)$ also determines X. More precisely, they showed that if ϕ is a surjective linear isometry from $C(X, E)$ onto $C(Y, E)$, then there is a homeomorphism $\tau : Y \to X$ and fiber surjective linear isometries J_y of E such that ϕ carries a weighted

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composition operator form

$$\phi(f)(y) = J_y(f(\tau(y))), \quad \forall\, y \in Y. \tag{1.1}$$

It is, however, not always the case, unless the Banach spaces E is uniformly non-square [19] or with trivial centralizers [9]. See also, e.g., [8, 10].

Some efforts in obtaining similar conclusions for bijective linear maps of continuous vector-valued functions preserving other properties have appeared in the literature. When $E = F$ is the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, it is well-known that every ring isomorphism $\phi : C(X) \to C(Y)$ gives rise to a homeomorphism $\tau : Y \to X$ such that $\phi(f) = \phi(1) f \circ \tau$ for all f in $C(X)$ (see, e.g., [16]). As a substitute for the multiplication preservers, which makes no sense for vector-valued functions, a linear map $\phi : C(X, E) \to C(Y, F)$ is said to be *separating* [15, 7, 3], or *disjointness preserving* [1, 2], if for any $f, g \in C(X, E)$,

$$\|f(x)\|\|g(x)\| = 0, \forall x \in X \quad \Longrightarrow \quad \|\phi(f)(y)\|\|\phi(g)(y)\| = 0, \forall y \in Y;$$

and ϕ is *biseparating* if the inverse implication also holds. If we let

$$z(f) = \{x \in X : f(x) = 0\}$$

be the *zero set* of f, then ϕ is biseparating exactly when

$$z(f) \cup z(g) = X \quad \Longleftrightarrow \quad z(\phi(f)) \cup z(\phi(g)) = Y, \quad \forall\, f, g \in C(X, E).$$

Without any additional assumption on E and F, surjective biseparating linear maps also provide homeomorphisms between the compact spaces X and Y (see, e.g., [3, 14, 15]). Moreover, ϕ carries the weighted composition operator form (1.1). The fiber bijective linear maps J_y are all bounded if and only if ϕ is bounded; indeed, $\|\phi\| = \sup_{y\in Y} \|J_y\|$ (see, e.g., [18, 14]).

When X, Y are realcompact and the Banach spaces E, F are *infinite* dimensional, surjective biseparating linear maps $\phi : C^b(X, E) \to C^b(Y, F)$ between bounded continuous vector-valued function spaces again gives rise to a homeomorphism $\tau : Y \to X$ and carries the form (1.1) as well (see, e.g., [4, 5, 6, 7]). Surprisingly, the following example from [16, 4M] shows that the algebra, the lattice, and the geometric structures of the Banach algebra $C^b(X)$ of bounded continuous functions altogether are still not enough to determine a realcompact space X.

Example 1.1. Let Σ be $\mathbb{N} \cup \{\sigma\}$ (where $\sigma \in \beta\mathbb{N}\backslash\mathbb{N}$). Then $\mathbb{N}$ is dense in Σ, and every function f in $C^b(\mathbb{N})$ can be extended uniquely to a function f^σ in $C^b(\Sigma)$. Although the bijective linear map ϕ from $C^b(\mathbb{N})$ onto $C^b(\Sigma)$ defined by $f \mapsto f^\sigma$ provides an isometric, algebraic and lattice isomorphism, the realcompact spaces $\mathbb{N}$ and Σ are not homeomorphic.

We are now looking for an ultimate condition to ensure a Banach-Stone type theorem for vector-valued functions on realcompact, or more generally, completely regular spaces in this paper. We see in Example 1.1 and Theorem 3.4 that the correct condition for the realcompact case is not being biseparating but that of *preserving zero set containments* (in two directions), i.e.,

$$z(f) \subseteq z(g) \quad \Longleftrightarrow \quad z(\phi(f)) \subseteq z(\phi(g)).$$

This condition ensures a homeomorphism $\tau : Y \to X$, and fiber bijective linear maps $J_y : E \to F$ such that (1.1) holds. An even weaker condition is that of *preserving nonvanishing functions* (in two directions), i.e.,

$$z(f) = \emptyset \quad \Longleftrightarrow \quad z(\phi(f)) = \emptyset.$$

In many interesting cases, we shall see that this condition also suffices to ensure the desired conclusion, as shown in Theorems 4.4 and 4.7.

Finally, we mention that our results work for the case E, F being locally convex spaces. Moreover, we develop our results in a general setting, which covers in particular also uniformly continuous vector-valued functions on metric spaces. Our results extend and generalize those mentioned above and also those in [4, 5, 28, 11, 13, 21, 25, 27], while our arguments are not usually seen in the literature. As an application, we show that every surjective local automorphism of $C(X)$ is an automorphism, where X is a completely regular space.

2. Topological preliminaries

Assume the underlying field is $\mathbb{R}$ in this section. We can describe the realcompactification υX of a completely regular space X by z-ultrafilters. For any set $\mathcal{A}$ of continuous functions on X, denote by

$$Z(\mathcal{A}) = \{z(f) : f \in \mathcal{A}\}$$

the family of zero sets of functions in $\mathcal{A}$. In particular, we write

$$Z(X) := Z(C(X)) = Z(C^b(X)).$$

A *z-filter* $\mathcal{F}$ on X is a filter of zero sets in $Z(X)$. Call $\mathcal{F}$ a *z-ultrafilter* if it is a maximal z-filter; and call $\mathcal{F}$ *prime* if $A \in \mathcal{F}$ or $B \in \mathcal{F}$ whenever $X = A \cup B$ and $A, B \in Z(X)$. Associated to each z-ultrafilter $\mathcal{F}$, a maximal ideal I of $C(X)$ consists of all continuous function f such that $z(f) \in \mathcal{F}$. Call $\mathcal{F}$ *fixed* if $\bigcap \mathcal{F}$ is a singleton, and call $\mathcal{F}$ *real* if the quotient field $C(X)/I$ is isomorphic to $\mathbb{R}$.

