GEOMETRIC UNITARIES IN JB-ALGEBRAS

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Abstract. In this note, we study geometric unitaries of JB-algebras (in particular, self-adjoint parts of C^*-algebras). By using order theoretical arguments, we show that the geometric unitaries of a JB-algebra are precisely the central algebraic unitaries (or central symmetries).

1. Introduction and Preliminaries

The celebrated result of Kadison in [12] shows that a surjective isometry between unital C^*-algebras that preserve identities is a Jordan isomorphism. The corresponding result in the case of JB-algebras was proved by Wright and Youngson in [18]. Indeed, further codifying the idea in [18], Isidro and Rodríguez [13] showed that the extreme points of the closed unit ball of a JB-algebra are exactly symmetries, and the isolated points of the set of symmetries are exactly central symmetries. Thus a surjective isometry of JB-algebras preserves symmetries and central symmetries. It is then routine to see that such a surjective isometry is a Jordan isomorphism by standard spectral arguments.

In the case of complex unital C^*-algebras, a norm characterization of their unitaries was given in [2, Theorem 2] (as pointed out in [15], this characterization can also be found in [7, 4.1] and [14, 9.5.16]). Motivated by this result, the notion of geometric unitaries was introduced in [3, 2.1] and there are a number of recent researches on geometric unitaries (see e.g. [4–6, 8–10, 15]).

Compared with geometric unitaries of complex Banach spaces, those of real Banach spaces are less well studied and these two are not directly related (see the discussion below). As far as we know, the paper [9] of Fernández-Polo, Moreno and Peralta is the only one that studies the real case. In particular, they showed that for real JB^*-triples, geometric unitaries are the same as vertices of the closed unit balls. They also gave a characterization in terms of the associated Jordan product (i.e. a real analogue of [16, 19.13] in the light of [15, 2.1]). However, this equivalent condition is not easy to check (even in the case of $\mathcal{L}(H)_{sa}$) and is not totally algebraical (some norm conditions need to be verified).

The aim of this article is to give some totally algebraical equivalent descriptions of the geometric unitaries of unital JB-algebras, and to reprove [9, 2.8] in the case of unital JB-algebras using an elementary and order theoretical argument. The referee has kindly pointed out to us the following theorem based on [9] and [13], which has some overlap with our main result. Therefore, this paper gives another autonomous proof of this theorem. We would like to mention that our proof does not depend on the results about vertices in general real JB*-triples given in [9].

Theorem 1.1. *A JB-algebra A has vertices in its unit ball (see Section 2 for their definitions) if and only if it has a unit. In this case, vertices of A are precisely its central symmetries.*

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Proof. It is well known that central symmetries in unital JB-algebras are vertices (a consequence of [11, 1.2.6 and 3.3.10]). Let u be a vertex of A. Since JB-algebras are real JB^*-triples, Proposition 2.8 of [9] applies. Therefore, the Banach space of A, endowed with a suitable product, becomes a JB-algebra (say A_u) with unit u. Now the mapping $a \mapsto a$ is a surjective linear isometry from A_u onto A, and hence, by Theorem 1.9 of [13], there exists a central symmetry b in the JB-algebra of multipliers of A, together with a surjective algebra isomorphism $\Phi : A_u \to A$, such that we have

$$a = b \cdot \Phi(a) \tag{1.1}$$

for every $a \in A$. It follows that A has a unit (namely, $\Phi(u)$), and then (by taking $a = u$ in (1.1)) that $u = b$ is indeed a central symmetry in A. $\square$

2. Geometric unitaries of JB-algebras

In the following, we will first recall the definition of geometric unitaries and give some remarks about them. However, we will not recall the definitions of (real and complex) JB^*-triples nor that of JB-algebras. Readers can find their definitions and basic properties in standard textbooks (e.g. [1], [11] and [16]).

Definition 2.1. Suppose that X is a real or complex normed space with unit sphere $\mathfrak{S}_1(X)$ and dual space X^*. An element $u \in \mathfrak{S}_1(X)$ is called a *vertex of the unit ball of* X (respectively, a *geometric unitary of* X) if its exposed face

$$S_{u,X} \ := \ \{f \in X^* : \|f\| = 1 = f(u)\}$$

separates points of X (respectively, spans X^*). We denote by $\mathcal{V}(X)$ (respectively, $\mathcal{GU}(X)$) the set of all vertices of the unit ball of X (respectively, geometric unitaries of X).

