FIXED POINT THEOREMS FOR THREE NEW NONLINEAR MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we first consider three classes of nonlinear mappings in Banach spaces which contain the class of 2-generalized hybrid mappings defined by Maruyama, Takahashi and Yao [22] in a Hilbert space. Then, we prove fixed point theorems for these classes of nonlinear mappings in Banach spaces.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let T be a mapping of C into itself. Then we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is said to be *nonexpansive*, *nonspreading* [20], and *hybrid* [28] if

$$\|Tx - Ty\| \le \|x - y\|,$$

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.1}$$

and

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.2}$$

for all $x, y \in C$, respectively. These mappings are deduced from a firmly nonexpansive mapping in a Hilbert space; see [28]. A mapping $F : C \to C$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \le \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$; see, for instance, Browder [5] and Goebel and Kirk [9]. From Baillon [3], and Takahashi and Yao [32], we know the following nonlinear ergodic theorem for nonlinear mappings in a Hilbert space.

Theorem 1.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself such that $F(T)$ is nonempty. Suppose that T satisfies one of the following:*

(i) *T is nonexpansive;*
(ii) *T is nonspreading;*
(iii) *T is hybrid;*
(iv) $2\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$

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Then, for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to a fixed point of T.

Motivated by such a theorem, Aoyama, Iemoto, Kohsaka and Takahashi [2] introduced a class of nonlinear mappings called λ-hybrid containing the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Kocourek, Takahashi and Yao [17] also introduced a more broad class of nonlinear mappings than the class of λ-hybrid mappings in a Hilbert space. A mapping $T : C \to C$ is called *generalized hybrid* [17] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Very recently, Maruyama, Takahashi and Yao [22] introduced a broad class of nonlinear mappings containing the class of generalized hybrid mappings defined by Kocourek, Takahashi and Yao [17] in a Hilbert space. A mapping $T : C \to C$ is called *2-generalized hybrid* if there are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|T^2x - Ty\|^2 &+ \alpha_2\|Tx - Ty\|^2 + (1-\alpha_1-\alpha_2)\|x - Ty\|^2 \\ &\le \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1-\beta_1-\beta_2)\|x - y\|^2\end{aligned}$$

for all $x, y \in C$. Then, they proved fixed point theorems and weak convergence theorems for 2-generalized hybrid mappings in a Hilbert space; see also Takahashi and Termwuttipong [30].

In this paper, motivated by Maruyama, Takahashi and Yao [22], we introduce three classes of nonlinear mappings in Banach spaces which contain the class of 2-generalized hybrid mappings in a Hilbert space. Then, we prove fixed point theorems for these classes of nonlinear mappings in Banach spaces.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to C$ is *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty closed convex subset of a strictly convex Banach space E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [15]. Let E be a Banach space. The *duality* mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called *smooth*. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm to norm continuous. If E is uniformly smooth, J is uniformly norm to norm continuous on each bounded subset of E. For more details, see [25, 26]. The following results are in [25, 26].

Lemma 2.1. *Let E be a Banach space and let J be the duality mapping on E. Then, for any $x, y \in E$,*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j\rangle,$$

where $j \in Jy$.

Lemma 2.2. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi\colon E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.2}$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [16]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.3}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Further, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.4}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then from Lemma 2.2 we have

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.5}$$

The following result was proved by Xu [35].

Lemma 2.3 (Xu [35])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For a proof of existence of a Banach limit and its other elementary properties, see [25].

Using Lemma 2.3 and properties of means, Takahashi and Jeong [29] proved the following result.

Lemma 2.4 (Takahashi and Jeong [29]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, let $\{x_n\}$ be a bounded sequence in E and let μ be a mean on l^∞. If $g : E \to \mathbb{R}$ is defined by*

$$g(z) = \mu_n \|x_n - z\|^2, \quad \forall z \in E,$$

then there exists a unique $z_0 \in C$ such that $g(z_0) = \min\{g(z) : z \in C\}$.

3. Fixed Point Theorem 1

Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to C$ is said to be *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \le \langle x - y, j\rangle,$$

for all $x, y \in C$, where $j \in J(Tx - Ty)$; see Bruck [6]. A mapping $T : C \to C$ is called *generalized hybrid* [11] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{3.1}$$

for all $x, y \in C$. Such a mapping T is also called an (α, β)-*generalized hybrid* mapping in a Banach space. Using Lemma 2.1, Takahashi and Yao [34] proved the following result.

Proposition 3.1. *Let E be a Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a firmly nonexpansive mapping and let $\lambda \in [0, 1]$. Then, T is $(2 - \lambda, 1 - \lambda)$-generalized hybrid, i.e.,*

$$(2 - \lambda)\|Tx - Ty\|^2 + (\lambda - 1)\|x - Ty\|^2 \le (1 - \lambda)\|Tx - y\|^2 + \lambda\|x - y\|^2$$

for all $x, y \in C$.

We notice from Proposition 3.1 that the classes of nonexpansive mappings, nonspreadind mappings and hybrid mappings in the sense of norm are deduced from the class of firmly nonexpansive mappings in a Banach space. Motivated by Bruck [6], Takahashi and Yao [34], and Maruyama, Takahashi and Yao [22], in this section, we introduce a broad class of nonlinear mappings in a Banach space containing the class of 2-generalized hybrid mappings defined by Maruyama, Takahashi and Yao

