Topologies and bornologies determined by operator ideals, II

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Abstract

Let $\mathfrak{A}$ be an operator ideal on LCS's. A continuous seminorm p of a LCS X is said to be $\mathfrak{A}$–*continuous* if $\widetilde{Q}_p \in \mathfrak{A}^{\text{inj}}(X, \widetilde{X}_p)$, where $\widetilde{X}_p$ is the completion of the normed space $X_p = X/p^{-1}(0)$ and $\widetilde{Q}_p$ is the canonical map. p is said to be a Groth($\mathfrak{A}$)–*seminorm* if there is a continuous seminorm q of X such that $p \leq q$ and the canonical map $\widetilde{Q}_{pq} : \widetilde{X}_q \longrightarrow \widetilde{X}_p$ belongs to $\mathfrak{A}(\widetilde{X}_q, \widetilde{X}_p)$. It is well-known that when $\mathfrak{A}$ is the ideal of absolutely summing (resp. precompact, weakly compact) operators, a LCS X is a nuclear (resp. Schwartz, infra–Schwartz) space if and only if every continuous seminorm p of X is $\mathfrak{A}$–continuous if and only if every continuous seminorm p of X is a Groth($\mathfrak{A}$)–seminorm. In this paper, we extend this equivalence to arbitrary operator ideals $\mathfrak{A}$ and discuss several aspects of these constructions which are initiated by A. Grothendieck and D. Randkte, respectively. A bornological version of the theory is obtained, too.

1 Introduction

Let X be a LCS (locally convex space) and p a continuous seminorm of X. Denote by X_p the quotient space $X/p^{-1}(0)$ equipped with the quotient seminorm (in fact, norm) $\|\cdot\|_p$. Q_p denotes the canonical map from X onto X_p and $\widetilde{Q}_p$ denotes the unique map induced by Q_p from X into the completion $\widetilde{X}_p$ of X_p. If q is a continuous seminorm of

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X such that $p \leq q$ (*i.e.* $p(x) \leq q(x), \forall x \in X$), the canonical map $Q_{pq} : X_q \longrightarrow X_p$ and $\widetilde{Q}_{pq} : \widetilde{X}_q \longrightarrow \widetilde{X}_p$ are continuous.

Let $\mathfrak{A}$ be an operator ideal on Banach spaces. Following A. Pietsch [10], we call a LCS X a Groth($\mathfrak{A}$)–space if for each continuous seminorm p of X there is a continuous seminorm q of X such that $p \leq q$ and $\widetilde{Q}_{pq} \in \mathfrak{A}(\widetilde{X}_q, \widetilde{X}_p)$. This amounts to say that the completion $\widetilde{X}$ of X is a topological projective limit $\varprojlim \widetilde{Q}_{pq}\widetilde{X}_q$ of Banach spaces of type $\mathfrak{A}$ (cf. [7]). A. Grothendieck's construction of nuclear spaces is a model of Groth($\mathfrak{A}$)–spaces. In fact, a LCS X is a nuclear (resp. Schwartz, infra–Schwartz) space if it is a Groth($\mathfrak{N}$)–space (resp. Groth($\mathfrak{K}_p$)–space, Groth($\mathfrak{W}$)–space), where $\mathfrak{N}$ (resp. $\mathfrak{K}_p, \mathfrak{W}$) is the ideal of nuclear (resp. precompact, weakly compact) operators. It was known that a locally convex space X is a Groth($\mathfrak{A}$)–space if and only if the identity operator id_X of X belongs to the right superior extension $\mathfrak{A}^{\mathrm{rup}}$ of $\mathfrak{A}$ to LCS's [10].

Another usual way to deal with these kind of spaces is due to D. Randkte [11]. A continuous seminorm p of a LCS X is said to be $\mathfrak{A}$–continuous if the canonical map $\widetilde{Q}_p : X \to \widetilde{X}_p$ belongs to the injective hull $\mathfrak{A}^{\mathrm{inj}}$ of $\mathfrak{A}$. X is said to be an $\mathfrak{A}$–topological space if every continuous seminorm of X is $\mathfrak{A}$–continuous. For example, a LCS X is nuclear (resp. Schwartz, infra–Schwartz) if X is a $\mathfrak{N}$– (resp. $\mathfrak{K}_p$–, $\mathfrak{W}$–) topological space.