Remark 2.2. Let X be a normed space over $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) and $u \in \mathfrak{S}_1(X)$.
(a) If Y is a subspace of X, then $\mathcal{GU}(X) \cap Y \subseteq \mathcal{GU}(Y)$ and $\mathcal{V}(X) \cap Y \subseteq \mathcal{V}(Y)$.
(b) $S_{u,X} \neq \emptyset$ and one defines a semi-norm $r_u(x) := \sup_{f \in S_{u,X}} |f(x)|$.
(c) Notice that $K_{u,X} := \bigcup_{\lambda \in \mathbb{R}_+} \lambda S_{u,X}$ is a proper cone, and $K_{u,X}$ is generating if and only if $u \in \mathcal{GU}(X)$. If we put

$$C_u \ := \ \{x \in X : f(x) \geq 0; f \in K_{u,X}\},$$

then $u \in \mathcal{V}(X)$ if and only if C_u is a proper cone.
(d) If $\mathbb{F} = \mathbb{R}$, then $u \in \mathcal{V}(X)$ if and only if there is a proper and generating closed cone C containing u as an order unit such that the order unit norm $\gamma_{u,C}$ is dominated by $\|\cdot\|$. If, in addition, X is a real Banach space, then $u \in \mathcal{GU}(X)$ if and only if there is a proper and generating closed cone C containing u as an order unit such that $\gamma_{u,C}$ is a complete norm dominated by $\|\cdot\|$. In fact, the forward implications of the above two statements follow from part (c) (and the completeness of γ_{u,C_u} in the second statement follows from a similar argument as [3, 3.1]). The backward implications follow from standard results concerning ordered normed spaces.
(e) Let $(E, \|\cdot\|, \{\cdots\})$ be a real JB^*-triple. If there exist a closed cone $C \subseteq E$ and $v \in \mathfrak{S}_1(E)$ which is an order unit in C such that the order unit norm $\gamma_{v,C}$ is dominated by $\|\cdot\|$, then there is a unital JB-algebra structure on E (with the same norm) such that the triple product is defined by the Jordan product in the canonical way. In fact, by part (d), v is a vertex of the unit ball of E and so [9, 2.8] tells us that the Jordan product $x \bullet y := \{x, v, y\}$, together

with the original norm, defines a JB-algebra structure on E with identity v such that $\{\cdot,\cdot,\cdot\}$ is given by $\bullet$.

(f) Let Ω be a locally compact Hausdorff space. If $\mathcal{V}(C_0(\Omega;\mathbb{R})) \neq \emptyset$, then Ω is compact. In this case, $\mathcal{V}(C(\Omega;\mathbb{R})) \subseteq \mathcal{U}(C(\Omega;\mathbb{R}))$. Indeed, if $u \in \mathcal{V}(C_0(\Omega;\mathbb{R}))$ and $\mu \in S_{u,X}$, we have $1 = |\mu|(\Omega) = \int_\Omega u\, d\mu = \int_\Omega uf\, d|\mu|$ (for a measurable function $f : \Omega \to \{1,-1\}$). As $uf = 1$ $|\mu|$-a.e., we have $|\mu|(W) = 0$ where $W := \{\omega \in \Omega : u(\omega) \notin \{1,-1\}\}$. We can then use the Urysohn lemma to show that $W = \emptyset$.

Notation 2.3. (a) Let A be a real Jordan Banach algebra. We denote by $\mathcal{Z}(A)$ the center of A and by $\mathrm{Aut}(A)$ (respectively, $\mathrm{Iso}(A)$) the sets of all isometric Jordan isomorphisms (respectively, surjective isometries) from A onto A. If A is unital, we denote by $\mathcal{U}(A)$ the set of all elements $u \in A$ with $u^2 = 1$.

(b) Let B be a JB^*-algebra. We denote by $\mathcal{U}(B)$ the set of all elements u with $u \circ u^* = 1$ and $u^2 \circ u^* = u$. Moreover, we define the following maps in $\mathcal{L}(B)$:

$$M_b(x) := b \circ x \qquad \text{and} \qquad U_b(x) := [b, x, b] \qquad\qquad (b, x \in B)$$

where $[x, y, z] := x \circ (y \circ z) - y \circ (z \circ x) + z \circ (x \circ y)$ $(x, y, z \in B)$.

In the following, we will study geometric unitaries of unital JB-algebras. By [15, 2.1], geometric unitaries of a unital JB^*-algebra are the same as its algebraic unitaries. Therefore, a naïve guess is that if A is a unital JB-algebra, then $\mathcal{GU}(A)$ coincides with the set $\mathcal{U}(A)$ of algebraic unitaries of A. Surprisingly, it is not the case. Geometric unitaries in A are, in fact, *central* algebraic unitaries. Another surprising fact is that one can use order theoretical argument to obtain this equivalence (which relates the norm structure with the algebraic structure).