[22] in a Hilbert space. Let E be a Banach space and let C be a nonempty subset of E. Then, a mapping $T : C \to C$ is called *2-generalized hybrid* if there are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|T^2x{-}Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \le \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2\end{aligned} \tag{3.2}$$

for all $x, y \in C$. We call such a mapping an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-*generalized hybrid* mapping. We observe that the mapping above covers several well-known mappings. For example, a $(0, \alpha_2, 0, \beta_2)$-generalized hybrid mapping is nonexpansive for $\alpha_2 = 1$ and $\beta_2 = 0$, nonspreading in the sense of norm for $\alpha_2 = 2$ and $\beta_2 = 1$, and hybrid for $\alpha_2 = \frac{3}{2}$ and $\beta_2 = \frac{1}{2}$; see (1.1) and (1.2). A $(0, \alpha_2, 0, \beta_2)$-generalized hybrid mapping is an (α_2, β_2)-generalized hybrid mapping in the sense of Hsu, Takahashi and Yao [11]. We can also show that if $x = Tx$, then for any $y \in C$,

$$\begin{aligned}\alpha_1\|x{-}Ty\|^2 + \alpha_2\|x - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \le \beta_1\|x - y\|^2 + \beta_2\|x - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2\end{aligned}$$

and hence $\|x - Ty\| \le \|x - y\|$. This means that a 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive in a Banach space. Now, we prove a fixed point theorem for 2-generalized hybrid mappings in a Banach space. Before proving it, we need the following lemma which was proved by Hsu, Takahashi and Yao [11]. This lemma was proved by using Lemma 2.4.

Lemma 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a mapping of C into itself. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on l^∞. If*

$$\mu_n\|x_n - Ty\|^2 \le \mu_n\|x_n - y\|^2$$

for all $y \in C$, then T has a fixed point in C.

Theorem 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $T : C \to C$ be a 2-generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

Proof. Since $T : C \to C$ is a 2-generalized hybrid mapping, there are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|T^2x{-}Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \le \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2\end{aligned}$$

for all $x, y \in C$. If $F(T) \ne \emptyset$, then $\{T^n z\} = \{z\}$ for $z \in F(T)$. So, $\{T^n z\}$ is bounded. Conversely, take $z \in C$ such that $\{T^n z\}$ is bounded. Let μ be a Banach limit. Then, for any $y \in C$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}\alpha_1\|T^{n+2}z{-}Ty\|^2 + \alpha_2\|T^{n+1}z - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|T^n z - Ty\|^2 \\ \le \beta_1\|T^{n+2}z - y\|^2 + \beta_2\|T^{n+1}z - y\|^2 + (1 - \beta_1 - \beta_2)\|T^n z - y\|^2\end{aligned}$$

for any $y \in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the above inequality. Then, we have

$$\begin{aligned}\mu_n(\alpha_1\|T^{n+2}z - Ty\|^2 + \alpha_2\|T^{n+1}z - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|T^n z - Ty\|^2) \\ \le \mu_n(\beta_1\|T^{n+2}z - y\|^2 + \beta_2\|T^{n+1}z - y\|^2 + (1 - \beta_1 - \beta_2)\|T^n z - y\|^2).\end{aligned}$$

So, we obtain

$$\begin{aligned}&\alpha_1\mu_n\|T^{n+2}z-Ty\|^2+\alpha_2\mu_n\|T^{n+1}z-Ty\|^2+(1-\alpha_1-\alpha_2)\mu_n\|T^nz-Ty\|^2\\&\quad\le\beta_1\mu_n\|T^{n+2}z-y\|^2+\beta_2\mu_n\|T^{n+1}z-y\|^2+(1-\beta_1-\beta_2)\mu_n\|T^nz-y\|^2\end{aligned}$$

and hence

$$\begin{aligned}&\alpha_1\mu_n\|T^nz-Ty\|^2+\alpha_2\mu_n\|T^nz-Ty\|^2+(1-\alpha_1-\alpha_2)\mu_n\|T^nz-Ty\|^2\\&\quad\le\beta_1\mu_n\|T^nz-y\|^2+\beta_2\mu_n\|T^nz-y\|^2+(1-\beta_1-\beta_2)\mu_n\|T^nz-y\|^2.\end{aligned}$$

This implies

$$\mu_n\|T^nz-Ty\|^2\le\mu_n\|T^nz-y\|^2$$

for all $y\in C$. By Lemma 3.1, T has a fixed point in C. □

As a direct consequence of Theorem 3.2, we have the following result.

Theorem 3.3. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a 2-generalized hybrid mapping from C to itself. Then T has a fixed point.*

Using Theorem 3.2, we can also prove the following well-known fixed point theorems. We first prove a fixed point theorem for nonexpansive mappings in a Banach space.

Theorem 3.4. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T:C\to C$ be a nonexpansive mapping, i.e.,*

$$\|Tx-Ty\|\le\|x-y\|,\quad\forall x,y\in C.$$

Suppose that there exists an element $x\in C$ such that $\{T^nx\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.2, a $(0,1,0,0)$-generalized hybrid mapping of C into itself is nonexpansive. By Theorem 3.2, T has a fixed point in C. □

The following is a fixed point theorem for nonspreading mappings in a Banach space.

Theorem 3.5 ([11])**.** *Let H be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T:C\to C$ be a nonspreading mapping, i.e.,*

$$2\|Tx-Ty\|^2\le\|Tx-y\|^2+\|Ty-x\|^2,\quad\forall x,y\in C.$$

Suppose that there exists an element $x\in C$ such that $\{T^nx\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.2, a $(0,2,0,1)$-generalized hybrid mapping of C into itself is nonspreading. By Theorem 3.2, T has a fixed point in C. □

The following is a fixed point theorem for hybrid mappings in a Banach space.

Theorem 3.6 ([11])**.** *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T:C\to C$ be a hybrid mapping, i.e.,*

$$3\|Tx-Ty\|^2\le\|x-y\|^2+\|Tx-y\|^2+\|Ty-x\|^2,\quad\forall x,y\in C.$$

Suppose that there exists an element $x\in C$ such that $\{T^nx\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.2, a $(0, \frac{3}{2}, 0, \frac{1}{2})$-generalized hybrid mapping of C into itself is hybrid in the sense of Takahashi [28]. By Theorem 3.2, T has a fixed point in C. □

We can also prove the following fixed point theorem in a Banach space.