The Stone-Čech compactification βX can be identified with the set of *all* z-ultrafilters on X. In this setting, X consists of all fixed z-ultrafilters. The Hewitt-Nachbin realcompactification υX consists of all *real* z-ultrafilters. It is worthwhile to remark that the realcompactification υX of X is the largest subspace of its Stone-Čech compactification βX such that every continuous real function on X extends uniquely to υX, while exactly every *bounded* continuous real function on X extends uniquely to the whole of βX.

Clearly, X is compact if and only if $X = \beta X$. Call X a *realcompact* space if $X = \upsilon X$. In fact, X is realcompact if and only if every prime z-filter with the countable intersection property is fixed. For instance, Linderlöf (and thus separable metric) spaces are realcompact, and discrete spaces of non-measurable cardinality are another examples. Especially, all subspaces of the Euclidean spaces $\mathbb{R}^n$ (and $\mathbb{C}^n$ as well) are realcompact. In general, a topological space X is completely regular if and only if X can be embedded into a product of real lines, and X is realcompact (resp. compact) if and only if X is homeomorphic to a closed (resp. compact) subspace of a product of real lines. However, the ordinal interval $[0, \omega_1)$ is not realcompact, where ω_1 is the first uncountable ordinal.

Since every G_δ-point forms a zero set, $\beta X \backslash X$ contains no G_δ-point in βX. As every zero set in υX meets X, we see that $\upsilon X \backslash X$ contains no G_δ-point in υX either. We refer the readers to the books [16] and [29] for more about z-ultrafilters and realcompact spaces.

3. A Banach-Stone theorem for linear zero set containment preservers

Suppose that X is a completely regular space, and E is a locally convex space with the topological dual space E^* over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If f is in $C(X)$ and e is a vector in E, denote by $f \otimes e$ the function $x \mapsto f(x)e$ in $C(X, E)$. In particular, $1 \otimes e$ denotes the constant function $x \mapsto e$ on X.

Let $\mathcal{A}(X, E)$ be a vector subspace of $C(X, E)$, and let

$$\mathcal{A}(X) := \{\psi \circ f : f \in \mathcal{A}(X, E), \psi \in E^*\},$$

be the subset of $C(X)$ consisting of coordinate functions of all f in $\mathcal{A}(X, E)$. Denote by $\mathcal{A}^b(X) = \mathcal{A}(X) \cap C^b(X)$.

Definition 3.1. A vector subspace $\mathcal{A}(X, E)$ of $C(X, E)$ is said to be *nicely regular* if the following conditions hold.

($\mathcal{A}1$) $\mathcal{A}(X)$ is self-adjoint if $\mathbb{K} = \mathbb{C}$, and its hermitian part $\mathrm{Re}\mathcal{A}(X)$ is a sublattice of $C(X)$ containing all constant functions.
($\mathcal{A}2$) For any h in $\mathcal{A}(X)$ and any e in E, the function $h \otimes e$ is in $\mathcal{A}(X, E)$.
($\mathcal{A}3$) $Z(X) = Z(\mathcal{A}(X)) = Z(\mathcal{A}(X, E))$.
($\mathcal{A}4$) If $h_n \geq 0$ is in $\mathcal{A}^b(X)$ for $n = 1, 2, \ldots$, then there is a strictly positive sequence $\{\alpha_n\}$ such that the sum $\sum_n \alpha_n h_n$ converges pointwisely to a function in $\mathcal{A}(X)$.

The basic models of nicely regular function spaces are $C(X, E)$ and $C^b(X, E)$. For a metric space X and a normed space E, the spaces $UC(X, E)$ and $UC^b(X, E)$ of uniformly and bounded uniformly continuous functions, respectively, are also nicely regular.

Lemma 3.2. *Let X, Y be completely regular spaces, and E, F be locally convex spaces. Assume both $\mathcal{A}(X, E)$ and $\mathcal{A}(Y, F)$ are nicely regular, and ϕ is a linear bijection from $\mathcal{A}(X, E)$ onto $\mathcal{A}(Y, F)$. If ϕ is nonvanishing preserving, i.e.,*

$$z(f) = \emptyset \quad \Longleftrightarrow \quad z(\phi(f)) = \emptyset, \quad \forall\, f \in \mathcal{A}(X, E),$$

then ϕ is biseparating, i.e.,

$$z(f) \cup z(g) = X \quad \Longleftrightarrow \quad z(\phi(f)) \cup z(\phi(g)) = Y, \quad \forall\, f, g \in \mathcal{A}(X, E).$$

Proof. Suppose that f and g are in $\mathcal{A}(X, E)$ with $z(f) \cup z(g) = X$, but $z(\phi(f)) \cup z(\phi(g)) \neq Y$. Let y_0 be in Y such that $(\phi f)(y_0) \neq 0$ and $(\phi g)(y_0) \neq 0$. Without loss of generality, we can assume that there exists a linear functional ψ in F^* such that $\psi((\phi f)(y_0)) = \psi((\phi g)(y_0)) = 1$.

Define h in $\mathcal{A}(Y)$ by

$$h(y) = \max\left\{0, \frac{1}{2} - \mathrm{Re}\,\psi((\phi f)(y)), \frac{1}{2} - \mathrm{Re}\,\psi((\phi g)(y))\right\}, \quad y \in Y.$$

Let

$$k = \phi^{-1}(h \otimes (\frac{(\phi f)(y_0) + (\phi g)(y_0)}{2})).$$

Claim: $z(\phi f + \phi k) = \emptyset$.