The advantage of the construction of Grothendieck is that we need to pay attention only on Banach spaces operators, while the construction of Randkte appears to be simpler and easier to apply. In this paper, we shall prove that these two constructions are in fact equivalent. Motivated by those examples of classical spaces, we define the notions of ideal topologies ($\mathfrak{A}$–topologies in §3) and Grothendieck topologies (Groth($\mathfrak{A}$)–topologies in §4) associated to an operator ideal $\mathfrak{A}$. Our main result, Theorem 5.1, says that Groth($\mathfrak{A}^{\mathrm{inj}}$)–topology = $\mathfrak{A}^{\mathrm{rup}}$–topology on LCS's. In particular, a LCS X is a Groth($\mathfrak{A}^{\mathrm{inj}}$)–spaces if and only if X is a $\mathfrak{A}^{\mathrm{rup}}$–topological space.

We also discuss dual concepts of Grothendieck spaces and $\mathfrak{A}$–topological spaces, *i.e.* co–Grothendieck spaces and $\mathfrak{A}$–bornological spaces, which also attract some research interests covering co–nuclear spaces, co–Schwartz spaces, semi–Montel spaces, and semi–reflexive spaces.

Finally, we refer the readers to [4, 5, 7–10, 20] concerning Groth($\mathfrak{A}$)–spaces and co–Groth($\mathfrak{A}$)–spaces, and to [6, 9, 11, 13–17, 19, 20] concerning $\mathfrak{A}$–topological spaces and $\mathfrak{A}$–bornological spaces for further information, and in particular, to [21] for a quick review of the theory of ideal topologies and bornologies.

2 Notations and preliminaries

We shall follow the terminologies in [21]. Let X and Y be LCS's. We denote by $\mathfrak{L}^b(X,Y)$, $\mathfrak{L}(X,Y)$, and $L^\times(X,Y)$ the collection of all operators from X into Y which are bounded (*i.e.* sending a 0-neighborhood to a bounded set), continuous, and locally bounded (*i.e.* sending bounded sets to bounded sets), respectively. Denote by $X_{\mathfrak{G}}$ a vector space X equipped with a locally convex (Hausdorff) topology $\mathfrak{G}$, and by $X^{\mathfrak{M}}$ a vector space X equipped with a convex vector (separated) bornology $\mathfrak{M}$. U_N always denotes the *closed* unit ball of a normed space N.

A subset B of a LCS X is said to be a *disk* if B is *absolutely convex, i.e.* $\lambda B+\beta B \subset B$ whenever $|\lambda|+|\beta| \leq 1$. A disk B is said to be a *σ–disk,* or *absolutely σ–convex* if $\Sigma_n \lambda_n b_n$ converges in X and the sum belongs to B whenever $(\lambda_n) \in U_{l_1}$ and $b_n \in B$, $n = 1, 2, \ldots$ Associated to each bounded disk B in X a normed space $X(B) = \cup_{\lambda>0}\lambda B$ equipped with the gauge γ_B of B as its norm, where $\gamma_B(x) = \inf\{\lambda > 0 : x \in \lambda B\}, \forall x \in X(B)$. The canonical map J_B sending x in $X(B)$ to x in X is continuous. Moreover, if A is a bounded disk in X such that $B \subset A$ then the canonical map J_{AB} sending x in the normed space $X(B)$ to x in the normed space $X(A)$ is bounded. A bounded disk B in X is said to be *infracomplete* if $X(B)$ is complete with respect to γ_B. It is known that a bounded and closed disk B in X is absolutely σ–convex if and only if B is infracomplete [21]. X is said to be *infracomplete* if the *von Neumann bornology* $\mathfrak{M}_{\mathrm{von}}(X)$, *i.e.* the bornology of all topologically bounded subsets of X, has a basis consisting of infracomplete subsets of X, or equivalently, σ–disked subsets of X. In other words, $(X, \mathfrak{M}_{\mathrm{von}}(X))$ is a complete convex bornological vector space (cf. [2]).