Theorem 3.7. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a mapping such that*

$$2\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.2, a $(0, 1, 0, \frac{1}{2})$-generalized hybrid mapping of C into itself is the mapping in our theorem. By Theorem 3.2, T has a fixed point in C. □

Finally, we prove the following fixed point theorem in a Banach space.

Theorem 3.8. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a mapping such that*

$$\|T^2x - Ty\|^2 + \|Tx - Ty\|^2 + \|x - Ty\|^2 \le 3\|x - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.2, consider a $(\frac{1}{3}, \frac{1}{3}, 0, 0)$-generalized hybrid mapping T of C into itself. Then, we have that

$$\frac{1}{3}\|T^2x - Ty\|^2 + \frac{1}{3}\|Tx - Ty\|^2 + \frac{1}{3}\|x - Ty\|^2 \le \|x - y\|^2, \quad \forall x, y \in C.$$

This is equivalent to the mapping in our theorem:

$$\|T^2x - Ty\|^2 + \|Tx - Ty\|^2 + \|x - Ty\|^2 \le 3\|x - y\|^2, \quad \forall x, y \in C.$$

By Theorem 3.2, T has a fixed point in C. □

Remark 1. Let E be a Banach space and let C be a nonempty closed convex subset of E. Let $n \in \mathbb{N}$. Then, a mapping $T : C \to C$ is called *n-generalized hybrid* if there are $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n \in \mathbb{R}$ such that

$$\begin{aligned} &\sum_{k=1}^{n} \alpha_k \|T^{n+1-k}x - Ty\|^2 + (1 - \sum_{k=1}^{n} \alpha_k)\|x - Ty\|^2 \\ &\qquad \le \sum_{k=1}^{n} \beta_k \|T^{n+1-k}x - y\|^2 + (1 - \sum_{k=1}^{n} \beta_k)\|x - y\|^2 \end{aligned} \tag{3.3}$$

for all $x, y \in C$. We call such a mapping an $(\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n)$-*generalized hybrid* mapping. As in the proof of Theorem 3.2, we can prove a fixed point theorem for n-generalized hybrid mappings in a uniformly convex Banach space.

4. Fixed Point Theorem 2

Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is called *2-generalized nonspreading* if there are $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1\phi(T^2x, Ty) + \alpha_2\phi(Tx, Ty) + (1 - \alpha_1 - \alpha_2)\phi(x, Ty) \\ &\quad + \gamma_1\{\phi(Ty, T^2x) - \phi(Ty, x)\} + \gamma_2\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\leq \beta_1\phi(T^2x, y) + \beta_2\phi(Tx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\ &\quad + \delta_1\{\phi(y, T^2x) - \phi(y, x)\} + \delta_2\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{4.1}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such a mapping an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$-*generalized nonspreading* mapping. Let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$-generalized nonspreading mapping. Observe that if $F(T) \neq \emptyset$, then $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. Indeed, putting $x = u \in F(T)$ in (4.1), we obtain

$$\begin{aligned} &\alpha_1\phi(u, Ty) + \alpha_2\phi(u, Ty) + (1 - \alpha_1 - \alpha_2)\phi(u, Ty) \\ &\quad + \gamma_1\{\phi(Ty, u) - \phi(Ty, u)\} + \gamma_2\{\phi(Ty, u) - \phi(Ty, u)\} \\ &\leq \beta_1\phi(u, y) + \beta_2\phi(u, y) + (1 - \beta_1 - \beta_2)\phi(u, y) \\ &\quad + \delta_1\{\phi(y, u) - \phi(y, u)\} + \delta_2\{\phi(y, u) - \phi(y, u)\}. \end{aligned}$$

So, we have that

$$\phi(u, Ty) \leq \phi(u, y) \tag{4.2}$$

for all $u \in F(T)$ and $y \in C$. Further, if E is a Hilbert space, then we have $\phi(x, y) = \|x - y\|^2$ for $x, y \in E$. So, from (4.1) we obtain the following:

$$\begin{aligned} &\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ &\quad + \gamma_1\{\|Ty - T^2x\|^2 - \|Ty - x\|^2\} + \gamma_2\{\|Ty - Tx\|^2 - \|Ty - x\|^2\} \\ &\leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2 \\ &\quad + \delta_1\{\|y - T^2x\|^2 - \|y - x\|^2\} + \delta_2\{\|y - Tx\|^2 - \|y - x\|^2\} \end{aligned}$$

for all $x, y \in C$. This implies that

$$\begin{aligned} &(\alpha_1 + \gamma_1)\|T^2x - Ty\|^2 + (\alpha_2 + \gamma_2)\|Tx - Ty\|^2 \\ &\quad + \{1 - (\alpha_1 + \gamma_1) - (\alpha_2 + \gamma_2)\}\|x - Ty\|^2 \\ &\leq (\beta_1 + \delta_1)\|T^2x - y\|^2 + (\beta_2 + \delta_2)\|Tx - y\|^2 \\ &\quad + \{1 - (\beta_1 + \delta_1) - (\beta_2 + \delta_2)\}\|x - y\|^2 \end{aligned}$$

for all $x, y \in C$. That is, T is a 2-generalized hybrid mapping [22] in a Hilbert space. Now, using the technique developed by [24], we prove a fixed point theorem for 2-generalized nonspreading mappings in a Banach space.