Assume on the contrary that y_1 belongs to $z(\phi f + \phi k)$, that is,

$$(\phi f)(y_1) + (\phi k)(y_1) = 0. \tag{3.1}$$

In particular,

$$\psi(\phi(f)(y_1)) + h(y_1) = 0.$$

This implies a contradiction

$$h(y_1) \geq \frac{1}{2} - \mathrm{Re}\,\psi(\phi(f)(y_1)) = \frac{1}{2} + h(y_1).$$

It follows from $z(\phi f + \phi k) = \emptyset$ that $z(f+k) = \emptyset$. In a similar way, $z(g+k) = \emptyset$. Because $z(f) \cap z(k) \subset z(f+k)$ and $z(g) \cap z(k) \subset z(g+k)$, we have $z(f) \cap z(k) = z(g) \cap z(k) = \emptyset$. By the assumption that $z(f) \cup z(g) = X$, we conclude $z(k) = \emptyset$. This is a contradiction since $(\phi k)(y_0) = 0$ and ϕ is nonvanishing preserving. Hence, $z(\phi f) \cup z(\phi g) = Y$, as asserted.

Similarly, we can derive that ϕ^{-1} is also separating. $\square$

We note that a biseparating map might not be nonvanishing preserving as shown in Example 1.1.

Remark 3.3. In [13, Theorem 2], it is mentioned that following a result in [14] a "biseparating" linear map $\phi : C(X,E) \to C(Y,F)$ between spaces of continuous Banach space vector-valued functions on compact spaces is a weighted composition operator. This is, however, not quite accurate. Indeed, the "biseparating" maps in [13] actually refer to maps "preserving pairs of functions without common zeros", i.e.,

$$z(f) \cap z(g) = \emptyset \quad \Leftrightarrow \quad z(\phi(f)) \cap z(\phi(g)) = \emptyset, \quad \forall\, f, g \in C(X,E).$$

As such maps automatically preserve nonvanishing functions (by setting $f = g$), in view of Lemma 3.2, they are also biseparating in the original sense in [14]. Therefore, this citation is correct anyway.

Recall that a linear map $\phi : \mathcal{A}(X,E) \to \mathcal{A}(Y,F)$ is continuous with respect to the topologies of uniform convergence if for any continuous seminorm q of F there is a continuous seminorm p of E such that

$$\sup_{y \in Y} q(\phi(f)(y)) \leq \sup_{x \in X} p(f(x)), \quad \forall\, f \in \mathcal{A}(X,E).$$

On the other hand, a family $J_y : E \to F$ of linear operators is equicontinuous if for any continuous seminorm q of F there is a continuous seminorm p of E such that

$$q(J_y(e)) \leq p(e), \quad \forall\, e \in E, y \in Y.$$

Theorem 3.4. *Suppose that X, Y are realcompact topological spaces and E, F are locally convex spaces. Assume both $\mathcal{A}(X,E)$ and $\mathcal{A}(Y,F)$ are nicely regular, and ϕ is a linear bijective map from $\mathcal{A}(X,E)$ onto $\mathcal{A}(Y,F)$ preserving zero set containments, i.e.,*

$$z(f) \subseteq z(g) \quad \Longleftrightarrow \quad z(\phi(f)) \subseteq z(\phi(g)), \quad \forall\, f, g \in \mathcal{A}(X,E).$$

Then there exist a homeomorphism $\tau : Y \to X$ and, for each y in Y, a bijective linear map $J_y : E \to F$ such that

$$(\phi f)(y) = J_y(f(\tau(y))), \quad \forall\, f \in \mathcal{A}(X,E), y \in Y.$$

Furthermore, if both $\mathcal{A}(X,E)$ and $\mathcal{A}(Y,F)$ are equipped with the topologies of uniform convergence, then the linear map ϕ is continuous if and only if the family of fiber linear maps $\{J_y\}$ is equicontinuous.

We will establish the proof of Theorem 3.4 in several lemmas.

Lemma 3.5. *The map ϕ is biseparating and preserves nonvanishing functions.*

Proof. In view of Lemma 3.2, it suffices to check that ϕ preserves nonvanishing functions. Suppose $z(f) = \emptyset$ for some f in $\mathcal{A}(X,E)$, then $z(f) \subseteq z(g)$ for all g in $\mathcal{A}(X,E)$. This implies that $z(\phi(f)) \subseteq z(\phi(g))$. Because ϕ is surjective and $\mathcal{A}(Y,F)$ is nicely regular, $z(\phi(f)) = \emptyset$ as asserted. $\square$

For any x_0 in X, let

$$\mathcal{K}_{x_0} = \{f \in \mathcal{A}(X,E) : f(x_0) = 0\},$$

and

$$\mathcal{Z}_{x_0} = Z(\phi(\mathcal{K}_{x_0})) = \{z(\phi f) : f \in \mathcal{K}_{x_0}\}.$$

Lemma 3.6. *$\mathcal{Z}_{x_0}$ is a prime z-filter in $Z(Y)$ with the countable intersection property.*

Proof. Note that $\mathcal{A}(Y,F)$ is nicely regular and ϕ is surjective, every zero set in $Z(Y)$ can be written as $z(\phi(f))$ for some f in $\mathcal{A}(X,E)$.