Let X and Y be LCS's. Q^1 in $\mathfrak{L}(X,Y)$ is said to be a *bornological surjection* if Q^1 is onto and induces the bornology of Y (*i.e.* for each bounded subset B of Y there is a bounded subset A of X such that $Q^1A = B$). Let $\mathfrak{C}$ be either the class $\mathbb{L}$ of locally convex spaces or the class $\mathbb{B}$ of Banach spaces. An operator ideal $\mathfrak{A}$ on $\mathfrak{C}$ is said to be *bornologically surjective* if whenever T is a continuous operator from X into Y and Q is a bornological surjection from X_0 onto X such that $TQ \in \mathfrak{A}(X_0, Y)$, we have $T \in \mathfrak{A}(X,Y)$, where $X, X_0, Y \in \mathfrak{C}$. The bornologically surjective hull $\mathfrak{A}^{\mathrm{bsur}}$ of $\mathfrak{A}$ is the intersection of all bornologically surjective operator ideals containing $\mathfrak{A}$. If $\mathfrak{C} = \mathbb{B}$, we have $\mathfrak{A}^{\mathrm{bsur}} = \mathfrak{A}^{\mathrm{sur}}$. But, if $\mathfrak{C} = \mathbb{L}$ then they are, in general, different objects, cf. [18]. We would like to mention that since a surjection is not always a bornological surjection (cf. [12, ex. 4.9 and 4.20] or [18]), Theorem 4.10(c) in [21] should be rewritten by replacing the word "surjective" by the phrase "bornologically surjective". All other results in [21] are unaffected.

We quote two recent results for later reference.

Proposition 2.1 ([1]) *We can associate to each LCS Y a LCS Y^∞ and an injection J_Y^∞ in $\mathfrak{L}(Y, Y^\infty)$ such that the injective hull $\mathfrak{A}^{\mathrm{inj}}$ of an operator ideal $\mathfrak{A}$ on LCS's is given by*

$$\mathfrak{A}^{\mathrm{inj}}(X,Y) = \{T \in \mathfrak{L}(X,Y) : J_Y^\infty T \in \mathfrak{A}(X, Y^\infty)\}.$$

Proposition 2.2 ([18]) *We can associate to each LCS X a LCS X^1 and a bornological surjection Q_X^1 in $\mathfrak{L}(X^1, X)$ such that the bornologically surjective hull of an operator ideal $\mathfrak{A}$ on LCS's is given by*

$$\mathfrak{A}^{\mathrm{bsur}}(X,Y) = \{T \in \mathfrak{L}(X,Y) : TQ_X^1 \in \mathfrak{A}(X^1, Y)\}.$$

3 $\mathfrak{A}$–topologies and $\mathfrak{A}$–bornologies

Let $\mathfrak{A}$ be an operator ideal on $\mathfrak{C}$, where $\mathfrak{C}$ is either the class of LCS's or the class of Banach spaces. The $\mathfrak{A}$–*topology* $\mathfrak{T}(\mathfrak{A})(X_0)$ of an X_0 in $\mathfrak{C}$ is defined to be the projective topology of X_0 with respect to the family $\{T \in \mathfrak{A}(X_0, Y) : Y \in \mathfrak{C}\}$ and the $\mathfrak{A}$–*bornology* $\mathfrak{B}(\mathfrak{A})(Y_0)$ of an Y_0 in $\mathfrak{C}$ is defined to be the inductive bornology of Y_0 with respect to the family $\{T \in \mathfrak{A}(X, Y_0) : X \in \mathfrak{C}\}$. In other words, $\mathfrak{T}(\mathfrak{A})(X_0)$ is the coarsest locally convex topology $\mathfrak{T}$ of X_0 such that all operators in $\mathfrak{A}$ with X_0 as domain are still continuous with respect to $\mathfrak{T}$, *i.e.* $\mathfrak{A}(X_0, Y) \subseteq \mathfrak{L}(X_{0\mathfrak{T}}, Y), \forall\, Y \in \mathfrak{C}$; and $\mathfrak{B}(\mathfrak{A})(Y_0)$ is the smallest convex vector bornology $\mathfrak{B}$ of Y_0 such that all operators in $\mathfrak{A}$ with Y_0 as range are still locally bounded with respect to $\mathfrak{B}$, *i.e.* $\mathfrak{A}(X, Y_0) \subseteq L^\times(X, Y_0^{\mathfrak{B}})$, $\forall X \in \mathfrak{C}$. Even more precisely, a seminorm p of X_0 is $\mathfrak{T}(\mathfrak{A})$–continuous (or simply $\mathfrak{A}$–continuous) if and only if there is a T in $\mathfrak{A}(X_0, Y)$ for some Y in $\mathfrak{C}$ and a continuous seminorm q of Y such that $p(x) \leq q(Tx)$, $\forall x \in X_0$; and a subset B of Y_0 is $\mathfrak{B}(\mathfrak{A})$–bounded (or simply $\mathfrak{A}$–bounded) if and only if there is a T in $\mathfrak{A}(X, Y_0)$ for some X in $\mathfrak{C}$ and a bounded subset A of X such that $B \subseteq TA$ (see [21]).