Theorem 4.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let T be a 2-generalized nonspreading mapping of C into itselt. Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Let T be a 2-generalized nonspreading mapping of C into itself. Then, there are $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_1 \phi(T^2x, Ty) + \alpha_2 \phi(Tx, Ty) + (1 - \alpha_1 - \alpha_2)\phi(x, Ty) \\
&\quad + \gamma_1\{\phi(Ty, T^2x) - \phi(Ty, x)\} + \gamma_2\{\phi(Ty, Tx) - \phi(Ty, x)\} \\
&\leq \beta_1 \phi(T^2x, y) + \beta_2 \phi(Tx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\
&\quad + \delta_1\{\phi(y, T^2x) - \phi(y, x)\} + \delta_2\{\phi(y, Tx) - \phi(y, x)\}
\end{aligned} \tag{4.3}$$

for all $x, y \in C$. If $F(T) \neq \emptyset$, then we have from (5.3) that $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. Taking a fixed point u of T, we have $\phi(u, T^n x) \leq \phi(u, x)$ for all $n \in \mathbb{N}$ and $x \in C$. This implies that for every $x \in C$, the sequence $\{T^n x\}$ is bounded. So, (a) $\Longrightarrow$ (b). Let us show (b) $\Longrightarrow$ (a). Suppose that there exists $x \in C$ such that $\{T^n x\}$ is bounded. Then replacing x by $T^k x$ in (4.3), where $k \in \mathbb{N} \cup \{0\}$, we have that for any $y \in C$,

$$\begin{aligned}
&\alpha_1 \phi(T^{k+2}x, Ty) + \alpha_2 \phi(T^{k+1}x, Ty) + (1 - \alpha_1 - \alpha_2)\phi(T^k x, Ty) \\
&\quad + \gamma_1\{\phi(Ty, T^{k+2}x) - \phi(Ty, T^k x)\} + \gamma_2\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\} \\
&\leq \beta_1 \phi(T^{k+2}x, y) + \beta_2 \phi(T^{k+1}x, y) + (1 - \beta_1 - \beta_2)\phi(T^k x, y) \\
&\quad + \delta_1\{\phi(y, T^{k+2}x) - \phi(y, T^k x)\} + \delta_2\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\} \\
&= \beta_1\{\phi(T^{k+2}x, Ty) + \phi(Ty, y) + 2\langle T^{k+2}x - Ty, JTy - Jy\rangle\} \\
&\quad + \beta_2\{\phi(T^{k+1}x, Ty) + \phi(Ty, y) + 2\langle T^{k+1}x - Ty, JTy - Jy\rangle\} \\
&\quad + (1 - \beta_1 - \beta_2)\{\phi(T^k x, Ty) + \phi(Ty, y) + 2\langle T^k x - Ty, JTy - Jy\rangle\} \\
&\quad + \delta_1\{\phi(y, T^{k+2}x) - \phi(y, T^k x)\} + \delta_2\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}.
\end{aligned} \tag{4.4}$$

This implies that

$$\begin{aligned}
0 &\leq (\beta_1 - \alpha_1)\{\phi(T^{k+2}x, Ty) - \phi(T^k x, Ty)\} \\
&\quad + (\beta_2 - \alpha_2)\{\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\} + \phi(Ty, y) \\
&\quad + 2\langle \beta_1 T^{k+2}x + \beta_2 T^{k+1}x + (1 - \beta_1 - \beta_2)T^k x - Ty, JTy - Jy\rangle \\
&\quad - \gamma_1\{\phi(Ty, T^{k+2}x) - \phi(Ty, T^k x)\} - \gamma_2\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\} \\
&\quad + \delta_1\{\phi(y, T^{k+2}x) - \phi(y, T^k x)\} + \delta_2\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\} \\
&= (\beta_1 - \alpha_1)\{\phi(T^{k+2}x, Ty) - \phi(T^k x, Ty)\} \\
&\quad + (\beta_2 - \alpha_2)\{\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\} + \phi(Ty, y) \\
&\quad + 2\langle T^k x - Ty + \beta_1(T^{k+2}x - T^k x) + \beta_2(T^{k+1}x - T^k x), JTy - Jy\rangle \\
&\quad - \gamma_1\{\phi(Ty, T^{k+2}x) - \phi(Ty, T^k x)\} - \gamma_2\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\} \\
&\quad + \delta_1\{\phi(y, T^{k+2}x) - \phi(y, T^k x)\} + \delta_2\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}.
\end{aligned} \tag{4.5}$$

Summing up these inequalities (4.5) with respect to $k = 0, 1, \dots, n-1$, we have

$$\begin{aligned}
0 \le\ & (\beta_1 - \alpha_1)\{\phi(T^{n+1}x, Ty) + \phi(T^n x, Ty) - \phi(Tx, Ty) - \phi(x, Ty)\} \\
& + (\beta_2 - \alpha_2)\{\phi(T^n x, Ty) - \phi(x, Ty)\} + n\phi(Ty, y) \\
& + 2\langle x + Tx + \cdots + T^{n-1}x - nTy, JTy - Jy\rangle \\
& + 2\langle \beta_1(T^{n+1}x + T^n x - Tx - x) + \beta_2(T^n x - x), JTy - Jy\rangle \\
& - \gamma_1\{\phi(Ty, T^{n+1}x) + \phi(Ty, T^n x) - \phi(Ty, Tx) - \phi(Ty, x)\} \\
& - \gamma_2\{\phi(Ty, T^n x) - \phi(Ty, x)\} \\
& + \delta_1\{\phi(y, T^{n+1}x) + \phi(y, T^n x) - \phi(y, Tx) - \phi(y, x)\} \\
& + \delta_2\{\phi(y, T^n x) - \phi(y, x)\}.
\end{aligned} \tag{4.6}$$