Since ϕ is nonvanishing preserving, the empty set $\emptyset$ is not in $\mathcal{Z}_{x_0}$. Let $f \in \mathcal{K}_{x_0}$ and $C = z(\phi(g)) \in Z(Y)$ such that $z(\phi(f)) \subseteq C$. Then $z(f) \subseteq z(g)$ since ϕ preserves zero set containments, and hence $g \in \mathcal{K}_{x_0}$. This means that $C \in \mathcal{Z}_{x_0}$. Let $\{f_n\}$ be a sequence of functions in $\mathcal{K}_{x_0}$. By the regularity of $\mathcal{A}(X,E)$, one can choose a non-negative real-valued function g_n from $\mathcal{A}^b(X)$ for each $n = 1, 2, \ldots$, with $z(g_n) = z(f_n)$, and also a strictly positive sequence $\{\alpha_n\}$ such that the pointwise sum $g = \sum_{n=1}^{\infty} \alpha_n g_n$ belongs to $\mathcal{A}(X)$. Fix a nonzero vector e in E, and define a vector-valued function h in $\mathcal{K}_{x_0}$ by

$$h = g \otimes e.$$

Clearly,

$$x_0 \in z(h) = \bigcap_{n=1}^{\infty} z(g_n) = \bigcap_{n=1}^{\infty} z(f_n).$$

It follows from the zero set containment preserving property of ϕ that

$$\emptyset \neq z(\phi h) \subseteq \bigcap_{n=1}^{\infty} z(\phi(f_n)).$$

Therefore, $\mathcal{Z}_{x_0}$ is a z-filter with the countable intersection property.

Finally, we show that the z-filter $\mathcal{Z}_{x_0}$ is prime. Suppose that A, B are two zero sets in $Z(Y)$ with $A \cup B = Y$. By the regularity assumption, there are f, g in $\mathcal{A}(X, E)$ such that $A = z(\phi(f))$ and $B = z(\phi(g))$. In particular, $z(\phi f) \cup z(\phi g) = Y$. Then $z(f) \cup z(g) = X$ since ϕ is biseparating by Lemma 3.2. As a result, x_0 must be in $z(f)$ or $z(g)$, and this means that f or g belongs to $\mathcal{K}_{x_0}$. Therefore, A or B is in $\mathcal{Z}_{x_0}$, as asserted. $\square$

Since Y is realcompact, from Lemma 3.6, we see that the intersection of $\mathcal{Z}_{x_0}$ is a singleton, and we denote it by $\sigma(x_0)$. In other words,

$$f(x_0) = 0 \quad \Longrightarrow \quad \phi(f)(\sigma(x_0)) = 0, \quad \forall f \in \mathcal{A}(X, E). \tag{3.2}$$

Lemma 3.7. *For any $f \in \mathcal{A}(X, E)$ and $x \in X$, we have*

$$(\phi f)(\sigma(x)) = \phi(1 \otimes f(x))(\sigma(x)).$$

Moreover, $f(x) = 0$ if and only if $(\phi f)(\sigma(x)) = 0$.

Proof. For any f in $\mathcal{A}(X, E)$ and x in X, the function $f - 1 \otimes f(x)$ belongs to $\mathcal{K}_x$. It follows from (3.2) that

$$\phi(f - 1 \otimes f(x))(\sigma(x)) = 0,$$

and thus $(\phi f)(\sigma(x)) = \phi(1 \otimes f(x))(\sigma(x))$.

Finally, if $\phi(f)(\sigma(x)) = 0$ then $z(\phi(1 \otimes f(x))) \neq \emptyset$. This gives $z(1 \otimes f(x)) \neq \emptyset$, and forces $f(x) = 0$. The reverse implication is trivial. $\square$

Proof of the Theorem 3.4. Since ϕ^{-1} also preserves zero set containment, there exists a map τ from Y into X such that

$$\{\tau(y)\} = \bigcap \{z(\phi^{-1} g) : g \in \mathcal{A}(Y, F), g(y) = 0\}, \quad \forall y \in Y.$$

For any x in X, we claim that $\tau(\sigma(x)) = x$. Indeed, if $\tau(\sigma(x)) = x' \neq x$, then there exists a function g_1 in $\mathcal{A}(X)$ such that $g_1(x') \neq 0$ and $g_1(x) = 0$. Define $f_1 = g_1 \otimes e$ for some nonzero vector e in E, by Lemma 3.7, one can conclude that $(\phi f_1)(\sigma(x)) = 0$. By Lemma 3.7 again, we also have $(\phi^{-1}(\phi f_1))(\tau(\sigma(x))) = 0$, that is, $f_1(x') = 0$. This is a contradiction. Similarly, we can also conclude that $\sigma(\tau(y)) = y$ for all y in Y. Therefore, $\tau = \sigma^{-1}$.

For each y in Y, define $J_y : E \to F$ by

$$J_y(e) = \phi(1 \otimes e)(y), \quad \forall e \in E.$$

Each J_y is linear and injective. By Lemma 3.7, we see that

$$\phi(f)(y) = \phi(1 \otimes f(\tau(y)))(y) = J_y(f(\tau(y))) \tag{3.3}$$

is true for all y in Y and f in $\mathcal{A}(X, E)$. In particular, all J_y are surjective.

We claim that τ is a homeomorphism from Y onto X. Indeed, suppose that $y_\lambda \to y$ in Y but $\{\tau(y_\lambda)\}$ does not approach to $\tau(y)$ in X. Then, by passing to a subnet and using the regularity, we can choose a function f_2 from $\mathcal{A}(X,E)$ such that $f_2(\tau(y_\lambda)) = 0$ for all λ but $f_2(\tau(y)) \neq 0$. However, by (3.3) and the continuity of $\phi(f_2)$, we derive a contradiction

$$0 \neq \phi(f_2)(y) = \lim_\lambda \phi(f_2)(y_\lambda) = 0.$$

Therefore, τ is continuous. Arguing with ϕ^{-1} we will see $\tau^{-1} = \sigma$ is also continuous, and thus τ is a homeomorphism.

Next, assume ϕ is continuous with respect to the topologies of uniform convergence. For every continuous seminorm q of F there is a continuous seminorm p of E such that

$$\sup_{y\in Y} q(\phi(f)(y)) \leq \sup_{x\in X} p(f(x)), \quad \forall\, f \in \mathcal{A}(X,E).$$

This implies

$$q(J_y(e)) = q(\phi(1\otimes e)y) \leq p(e), \quad \forall\, e \in E, \forall\, y \in Y.$$

Hence, the family $\{J_y\}$ of fiber linear maps is equicontinuous.