3.1 $\mathfrak{A}$–topologies and $\mathfrak{A}$–topological spaces

Proposition 3.1 *Let $\mathfrak{A}$ be an operator ideal on LCS's. The $\mathfrak{A}$–topology coincides with the $\mathfrak{A}^{\mathrm{inj}}$–topology on every LCS X. Moreover, a continuous seminorm p of X is $\mathfrak{A}$–continuous if and only if $Q_p \in \mathfrak{A}^{\mathrm{inj}}(X, X_p)$ if and only if $\widetilde{Q}_p \in \mathfrak{A}^{\mathrm{inj}}(X, \widetilde{X}_p)$.*

Proof. It is obvious that the $\mathfrak{A}^{\mathrm{inj}}$–topology is always finer than the $\mathfrak{A}$–topology on X. It suffices to show that for every LCS Y and T in $\mathfrak{A}^{\mathrm{inj}}(X, Y)$, T is also continuous with respect to the $\mathfrak{A}$–topology of X. By Proposition 2.1, $J_Y^\infty T \in \mathfrak{A}(X, Y^\infty)$. Since J_Y^∞ is an injection, the first assertion follows. On the other hand, we have shown in [21] that $Q_p \in \mathfrak{A}^{\mathrm{inj}}(X, X_p)$ if and only if p is $\mathfrak{A}^{\mathrm{inj}}$–continuous, and thus if and only if p is $\mathfrak{A}$–continuous. We are done as the canonical map $J_p : Q_p \to \widetilde{Q}_p$ is an injection and $\widetilde{Q}_p = J_p Q_p$. $\square$

Recall that a LCS X is said to be $\mathfrak{A}$–*topological* if its original topology $\mathfrak{G}_{\mathrm{ori}}(X)$ coincides with the $\mathfrak{A}$–topology, *i.e.* $\mathfrak{G}_{\mathrm{ori}}(X) = \mathfrak{T}(\mathfrak{A})(X)$ (cf. [21]).

Corollary 3.2 *Let $\mathfrak{A}$ be an operator ideal on LCS's and X a LCS. The following are all equivalent.*

(1) *X is $\mathfrak{A}$–topological.*

(2) *$\mathfrak{L}^b(X, Y) \subset \mathfrak{A}^{\mathrm{inj}}(X, Y)$ for every LCS Y.*

(3) *$\mathfrak{L}(X, F) = \mathfrak{A}^{\mathrm{inj}}(X, F)$ for every normed (or Banach) space F.*

Example 3.3 When $\mathfrak{A}$ is the ideal $\mathfrak{N}$ of nuclear operators or the ideal $\mathfrak{P}$ of absolutely summing operators (resp. the ideal $\mathfrak{K}_p$ of precompact operators, the ideal $\mathfrak{W}$ of weakly compact operators), the corresponding $\mathfrak{A}$–topological spaces are nuclear spaces (resp. Schwartz spaces, infra–Schwartz spaces). Corollary 3.2 serves as a prototype of a class of theorems concerning these spaces (see *e.g.* [20, pp. 17, 26, 149 and 157]).

In sequel, $\mathfrak{C}$ denotes either the class of LCS's or the class of Banach spaces. The following includes a result of Jarchow [6, Proposition 3] in the context of Banach spaces.

Theorem 3.4 *Let $\mathfrak{A}$ be a surjective operator ideal on $\mathfrak{C}$ and $X, Y \in \mathfrak{C}$. If Y is a (topological) quotient space of X then the $\mathfrak{A}$–topology of Y is the quotient topology induced by the $\mathfrak{A}$–topology of X.*