Dividing by n in (4.6), we have

$$\begin{aligned}
0 \le\ & \frac{1}{n}(\beta_1 - \alpha_1)\{\phi(T^{n+1}x, Ty) + \phi(T^n x, Ty) - \phi(Tx, Ty) - \phi(x, Ty)\} \\
& + \frac{1}{n}(\beta_2 - \alpha_2)\{\phi(T^n x, Ty) - \phi(x, Ty)\} + \phi(Ty, y) \\
& + 2\langle S_n x - Ty, JTy - Jy\rangle \\
& + \frac{1}{n}2\langle \beta_1(T^{n+1}x + T^n x - Tx - x) + \beta_2(T^n x - x), JTy - Jy\rangle \\
& - \frac{1}{n}\gamma_1\{\phi(Ty, T^{n+1}x) + \phi(Ty, T^n x) - \phi(Ty, Tx) - \phi(Ty, x)\} \\
& - \frac{1}{n}\gamma_2\{\phi(Ty, T^n x) - \phi(Ty, x)\} \\
& + \frac{1}{n}\delta_1\{\phi(y, T^{n+1}x)\phi(y, T^n x) - \phi(y, Tx) - \phi(y, x)\} \\
& + \frac{1}{n}\delta_2\{\phi(y, T^n x) - \phi(y, x)\},
\end{aligned} \tag{4.7}$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Since $\{T^n x\}$ is bounded by assumption, $\{S_n x\}$ is bounded. Thus we have a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ such that $\{S_{n_i} x\}$ converges weakly to a point $u \in C$. Letting $n_i \to \infty$ in (4.7), we obtain

$$0 \le \phi(Ty, y) + 2\langle u - Ty, JTy - Jy\rangle.$$

Putting $y = u$, we obtain

$$\begin{aligned}
0 &\le \phi(Tu, u) + 2\langle u - Tu, JTu - Ju\rangle \\
&= \phi(Tu, u) + \phi(u, u) + \phi(Tu, Tu) - \phi(u, Tu) - \phi(Tu, u) \\
&= -\phi(u, Tu).
\end{aligned} \tag{4.8}$$

Hence we have $\phi(u, Tu) \le 0$ and then $\phi(u, Tu) = 0$. Since E is strictly convex, we obtain $u = Tu$. Therefore $F(T)$ is nonempty. This completes the proof. □

Using Theorem 4.1, we have the following fixed point theorems in a Banach space.

Theorem 4.2 (Kohsaka and Takahashi [20]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonspreading mapping, i.e.,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. *Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$, $\alpha_2 = \beta_2 = \gamma_2 = 1$ and $\delta_2 = 0$ in (4.3), we obtain that

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. So, we have the desired result from Theorem 4.1. □

Theorem 4.3. *Let* E *be a smooth, strictly convex and reflexive Banach space and let* C *be a nonempty closed convex subset of* E. *Let* $T : C \to C$ *be a hybrid mapping* [28], *i.e.,*

$$2\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) + \phi(x, y)$$

for all $x, y \in C$. *Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$, $\alpha_2 = 1$, $\beta_2 = \gamma_2 = \frac{1}{2}$ and $\delta_2 = 0$ in (4.3), we obtain that

$$2\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) + \phi(x, y)$$

for all $x, y \in C$. So, we have the desired result from Theorem 4.1. □

Theorem 4.4. *Let* E *be a smooth, strictly convex and reflexive Banach space and let* C *be a nonempty closed convex subset of* E. *Let* $T : C \to C$ *be a mapping such that*

$$\alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y)$$

for all $x, y \in C$. *Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$, $\alpha_2 = \alpha$, $\beta_2 = \beta$ and $\gamma_2 = \delta_2 = 0$ in (4.3), we obtain that

$$\alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y)$$

for all $x, y \in C$. So, we have the desired result from Theorem 4.1. □

As a direct consequence of Theorem 5.5, we have the following Kocourek, Takahashi and Yao fixed point theorem [17] in a Hilbert space.

Theorem 4.5 (Kocourek, Takahashi and Yao [17])**.** *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and let* $T : C \to C$ *be a generalized hybrid mapping, i.e., there are* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. *Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. We know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in C$ in Theorem 5.5. So, we have the desired result from Theorem 5.5. □

Finally, we prove the following fixed point theorem in a Banach space.

Theorem 4.6. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a mapping such that*

$$\phi(T^2x, Ty) + \phi(Tx, Ty) + \phi(x, Ty) \le 3\phi(x, y)$$

for all $x, y \in C$. Then, the following are equivalent:

(a) $F(T) \neq \emptyset$;

(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha_1 = \alpha_2 = \frac{1}{3}$, $\beta_1 = \beta_2 = 0$, and $\gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0$ in (4.3), we have that

$$\frac{1}{3}\phi(T^2x, Ty) + \frac{1}{3}\phi(Tx, Ty) + \frac{1}{3}\phi(x, Ty) \le \phi(x, y)$$

for all $x, y \in C$. This is equivalent to the mapping in our theorem:

$$\phi(T^2x, Ty) + \phi(Tx, Ty) + \phi(x, Ty) \le 3\phi(x, y)$$

for all $x, y \in C$. So, we have the desired result from Theorem 4.1. □

Remark 2. Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. Let $n \in \mathbb{N}$. Then, a mapping $T : C \to C$ is called *n-generalized nonspreading* if there are $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n, \gamma_1, \gamma_2, \dots, \gamma_n, \delta_1, \delta_2, \dots, \delta_n \in \mathbb{R}$ such that

$$\begin{aligned}
&\sum_{k=1}^{n}\alpha_k\phi(T^{n+1-k}x, Ty) + (1 - \sum_{k=1}^{n}\alpha_k)\phi(x, Ty) \\
&\quad + \sum_{k=1}^{n}\gamma_k\{\phi(Ty, T^{n+1-k}x) - \phi(Ty, x)\} \\
&\le \sum_{k=1}^{n}\beta_k\phi(T^{n+1-k}x, y) + (1 - \sum_{k=1}^{n}\beta_k)\phi(x, y) \\
&\quad + \sum_{k=1}^{n}\delta_k\{\phi(y, T^{n+1-k}x) - \phi(y, x)\}
\end{aligned} \tag{4.9}$$

for all $x, y \in C$. Such a mapping is called an $(\alpha_1, \alpha_2, \dots, \alpha_n,\ \beta_1, \beta_2, \dots, \beta_n,\ \gamma_1, \gamma_2, \dots, \gamma_n,\ \delta_1, \delta_2, \dots, \delta_n)$*-generalized nonspreading* mapping. As in the proof of Theorem 4.1, we can prove a fixed point theorem for n-generalized nonspreading mappings in a smooth, strictly convex and reflexive Banach space.