Conversely, assume $\{J_y\}$ is equicontinuous. By (3.3), for any continuous seminorm q of F there exists a continuous seminorm p of E such that

$$q(\phi(f)(y)) = q(J_y(f(\tau(y)))) \leq p(f(\tau(y))) \leq \sup_{x\in X} p(f(x)), \quad \forall\, f \in \mathcal{A}(X,E), \forall\, y \in Y.$$

Thus, ϕ is continuous with respect to topologies of uniform convergence. $\square$

The following theorem arises when we consider the nicely regular space $C^b(X,E)$.

Theorem 3.8. *Suppose that X, Y are realcompact, E, F are Banach spaces, and ϕ is a bijective linear map from $C^b(X,E)$ onto $C^b(Y,F)$ preserving zero set containments. Then there exist a homeomorphism $\tau : Y \to X$ and, for each y in Y, a bijective linear map $J_y : E \to F$ such that*

$$(\phi f)(y) = J_y(f(\tau(y))), \quad \forall\, f \in C^b(X,E), y \in Y.$$

Moreover, ϕ is norm bounded if and only if all fiber linear maps J_y are bounded. In this case, we have

$$\|\phi\| = \sup_{y\in Y} \|J_y\|,$$

and J is a continuous map from Y into $(\mathcal{L}(E,F), SOT)$.

Proof (modified on [23, Lemma 2.4]*).* By Theorem 3.4, it suffices to prove the "moreover" part. Suppose that ϕ is bounded, then for any e in E, we have

$$\|J_y(e)\| = \|\phi(1\otimes e)(y)\| \leq \|\phi(1\otimes e)\| \leq \|\phi\|\|e\|.$$

Thus, $\|J_y\| \leq \|\phi\|$ for all y in Y.

Next, assume that all fiber linear maps J_y are bounded.

Claim. $\sup_{y\in Y} \|J_y\| < +\infty$.

Suppose on the contrary that there exists a y_n in Y and an f_n in $C^b(X,E)$ such that $\|f_n\| \leq 1$ and $\|\phi(f_n)(y_n)\| > n^3$ for $n = 1,2,\ldots$. Let $x_n = \sigma(y_n)$ and V_n be a neighborhood of x_n in X $(n = 1,2,\ldots)$ such that the family $\{V_n\}$ are pairwise disjoint. By regularity, we can choose a g_n in $C^b(X)$ such that $0 \leq g \leq 1$, $g_n(x_n) = 1$ and $g_n = 0$ outside V_n for any $n = 1,2,\ldots$. Observe that

$$\begin{aligned}\phi(f_n)(y_n) &= \phi(g_nf_n)(y_n) + \phi((1-g_n)f_n)(y_n)\\ &= \phi(g_nf_n)(y_n),\end{aligned}$$

as $((1-g_n)f_n)(x_n) = 0$. So we can assume f_n is supported in V_n for $n = 1,2,\ldots$. Let

$$f = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n \in C^b(X,E).$$

Since $(n^2f - f_n)(x_n) = 0$, we have that $n^2\phi(f)(y_n) = \phi(f_n)(y_n)$, and thus $\|\phi(f)(y_n)\| > n$ for any $n = 1,2,\ldots$. As $\phi(f)$ is a bounded vector-valued function on Y, we arrive at a contradiction. For any f in $C^b(X,E)$ and y in Y, we have $\|(\phi f)(y)\| = \|J_y(f(\tau(y)))\| \leq \|J_y\|\|f\|$. This implies $\|\phi\| \leq \sup_{y\in Y}\|J_y\|$. Therefore, $\|\phi\| = \sup_{y\in Y}\|J_y\|$.

Finally, if a net $\{y_\lambda\}$ converges to y in Y, then, for any e in E,

$$\|J_{y_\lambda}(e) - J_y(e)\| = \|\phi(1\otimes e)(y_\lambda) - \phi(1\otimes e)(y)\| \to 0$$

since $\phi(1\otimes e)$ is continuous on Y. Therefore, J is a continuous map from Y into $\mathcal{L}(E,F)$ with respect to the strong operator topology. □

Remark 3.9. We note that in the above theorem, a bijective linear zero set containment preserver ϕ between bounded continuous vector-valued function spaces on even compact spaces can be unbounded in general (see, e.g., [14, Example 2.4]).

4. A Banach-Stone theorem for linear nonvanishing preservers

Lemma 4.1. *Let X, Y be completely regular spaces, and E, F be locally convex vector spaces. Assume that both $\mathcal{A}(X,E)$ and $\mathcal{A}(Y,F)$ are nicely regular, and $\phi : \mathcal{A}(X,E) \to \mathcal{A}(Y,F)$ is a bijective linear map preserving nonvanishing functions.*

(1) If $\dim E = n$ *is finite then* $\dim F = n$.

(2) If E, F are of finite dimensional, then ϕ sends functions without common zeros to functions without common zeros. That is, for any $m \in \mathbb{N}$ and $f_1, \ldots, f_m$ in $\mathcal{A}(X, E)$, we have

$$\bigcap_{k=1}^{m} z(f_k) = \emptyset \quad \Longleftrightarrow \quad \bigcap_{k=1}^{m} z(\phi(f_k)) = \emptyset. \tag{4.1}$$

Proof. (1) Fix a basis $\{e_1, \ldots, e_n\}$ of E, and let $g_k = \phi(1 \otimes e_k)$ in $\mathcal{A}(Y, F)$ for $k = 1, 2, \ldots, n$.

Claim 1. $\{g_1(y), \ldots, g_n(y)\}$ is a basis of F for all y in Y.