Our next lemma is a crucial result that gives certain algebraic properties of vertices of closed unit balls of JB-algebras. The last statement of this lemma tells us that the set of geometric unitaries is transitive.

Lemma 2.4. *Let B be a unital JB^*-algebra and $A = B_{sa}$. For any $u \in \mathcal{V}(A)$, there exists $v \in \mathcal{U}(B)$ such that $v^2 = u$, $U_v(A) = A$ and $U_v^2 = \mathrm{id}$. Consequently, $\mathcal{GU}(A) = \{\Psi(1) : \Psi \in \mathrm{Iso}(A)\} = \mathcal{V}(A) \subseteq \mathcal{U}(A)$.*

Proof: Define $\psi : A^* \to B^*$ by $\psi(f)(x + iz) = f(x) + if(z)$. Then ψ is an isometry. Moreover,

$$\psi(A^*) = B^*_h := \{f \in B^* : f(A) \subseteq \mathbb{R}\} \qquad \text{and} \qquad \psi(A^*_+) = B^*_+.$$

For any $f \in K_{u,A}$, we have $\|\psi(f)\| = \|f\| = f(u) = \psi(f)(u)$ which means that

$$\psi(K_{u,A}) = K_{u,B} \cap B^*_h. \tag{2.1}$$

Since elements of $\mathcal{V}(A)$ are extreme points of the closed unit ball of A, [13, Lemma 1.2] applies to deduce that u belongs to $\mathcal{U}(A)$. Put $v := \frac{1+u}{2} + i\frac{1-u}{2} \in B$. Then v lies in $\mathcal{U}(B)$ and $v^2 = u$. As v is a unitary, $U_{v^*} \circ U_v = \mathrm{id}$ (see e.g. [16, 19.18]) and $U_v^* : B^* \to B^*$ is a surjective isometry. Thus, for any $g \in B^*$, we have $\|U_v^*(g)\| = \|g\|$ and $U_v^*(g)(1) = g(u)$ which implies that

$$U_v^*(K_{u,B}) = K_{1,B} = B^*_+. \tag{2.2}$$

Since $A^* = \overline{K_{u,A} - K_{u,A}}^{\sigma(A^*,A)}$ (a reformulation of the fact that u belongs to $\mathcal{V}(A)$), and ψ is weak-*-continuous,

$$B^*_h \subseteq \overline{\psi(K_{u,A}) - \psi(K_{u,A})}^{\sigma(B^*,B)} \subseteq \overline{K_{u,B} - K_{u,B}}^{\sigma(B^*,B)}. \tag{2.3}$$

Using the weak-*-continuity of U_v^*, relations (2.3) and (2.2) tell us that $U_v^*(B_h^*) \subseteq B_h^*$. Thus, $f([v,a,v]) \subseteq \mathbb{R}$ ($a \in A$; $f \in B_h^*$) and so $U_v(A) \subseteq A$. This is the same as $[v^*,a,v^*] = [v,a,v]^* = [v,a,v]$ ($a \in A$), i.e. $U_v = U_{v^*}$. This gives

$$(U_v)^2(b) \ = \ b \qquad (b \in B). \tag{2.4}$$

Consequently, $U_v(A) = A$. The last statement follows from the above consideration. □

Corollary 2.5. *Let A be a unital JB-algebra and $u \in \mathfrak{S}_1(A)$.*

(a) The seminorm r_u in Remark 2.2(b) is a norm if and only if r_u coincides with the original norm of A.

(b) The closed cone C_u in Remark 2.2(c) is proper if and only if there exists $\Psi \in \mathrm{Iso}(A)$ such that $\Psi(A_+) = C_u$ and $\Psi(1) = u$.

Proof: (a) If r_u is a norm, then $u \in \mathcal{V}(A)$. By Lemma 2.4 (note that A is the self-adjoint part of a JB^*-algebra by [17]), there is a surjective isometry U that sends u to 1. Hence, $U^*(S_{1,A}) = S_{u,A}$ and so $r_u(a) = \|U(a)\| = \|a\|$ ($a \in A$).

(b) Suppose that C_u is proper. Then $u \in \mathcal{V}(A)$ (because of Remark 2.2(d)). The existence of $\Psi \in \mathrm{Iso}(A)$ with $\Psi(1) = u$ is ensured by Lemma 2.4, and clearly Ψ preserves the corresponding cones. □

The following is our main result, which covers especially Theorem 1.1. Note that conditions (iv), (v) and (vi) are totally algebraical and the equivalences of (i), (ii) and (vii) are precisely those in [9, 2.8]. Notice also that (vi) is a disguised form of $A = A^1(u)$ (in the notation of [9]) and "$E = E^1(u)$" is an intermediate step in the proof of [9, 2.8]. Hence statement (vi) is actually included implicitly in [9]. However, we will not use [9, 2.8] but reprove it here in this case of unital JB-algebras. Our proof is order theoretical and elementary.