5. Fixed Point Theorem 3

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into itself. Define a mapping T^* as follows:

$$T^*x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the *duality* mapping of T; see [33] and [10]. It is easy to show that T^* is a mapping of JC into itself. In fact, for $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$. So, we have

$$T^*x^* = JTJ^{-1}x^* \in JC.$$

Then, T^* is a mapping of JC into itself. Furthermore, we define the duality mapping T^{**} of T^* as follows:

$$T^{**}x = J^{-1}T^*Jx, \quad \forall x \in C.$$

It is easy to show that T^{**} is a mapping of C into itself. In fact, for $x \in C$, we have

$$T^{**}x = J^{-1}T^*Jx = J^{-1}JTJ^{-1}Jx = Tx \in C.$$

So, T^{**} is a mapping of C into itself. We know the following result in a Banach space; see [8] and [33].

Lemma 5.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into itself and let T^* be the duality mapping of JC into itself. Then, the following hold:*

(1) $JF(T) = F(T^*)$;
(2) $\|T^n x\| = \|(T^*)^n Jx\|$ *for each* $x \in C$ *and* $n \in \mathbb{N}$.

Let E be a smooth Banach space, let J be the duality mapping from E into E^* and let C be a nonempty subset of E. Then, a mapping $T : C \to C$ is called *2-skew-generalized nonspreading* if there are $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1\phi(Ty, T^2x) + \alpha_2\phi(Ty, Tx) + (1 - \alpha_1 - \alpha_2)\phi(Ty, x) \\ &\quad + \gamma_1\{\phi(T^2x, Ty) - \phi(x, Ty)\} + \gamma_2\{\phi(Tx, Ty) - \phi(x, Ty)\} \\ &\le \beta_1\phi(y, T^2x) + \beta_2\phi(y, Tx) + (1 - \beta_1 - \beta_2)\phi(y, x) \\ &\quad + \delta_1\{\phi(T^2x, y) - \phi(x, y)\} + \delta_2\{\phi(Tx, y) - \phi(x, y)\} \end{aligned} \tag{5.1}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such a mapping an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$-*skew-generalized nonspreading* mapping. Let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$-skew-generalized nonspreading mapping. Observe that if $F(T) \neq \emptyset$, then $\phi(Ty, u) \le \phi(y, u)$ for all $u \in F(T)$ and $y \in C$. Indeed, putting $x = u \in F(T)$ in (5.1), we obtain

$$\begin{aligned} &\alpha_1\phi(Ty, u) + \alpha_2\phi(Ty, u) + (1 - \alpha_1 - \alpha_2)\phi(Ty, u) \\ &\quad + \gamma_1\{\phi(u, Ty) - \phi(u, Ty)\} + \gamma_2\{\phi(u, Ty) - \phi(u, Ty)\} \\ &\le \beta_1\phi(y, u) + \beta_2\phi(y, u) + (1 - \beta_1 - \beta_2)\phi(y, u) \\ &\quad + \delta_1\{\phi(u, y) - \phi(u, y)\} + \delta_2\{\phi(u, y) - \phi(u, y)\}. \end{aligned} \tag{5.2}$$

So, we have that

$$\phi(Ty, u) \le \phi(y, u) \tag{5.3}$$

for all $u \in F(T)$ and $y \in C$. Further, if E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. So, from (5.1) we obtain the following:

$$\begin{aligned} &\alpha_1\|Ty - T^2x\|^2 + \alpha_2\|Ty - Tx\|^2 + (1 - \alpha_1 - \alpha_2)\|Ty - x\|^2 \\ &\quad + \gamma_1\{\|T^2x - Ty\|^2 - \|x - Ty\|^2\} + \gamma_2\{\|Tx - Ty\|^2 - \|x - Ty\|^2\} \\ &\le \beta_1\|y - T^2x\|^2 + \beta_2\|y - Tx\|^2 + (1 - \beta_1 - \beta_2)\|y - x\|^2 \\ &\quad + \delta_1\{\|T^2x - y\|^2 - \|x - y\|^2\} + \delta_2\{\|Tx - y\|^2 - \|x - y\|^2\} \end{aligned} \tag{5.4}$$

for all $x, y \in C$. This implies that

$$\begin{aligned}(\alpha_1+\gamma_1)\|T^2x-Ty\|^2+(\alpha_2+\gamma_2)\|Tx-Ty\|^2\\ +\{1-(\alpha_1+\gamma_1)-(\alpha_2+\gamma_2)\}\|x-Ty\|^2\\ \le(\beta_1+\delta_1)\|T^2x-y\|^2+(\beta_2+\delta_2)\|Tx-y\|^2\\ +\{1-(\beta_1+\delta_1)-(\beta_2+\delta_2)\}\|x-y\|^2\end{aligned}$$

for all $x, y \in C$. That is, T is a 2-generalized hybrid mapping [22] in a Hilbert space. Now, we prove a fixed point theorem for 2-skew-generalized nonspreading mappings in a Banach space. Before proving the theorem, we need the following definition: Let $\phi_*\colon E^* \times E^* \to (-\infty,\infty)$ be the function defined by

$$\phi_*(x^*,y^*)=\|x^*\|^2-2\langle J^{-1}y^*,x^*\rangle+\|y^*\|^2$$

for all $x^*, y^* \in E^*$, where J is the duality mapping of E. It is easy to see that

$$\phi(x,y)=\phi_*(Jy,Jx) \tag{5.5}$$

for all $x, y \in E$.