Suppose that $\lambda_1, \ldots, \lambda_n$ are scalars such that $\sum_{k=1}^n \lambda_k g_k(y) = 0$ for some y in Y. Then $z(\sum_{k=1}^n \lambda_k g_k) \neq \emptyset$ implies that $z(\sum_{k=1}^n \lambda_k(1 \otimes e_k)) \neq \emptyset$, and thus $\lambda_1 = \ldots = \lambda_n = 0$. Therefore, $\{g_1(y), \ldots, g_n(y)\}$ is linearly independent in F for all y in Y. Consequently, $\dim F \geq n$.

If F has $n+1$ linearly independent vectors, then by arguing with ϕ^{-1} in a similar way, one can see that $\dim E \geq n+1$. This contradiction tells us that $\dim F = n$ and $\{g_1(y), \ldots, g_n(y)\}$ is a basis of F for every y in Y.

(2) First note that, by Lemma 3.2, ϕ is biseparating. Composing ϕ with any linear topological isomorphism between the n-dimensional locally convex spaces E and F, we can assume that $E = F$ and ϕ is a linear biseparating map from $\mathcal{A}(X, E)$ into $C(Y, E)$ sending nonvanishing functions to nonvanishing functions. Let $\{e_1', \ldots, e_n'\}$ be the basis of E^* dual to $\{e_1, \ldots, e_n\}$. It follows from Claim 1 that the inverse $G(y)$ of the $n \times n$ scalar matrix $\begin{bmatrix} g_1(y) & g_2(y) & \cdots & g_n(y) \end{bmatrix}$, with respect to the basis $\{e_1, \ldots, e_n\}$ of $E = F$, exists for all y in Y. All entries in $G(y)$ give rise to continuous functions in $C(Y)$.

Define $\phi' : \mathcal{A}(X, E) \to C(Y, E)$ by

$$\phi'(f)(y) := G(y)\phi(f)(y), \quad \forall y \in Y.$$

Note that

$$z(\phi'(f)) = z(\phi(f)), \quad \forall f \in \mathcal{A}(X, E),$$

and

$$\phi'(1 \otimes e) = 1 \otimes e, \quad \forall e \in E.$$

Moreover, ϕ' is also nonvanishing preserving.

Claim 2. Let $f \otimes e \in \mathcal{A}(X, E)$ for some e in E. Then $\phi'(f \otimes e) = g \otimes e$ such that the ranges $f(X)$ and $g(Y)$ coincide.

For any u in F independent of e, we see that $f\otimes e+\alpha\otimes u$ is nonvanishing for every nonzero scalar α. Thus $\phi'(f\otimes e+\alpha\otimes u)$ is nonvanishing as well. This shows that $\phi'(f\otimes e)(y)$ is never equal to any nonzero multiple of u. In other words, $\phi'(f\otimes e)=g\otimes e$ for some g in $C(Y)$. Furthermore, let $\lambda\in\mathbb{K}\setminus f(X)$. Then $(f-\lambda)\otimes e$ is nonvanishing. It follows that $\phi'((f-\lambda)\otimes e)=g\otimes e-\lambda\otimes e$ is also nonvanishing. Consequently, $\lambda\notin g(Y)$. The reverse inclusion follows similarly.

For any f in $\mathcal{A}(X,E)$ we can write

$$f(x)=e_1'(f(x))\otimes e_1+\cdots+e_n'(f(x))\otimes e_n,$$

and all the coordinate functions $e_k'(f)$ are in $\mathcal{A}(X)$. On the other hand, every continuous scalar function h in $\mathcal{A}(X)$ can be written uniquely as a sum of four non-negative continuous real functions in $\mathcal{A}(X)$,

$$h=h_1-h_2+i(h_3-h_4),$$

such that $h_1h_2=h_3h_4=0$. Consequently, we can write

$$f=\sum_{k=1}^n(f_{k1}-f_{k2}+i(f_{k3}-f_{k4}))\otimes e_k$$

such that all f_{kj} are continuous non-negative real functions in $\mathcal{A}(X)$, and $f_{k1}f_{k2}=f_{k3}f_{k4}=0$ for $k=1,2,\dots,n$. Accordingly, we associate a function $|f|$ in $\mathcal{A}(X,E)$ to f by defining

$$|f|=\sum_{k=1}^n(f_{k1}+f_{k2}+f_{k3}+f_{k4})\otimes e_k.$$

Note that $z(f)=z(|f|)$.

Claim 3. $|\phi'(f)|=\phi'(|f|)$ for all f in $\mathcal{A}(X,E)$.

It follows from Claim 2 that we can write

$$\phi'(f)=\sum_{k=1}^n(g_{k1}-g_{k2}+i(g_{k3}-g_{k4}))\otimes e_k$$

such that all g_{kj} are non-negative continuous real functions. Inherited from ϕ, on the other hand, ϕ' is separating. Consequently, $g_{k1}g_{k2}=g_{k3}g_{k4}=0$ for $k=1,2,\dots,n$. As a result,

$$|\phi'(f)|=\sum_{k=1}^n(g_{k1}+g_{k2}+g_{k3}+g_{k4})\otimes e_k=\phi'(|f|).$$

Now, let $f_1,\dots,f_m$ be in $\mathcal{A}(X,E)$ such that

$$\emptyset=\bigcap_{i=1}^m z(f_i)=\bigcap_{i=1}^m z(|f_i|)=z(\sum_{i=1}^m|f_i|).$$

Observe that

$$\emptyset = z(\phi'(\sum_{i=1}^{m} |f_i|)) = z(\sum_{i=1}^{m} \phi'(|f_i|)) = z(\sum_{i=1}^{m} |\phi'(f_i)|)$$
$$= \bigcap_{i=1}^{m} z(|\phi'(f_i)|) = \bigcap_{i=1}^{m} z(\phi'(f_i)) = \bigcap_{i=1}^{m} z(\phi(f_i)).$$

Therefore, ϕ sends functions without common zeros to functions without common zeros. Arguing with ϕ^{-1} similarly, we will establish the reverse preservation, and the proof is thus complete. □

When putting $m = 1$ in (4.1) we see that a linear map preserving functions without common zeros is nonvanishing preserving. Employing an argument similar as in the last part of the proof of Lemma 4.1, we can establish the following result in [25] (see, also, [12]).