Theorem 2.6. *Suppose that A is a unital JB-algebra and $u \in A$. Then the following statements are equivalent.*

(i). $u \in \mathcal{GU}(A)$.
(ii). $u \in \mathcal{V}(A)$.
(iii). *u is an isolated point of $\mathcal{U}(A)$ (endowed with the norm topology).*
(iv). $u \in \mathcal{U}(A) \cap \mathcal{Z}(A)$.
(v). $M_u^2 = \mathrm{id}$.
(vi). $U_u = \mathrm{id}$.
(vii). *If $a \bullet b := [a,u,b]$, then $(A,\bullet)$ is a JB-algebra with identity u.*

Proof: Let B be the JB^*-algebra with $A = B_{sa}$ (see [17]).

(i)⇒(ii). This implication is clear.

(ii)⇒(iii). By Lemma 2.4, there exists $\Psi \in \mathrm{Iso}(A)$ such that $u = \Psi(1)$. Since elements of $\mathcal{U}(A)$ can be geometrically characterized (indeed, by [13, 1.2], they are the extreme points of the closed unit ball of A), and 1 is an isolated point of $\mathcal{U}(A)$ (by the "if" part of [13, 1.3]), the above implies Statement (iii).

(iii)⇒(iv). By the only if part of [13, 1.3].

(iv)⇒(v). For any $a \in A$, the operator commutativity of u and a implies that $M_u^2(a) = M_u(M_u(a)) = M_a(M_u(u)) = a$ (as $u^2 = 1$).

(v)⇒(vi). Note that $u^2 = M_u^2(1) = 1$ and $U_u = 2M_u^2 - M_{u^2} = \mathrm{id}$.

(vi)⇒(vii). As $[u, 1, u] = 1$, we see that $u \in \mathcal{U}(A)$. The involution on B is given by $\overline{a + bi} := a - bi$ $(a, b \in A)$. Note that, as $u \in \mathcal{U}(B)$, B becomes another JB^*-algebra under the new product $x \bullet y := [x, u, y]$ and the new involution $x^* := [u, \bar{x}, u]$ (see e.g. [16, 19.13]). Since $U_u = \mathrm{id}$, we see that $(a + bi)^* = a - bi$ and so A is the selfadjoint part of $(B, \bullet, {}^*)$. Therefore, A is a JB-algebra under this new product $\bullet$.

(vi)⇒(i). This follows directly from [11, 3.3.10] and Remark 2.2(d). □

As an application of this theorem, we have the following corollary. Note that it is pretty hard to determine directly those elements $u \in \mathfrak{S}_1(\mathcal{L}(H)_{sa})$ with $(\mathcal{L}(H)_{sa}, \|\cdot\|, \bullet)$ being a JB-algebra (where $a \bullet b := (aub + bua)/2$), and it is not easy to obatin $\mathcal{GU}(\mathcal{L}(H)_{sa}) = \{1, -1\}$ from [9, 2.8] directly.

Corollary 2.7. *If H is a Hilbert space, then $\mathcal{GU}(\mathcal{L}(H)_{sa}) = \{1, -1\}$.*

It is well known that $u \mapsto (1 - u)/2$ gives a bijective correspondence between central symmetries and central projections of a unital JB-algebra. Since a JBW-algebra is generated by its projections (see e.g. [11, 4.2.3]), we obtain directly part (a) of the following interesting application of geometric unitaries. Note that part (b) is a generalization of Corollary 2.7.

Corollary 2.8. *Let A be a JBW-algebra. Then we have*

(a) The closed linear span of $\mathcal{GU}(A)$ is the center of A.

(b) A is a JBW-factor if and only if $\mathcal{GU}(A) = \{1, -1\}$.

Remark 2.9. Given a dual Banach space X (with its predual denoted by X_*), one can define weak*-geometric unitaries of X as those norm-one elements u of X such that X_* equals the linear span of the set $\{\omega \in X_* : \|\omega\| = 1 = \omega(u)\}$. Since weak*-geometric unitaries are vertices of the closed unit ball, it follows from Theorem 2.6 that weak*-geometric unitaries of a JBW-algebra are geometric unitaries. The converse is also true. Indeed, by [11, 4.5.3], the unit 1 of a JBW-algebra A is a weak*-geometric unitary of A, and, if u is any geometric unitary of A, then, again by Theorem 2.6, the mapping $a \mapsto ua$ becomes a weak*-continuous surjective linear isometry on A sending 1 to u.

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