Theorem 5.2. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let T be a 2-skew-generalized nonspreading mapping of C into itself. Then, the following are equivalent:*

(a) *$F(T) \neq \emptyset$;*
(b) *$\{T^n x\}$ is bounded for some $x \in C$.*

Proof. Let T be a 2-skew-generalized nonspreading mapping of C into itself. Then, there are $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\phi(Ty,T^2x)+\alpha_2\phi(Ty,Tx)+(1-\alpha_1-\alpha_2)\phi(Ty,x)\\ +\gamma_1\{\phi(T^2x,Ty)-\phi(x,Ty)\}+\gamma_2\{\phi(Tx,Ty)-\phi(x,Ty)\}\\ \le\beta_1\phi(y,T^2x)+\beta_2\phi(y,Tx)+(1-\beta_1-\beta_2)\phi(y,x)\\ +\delta_1\{\phi(T^2x,y)-\phi(x,y)\}+\delta_2\{\phi(Tx,y)-\phi(x,y)\}\end{aligned} \tag{5.6}$$

for all $x, y \in C$. If $F(T) \neq \emptyset$, then $\phi(Ty,u) \le \phi(y,u)$ for all $u \in F(T)$ and $y \in C$. Taking a fixed point u of T, we have $\phi(T^n x,u) \le \phi(x,u)$ for all $n \in \mathbb{N}$ and $x \in C$. This implies (a) $\Longrightarrow$ (b). Let us show (b) $\Longrightarrow$ (a). Suppose that there exists $x \in C$ such that $\{T^n x\}$ is bounded. Then for any $x^*, y^* \in JC$ with $x^* = Jx$ and $y^* = Jy$, we have from (5.5) and $T^* = JTJ^{-1}$ that

$$\begin{aligned}\phi_*((T^*)^2x^*,T^*y^*)&=\phi_*(T^*T^*x^*,T^*y^*)\\ &=\phi_*(JTJ^{-1}JTJ^{-1}Jx,JTJ^{-1}Jy)\\ &=\phi_*(JTTx,JTy)\\ &=\phi_*(JT^2x,JTy)\\ &=\phi(Ty,T^2x).\end{aligned}$$

Similarly, we have that

$$\phi_*(T^*x^*,T^*y^*)=\phi_*(JTJ^{-1}Jx,JTJ^{-1}Jy)=\phi_*(JTx,JTy)=\phi(Ty,Tx).$$

Thus, we have that

$$\begin{aligned}
&\alpha_1\phi_*((T^*)^2x^*, T^*y^*) + \alpha_2\phi_*(T^*x^*, T^*y^*) + (1-\alpha_1-\alpha_2)\phi_*(x^*, T^*y^*) \\
&\quad + \gamma_1\{\phi_*(T^*y^*, (T^*)^2x^*) - \phi_*(T^*y^*, x^*)\} \\
&\quad + \gamma_2\{\phi_*(T^*y^*, T^*x^*) - \phi_*(T^*y^*, x^*)\} \\
&= \alpha_1\phi_*(JT^2x, JTy) + \alpha_2\phi_*(JTx, JTy) + (1-\alpha_1-\alpha_2)\phi_*(Jx, JTy) \\
&\quad + \gamma_1\{\phi_*(JTy, JT^2x) - \phi_*(JTy, Jx)\} \\
&\quad + \gamma_2\{\phi_*(JTy, JTx) - \phi_*(JTy, Jx)\} \\
&= \alpha_1\phi(Ty, T^2x) + \alpha_2\phi(Ty, Tx) + (1-\alpha_1-\alpha_2)\phi(Ty, x) \\
&\quad + \gamma_1\{\phi(T^2x, Ty) - \phi(x, Ty)\} + \gamma_2\{\phi(Tx, Ty) - \phi(x, Ty)\}.
\end{aligned}$$

We also have that

$$\begin{aligned}
&\beta_1\phi_*((T^*)^2x^*, y^*) + \beta_2\phi_*(T^*x^*, y^*) + (1-\beta_1-\beta_2)\phi_*(x^*, y^*) \\
&\quad + \delta_1\{\phi_*(y^*, (T^*)^2x^*) - \phi_*(y^*, x^*)\} \\
&\quad + \delta_2\{\phi_*(y^*, T^*x^*) - \phi_*(y^*, x^*)\} \\
&= \beta_1\phi_*(JT^2x, Jy) + \beta_2\phi_*(JTx, Jy) + (1-\beta_1-\beta_2)\phi_*(Jx, Jy) \\
&\quad + \delta_1\{\phi_*(Jy, JT^2x) - \phi_*(Jy, Jx)\} + \delta_2\{\phi_*(Jy, JTx) - \phi_*(Jy, Jx)\} \\
&= \beta_1\phi(y, T^2x) + \beta_2\phi(y, Tx) + (1-\beta_1-\beta_2)\phi(y, x) \\
&\quad + \delta_1\{\phi(T^2x, y) - \phi(x, y)\} + \delta_2\{\phi(Tx, y) - \phi(x, y)\}.
\end{aligned}$$

Since T is 2-skew-generalized nonspreading, we have from (5.6) that

$$\begin{aligned}
&\alpha_1\phi_*((T^*)^2x^*, T^*y^*) + \alpha_2\phi_*(T^*x^*, T^*y^*) + (1-\alpha_1-\alpha_2)\phi_*(x^*, T^*y^*) \\
&\quad + \gamma_1\{\phi_*(T^*y^*, (T^*)^2x^*) - \phi_*(T^*y^*, x^*)\} \\
&\quad + \gamma_2\{\phi_*(T^*y^*, T^*x^*) - \phi_*(T^*y^*, x^*)\} \\
&\le \beta_1\phi_*((T^*)^2x^*, y^*) + \beta_2\phi_*(T^*x^*, y^*) + (1-\beta_1-\beta_2)\phi_*(x^*, y^*) \\
&\quad + \delta_1\{\phi_*(y^*, (T^*)^2x^*) - \phi_*(y^*, x^*)\} \\
&\quad + \delta_2\{\phi_*(y^*, T^*x^*) - \phi_*(y^*, x^*)\}.
\end{aligned}$$

This implies that T^* is a 2-generalized nonspreading mapping of JC into itself. We know from Lemma 5.1 and Theorem 4.1 that T^* has a fixed point in JC. We also have from Lemma 5.1 that $F(T^*) = JF(T)$. Therefore $F(T)$ is nonempty. This completes the proof. □

Using Theorem 5.2, we have the following fixed point theorems in a Banach space.