Lemma 4.2. *Suppose that X, Y are completely regular spaces, and E, F are locally convex Riesz spaces and $\phi : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ sends exactly positive elements to positive elements. Then ϕ preserves functions without common zeros if and only if ϕ is nonvanishing preserving.*

Lemma 4.3. *Let X, Y be completely regular spaces, and E, F be locally convex spaces. Assume both $\mathcal{A}(X, E)$ and $\mathcal{A}(Y, F)$ are nicely regular. Suppose that a linear bijective map $\phi : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ preserves pairs without common zeros, i.e.,*

$$z(h_1) \cap z(h_2) = \emptyset \quad \Longleftrightarrow \quad z(\phi(h_1)) \cap z(\phi(h_2)) = \emptyset, \quad \forall\, h_1, h_2 \in \mathcal{A}(X, E).$$

Then ϕ preserves zero set containments, i.e.,

$$z(f) \subseteq z(g) \quad \Longleftrightarrow \quad z(\phi(f)) \subseteq z(\phi(g)), \quad \forall\, f, g \in \mathcal{A}(X, E).$$

Proof. Suppose that $z(f) \subset z(g)$ and $y \in Y$ satisfies $\phi(g)(y) \neq 0$. As in the proof of Lemma 3.2, we can find a function k in $\mathcal{A}(X, E)$ such that

$$z(\phi(g) + \phi(k)) = \emptyset \quad \text{and} \quad \phi(k)(y) = 0.$$

By the assumption,

$$z(f) \cap z(k) \subseteq z(g) \cap z(k) \subseteq z(g + k) = \emptyset.$$

This implies

$$z(\phi(f)) \cap z(\phi(k)) = \emptyset,$$

and thus $\phi(f)(y) \neq 0$, as asserted. The other direction is similar. □

Case 3 in the following theorem extends [25, Theorem 10].

Theorem 4.4. *Suppose that X, Y are realcompact spaces, E, F are locally convex spaces, and both $\mathcal{A}(X,E)$ and $\mathcal{A}(Y,F)$ are nicely regular. Let $\phi : \mathcal{A}(X,E) \to \mathcal{A}(Y,F)$ be a bijective linear map preserving nonvanishing functions. Assume that any one of the following conditions holds.*

(1) *E or F (and thus both) is of finite dimension.*
(2) *E and F are locally convex Riesz spaces, and ϕ sends exactly positive functions to positive functions.*
(3) *ϕ preserves pairs of functions without common zeros.*

Then ϕ carries the form

$$(\phi f)(y) = J_y(f(\tau(y))), \quad \forall f \in \mathcal{A}(X,E), y \in Y.$$

Here, τ is a homeomorphism from Y onto X, and all fiber linear maps $J_y : E \to F$ are bijective. When $\mathcal{A}(X,E)$ and $\mathcal{A}(Y,F)$ are equipped with the topologies of uniform convergence, ϕ is continuous if and only if the family $\{J_y\}$ is equicontinuous.

Proof. The conclusions follow from Theorem 3.4, and Lemmas 4.1, 4.2 and 4.3. □

The following special case of Theorem 4.4(2) extends [11] and [13], in which X and Y are assumed to be compact Hausdorff spaces.

Corollary 4.5. Suppose that X, Y are realcompact spaces, and E and F are Banach lattices. Let ϕ be a linear bijective map from $C(X,E)$ (resp. $C^b(X,E)$) onto $C(Y,F)$ (resp. $C^b(Y,F)$). Assume that ϕ is nonvanishing preserving, and sends exactly positive functions to positive functions. Then there exist a homeomorphism τ from Y onto X and, for any y in Y, an (automatically bounded) linear Riesz isomorphism J_y from E onto F such that

$$(\phi f)(y) = J_y(f(\tau(y)))$$

for all f in $C(X,E)$ ($C^b(X,E)$, respectively) and y in Y.

In Corollary 4.5, we assume that ϕ is nonvanishing preserving. The following example shows that the theorem is no longer valid if ϕ is not nonvanishing preserving.

Example 4.6. Let X be $\{1,2\}$ in the discrete topology and Y be the one-point topological space $\{0\}$. Equip the spaces $C(X,\mathbb{R})$ and $C(Y,\mathbb{R}^2)$ with the usual pointwise ordering and sup norm. Suppose that ϕ is a map from $C(X,\mathbb{R})$ into $C(Y,\mathbb{R}^2)$, defined by $(\phi f)(0) = (f(1), f(2))$ for all f in $C(X,\mathbb{R})$. Then ϕ is a Riesz isomorphism but it is not nonvanishing preserving. Note that the compact spaces X and Y are not homeomorphic.

Denote by $UC(X,E)$ (resp. $UC^b(X,E)$) the nicely regular spaces of (resp. bounded) uniformly continuous functions from a metric space X into a normed space E.