Theorem 5.3 (Dhompongsa, Fupinwong, Takahashi and Yao [8])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a skew-nonspreading mapping, i.e.,*

$$\phi(Ty, Tx) + \phi(Tx, Ty) \le \phi(y, Tx) + \phi(x, Ty)$$

for all $x, y \in C$. Then, the following are equivalent:

(a) *$F(T) \neq \emptyset$;*

(b) *$\{T^n x\}$ is bounded for some $x \in C$.*

Proof. Putting $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$, $\alpha_2 = \beta_2 = \gamma_2 = 1$ and $\delta_2 = 0$ in (5.1), we obtain that

$$\phi(Ty, Tx) + \phi(Tx, Ty) \le \phi(y, Tx) + \phi(x, Ty)$$

for all $x, y \in C$. So, we have the desired result from Theorem 5.2. □

Theorem 5.4. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a mapping such that*

$$2\phi(Ty, Tx) + \phi(Tx, Ty) \le \phi(y, Tx) + \phi(x, Ty) + \phi(y, x)$$

for all $x, y \in C$. Then, the following are equivalent:

(a) *$F(T) \neq \emptyset$;*
(b) *$\{T^n x\}$ is bounded for some $x \in C$.*

Proof. Putting $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$, $\alpha_2 = 1$, $\beta_2 = \gamma_2 = \frac{1}{2}$ and $\delta_2 = 0$ in (5.1), we obtain that

$$2\phi(Ty, Tx) + \phi(Tx, Ty) \le \phi(y, Tx) + \phi(x, Ty) + \phi(y, x)$$

for all $x, y \in C$. So, we have the desired result from Theorem 5.2. □

Theorem 5.5. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a mapping such that*

$$\alpha\phi(Ty, Tx) + (1 - \alpha)\phi(Ty, x) \le \beta\phi(y, Tx) + (1 - \beta)\phi(y, x)$$

for all $x, y \in C$. Then, the following are equivalent:

(a) *$F(T) \neq \emptyset$;*
(b) *$\{T^n x\}$ is bounded for some $x \in C$.*

Proof. Putting $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$, $\alpha_2 = \alpha$, $\beta_2 = \beta$ and $\gamma_2 = \delta_2 = 0$ in (5.1), we obtain that

$$\alpha\phi(Ty, Tx) + (1 - \alpha)\phi(Ty, x) \le \beta\phi(y, Tx) + (1 - \beta)\phi(y, x)$$

for all $x, y \in C$. So, we have the desired result from Theorem 5.2. □

Finally, we prove the following fixed point theorem in a Banach space.

Theorem 5.6. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a mapping such that*

$$\phi(Ty, T^2x) + \phi(Ty, Tx) + \phi(Ty, x) \le 3\phi(y, x)$$

for all $x, y \in C$. Then, the following are equivalent:

(a) *$F(T) \neq \emptyset$;*
(b) *$\{T^n x\}$ is bounded for some $x \in C$.*

Proof. Putting $\alpha_1 = \alpha_2 = \frac{1}{3}$, $\beta_1 = \beta_2 = 0$, and $\gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0$ in (5.1), we have that

$$\frac{1}{3}\phi(Ty, T^2x) + \frac{1}{3}\phi(Ty, Tx) + \frac{1}{3}\phi(Ty, x) \le \phi(y, x)$$

for all $x, y \in C$. This is equivalent to the mapping in our theorem:

$$\phi(T^2x, Ty) + \phi(Tx, Ty) + \phi(x, Ty) \le 3\phi(x, y)$$

for all $x, y \in C$. So, we have the desired result from Theorem 5.2. □

Remark 3. Let E be a smoth Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $n \in \mathbb{N}$. Then, a mapping $T : C \to C$ is called *n-skew-generalized nonspreading* if there are $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\beta_1, \beta_2, \ldots, \beta_n$, $\gamma_1, \gamma_2, \ldots, \gamma_n$, $\delta_1, \delta_2, \ldots, \delta_n \in \mathbb{R}$ such that

$$\begin{aligned}
&\sum_{k=1}^{n}\alpha_k \phi(Ty, T^{n+1-k}x) + (1 - \sum_{k=1}^{n}\alpha_k)\phi(Ty, x) \\
&\quad + \sum_{k=1}^{n}\gamma_k\{\phi(T^{n+1-k}x, Ty) - \phi(x, Ty)\} \\
&\le \sum_{k=1}^{n}\beta_k \phi(y, T^{n+1-k}x) + (1 - \sum_{k=1}^{n}\beta_k)\phi(y, x) \\
&\quad + \sum_{k=1}^{n}\delta_k\{\phi(T^{n+1-k}x, y) - \phi(x, y)\}
\end{aligned} \tag{5.7}$$

for all $x, y \in C$. Such a mapping is called an $(\alpha_1, \alpha_2, \ldots, \alpha_n,\ \beta_1, \beta_2, \ldots, \beta_n,$ $\gamma_1, \gamma_2, \ldots, \gamma_n,\ \delta_1, \delta_2, \ldots, \delta_n)$-*skew-generalized nonspreading* mapping. As in the proof of Theorem 5.2, we can prove a fixed point theorem for n-skew-generalized nonspreading mappings in a smooth, strictly convex and reflexive Banach space.

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