Theorem 4.7. *Suppose that X, Y are realcompact spaces and E, F are Banach spaces. Let ϕ be a linear bijective map between the following nicely regular function spaces preserving nonvanishing functions.*

Case 1. $\phi : C(X,E) \to C(Y,F)$.
Case 2. $\phi : C^b(X,E) \to C^b(Y,F)$.
Case 3. $\phi : UC(X,E) \to UC(Y,F)$, *where X, Y are metric spaces.*
Case 4. $\phi : UC^b(X,E) \to UC^b(Y,F)$, *where X, Y are metric spaces.*

Then ϕ carries the form

$$(\phi f)(y) = J_y(f(\tau(y))), \quad \forall f \in \mathcal{A}(X,E), y \in Y.$$

Here, τ is a homeomorphism from Y onto X and all fiber linear maps $J_y : E \to F$ are bijective. When the vector-valued function spaces are equipped with the topology of uniform convergence, then ϕ is continuous if and only if the family $\{J_y\}$ is equicontinuous.

In Cases 2 and 4, ϕ is bounded if and only if all fiber linear maps J_y are bounded, and

$$\|\phi\| = \sup_{y \in Y} \|J_y\|. \tag{4.2}$$

Moreover, in Cases 3 and 4, τ is a uniform homeomorphism.

Proof. By Lemma 3.2, ϕ is biseparating. Cases 1 and 3, follow from [5, Theorem 3.5] (see also [4]). When E, F are of infinite dimension, Cases 2 and 4 follow from [5, Theorem 3.5]. When one (and thus both) of E, F is of finite dimension, all cases follow from Theorem 4.4. The uniform continuity of τ and τ^{-1} follow from the arguments in [24, Theorem 2.3]. The equality (4.2) follows from the arguments in Theorem 3.8. ☐

On the other hand, as in the next example, we can see that the requirement of realcompactness of the topological spaces X and Y is necessary in above theorems.

Example 4.8. Let ω_1 be the first uncountable ordinal. It is well-known that the ordinal interval $X = [0, \omega_1)$ is not realcompact, while $Y = [0, \omega_1]$ is compact and hence realcompact. Since every continuous function in $C(X)$ is eventually constant, we have $C(X) = C^b(X)$ (see [16, Section 5.12]). For any f in $C(X)$, we can extend it to a unique function $\phi(f)$ in $C(Y)$. Then ϕ is a linear lattice isomorphism from $C(X)$ onto $C(Y)$ preserving nonvanishing functions. Nevertheless, X is not homeomorphic to Y.

5. Banach-Stone theorems for completely regular spaces

We now discuss the general case when X is a completely regular space and E is a realcompact locally convex space, e.g., E is a separable Banach or Frechet space. As noted in [16, Chapter 8], every function f in $C(X, E)$ has a unique extension f^υ in $C(\upsilon X, E)$, and $z(f) = \emptyset$ if and only if $z(f^\upsilon) = \emptyset$. Therefore, if X, Y are completely regular spaces and E, F are realcompact locally convex spaces, every linear bijection ϕ from $C(X, E)$ onto $C(Y, F)$ has a linear extension ϕ^υ from $C(\upsilon X, E)$ onto $C(\upsilon Y, F)$, defined in the canonical manner $\phi^\upsilon : f^\upsilon \mapsto (\phi f)^\upsilon$. Moreover, ϕ preserves zero set containments or nonvanishing functions if and only if ϕ^υ does. Note that if $\mathcal{A}(X, E)$ is a nicely regular function space then

$$\mathcal{A}^\upsilon(\upsilon X, E) := \{f^\upsilon : f \in \mathcal{A}(X, E)\}$$

is also nicely regular. The following theorem holds for example when X, Y are metrizable spaces.

Theorem 5.1. *Suppose that X, Y are completely regular spaces and E, F are realcompact locally convex spaces. If there exists a linear bijection $\phi : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ between nicely regular function spaces preserving zero set containments, then the realcompatifications υX and υY are homeomorphic by a homeomorphism τ. In particular, if X, Y are realcompact, or all points in X, Y are G_δ, then X, Y are homeomorphic and $\tau(Y) = X$. Moreover,*

$$\phi(f)^\upsilon(y) = J_y(f^\upsilon(\tau(y))), \quad \forall f \in C(X, E), \forall\, y \in \upsilon Y.$$

Here, all the fiber maps $J_y : E \to F$ are bijective and linear. Furthermore, ϕ is continuous with respect to the topologies of uniform convergence if and only if the family $\{J_y\}$ is equicontinuous.

The same conclusions hold provided that ϕ preserves nonvanishing functions instead and any one of the conditions in Theorems 4.4 or 4.7 is assumed.

Proof. The results follows from Theorems 3.4, 4.4 and 4.7, and the fact that no point in $\upsilon X \setminus X$ is G_δ. □

Let $\mathcal{A}$ be an algebra and ϕ be a map from $\mathcal{A}$ into itself. Recall that ϕ is an *automorphism* if ϕ is bijective, linear and multiplicative; and ϕ is a *local automorphism* if ϕ agrees at each point a in $\mathcal{A}$ with an automorphism ϕ_a. Equipped with Theorem 5.1, we investigate when a local automorphism of $C(X)$ is an automorphism. This is nontrivial even in the case X is compact, as we cannot use the Gleason-Kahane-Zelazko Theorem when the underlying field is the real $\mathbb{R}$. For more "preserver problems" of a similar nature, readers are referred to [17, 26].

Corollary 5.2. Suppose that X is a completely regular space. Then every surjective linear local automorphism ϕ of $C(X)$ is an automorphism.

Proof. Since ϕ is a local automorphism, ϕ is injective, $\phi(1) = 1$, and sends exactly invertible elements to invertible elements. As invertible elements in $C(X)$ are exactly nonvanishing functions, ϕ is nonvanishing preserving. By Theorem 5.1, ϕ^v is a composition operator arising from a homeomorphism. In particular, inherited from ϕ^v, the bijective linear map ϕ is multiplicative, and hence an automorphism. $\square$

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School of Mathematical Sciences and LPMC, Nankai University, Tianjin, 300071, China

E-mail address: leilee@nankai.edu.cn

Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan

E-mail address: wong@math.nsysu.edu.tw