PROOF. Let Q be the quotient map from X onto Y. Let $X_{\mathfrak{A}}$ (resp. $Y_{\mathfrak{A}}$) denote the LCS X (resp. Y) equipped with the $\mathfrak{A}$–topology. We have $Q \in \mathfrak{L}(X_{\mathfrak{A}}, Y_{\mathfrak{A}})$ [21, Theorem 3.8]. It implies that the $\mathfrak{A}$–topology of Y is weaker than the quotient topology induced by the $\mathfrak{A}$–topology of X. Let p be an $\mathfrak{A}$–continuous seminorm of X and q the quotient seminorm of Y induced by p. Let $\widetilde{Q}_p : X \to \widetilde{X}_p$, $\widetilde{Q}_q : Y \to \widetilde{Y}_q$ and $\widetilde{Q}_{qp} : \widetilde{X}_p \to \widetilde{Y}_q$ be the canonical maps. By Proposition 3.1 (or [21, Lemma 3.3] for the Banach space version), $\widetilde{Q}_p \in \mathfrak{A}^{\mathrm{inj}}(X, \widetilde{X}_p)$. Now $\widetilde{Q}_q Q = \widetilde{Q}_{qp}\widetilde{Q}_p \in \mathfrak{A}^{\mathrm{inj}}(X, \widetilde{Y}_q)$ implies $\widetilde{Q}_q \in (\mathfrak{A}^{\mathrm{inj}})^{\mathrm{sur}}(Y, \widetilde{Y}_q)$ since Q is a surjection. However, $(\mathfrak{A}^{\mathrm{sur}})^{\mathrm{inj}}$ is always surjective, by Proposition 2.1. As a result, $(\mathfrak{A}^{\mathrm{inj}})^{\mathrm{sur}} \subset (\mathfrak{A}^{\mathrm{sur}})^{\mathrm{inj}}$. Thus $\widetilde{Q}_q \in (\mathfrak{A}^{\mathrm{sur}})^{\mathrm{inj}}(Y, \widetilde{Y}_q) = \mathfrak{A}^{\mathrm{inj}}(Y, \widetilde{Y}_q)$ since $\mathfrak{A}$ is surjective. It implies that q is $\mathfrak{A}$–continuous. Therefore, the $\mathfrak{A}$–topology of Y coincides with the quotient topology induced by the $\mathfrak{A}$–topology of X. $\square$

Corollary 3.5 *Let $\mathfrak{A}$ be a surjective operator ideal on $\mathfrak{C}$. Then a quotient space of an $\mathfrak{A}$–topological space is again an $\mathfrak{A}$–topological space.*

3.2 $\mathfrak{A}$–bornologies and $\mathfrak{A}$–bornological spaces

Proposition 3.6 *Let $\mathfrak{A}$ be an operator ideal on LCS's. The $\mathfrak{A}$–bornology coincides with the $\mathfrak{A}^{\mathrm{bsur}}$–bornology on every LCS. Moreover, a bounded subset B_0 of a LCS Y is $\mathfrak{A}$–bounded if and only if $J_B \in \mathfrak{A}^{\mathrm{bsur}}(Y(B), Y)$, where B is the absolutely convex hull of B_0 and J_B is the canonical map. When $\mathfrak{A}$ is surjective, we can replace $\mathfrak{A}^{\mathrm{bsur}}$ by $\mathfrak{A}$.*

PROOF. The first part is similar to Proposition 3.1. For the rest, W.O.L.G. we can assume that $\mathfrak{A}$ is bornologically surjective. If $J_B \in \mathfrak{A}(Y(B), Y)$ then $B_0 \subset J_B U_{Y(B)}$ is, by definition, $\mathfrak{A}$–bounded in Y. Conversely, if B_0 is $\mathfrak{A}$–bounded in Y, B is also $\mathfrak{A}$–bounded in Y and we can choose a bounded disk A in a LCS X and a T in $\mathfrak{A}(X, Y)$ such that $TA = B$. So we have a T_0 in $\mathfrak{L}(X(A), Y(B))$ such that $TJ_A = J_B T_0$. Now $TJ_A \in \mathfrak{A}(X(A), Y)$ and the bornological surjectivity of T_0 implies $J_B \in \mathfrak{A}(Y(B), Y)$. The last assertion follows from [18, Corollary 2.6] which says that $\mathfrak{A}^{\mathrm{bsur}}(N, Y) = \mathfrak{A}(N, Y)$ for every normed space N and every LCS Y if $\mathfrak{A}$ is surjective. $\square$

Recall that a LCS Y is said to be $\mathfrak{A}$–*bornological* if its von Neumann bornology of Y (*i.e.* the family of all topologically bounded subsets of Y) coincides with the $\mathfrak{A}$–bornology (cf. [21]).

Corollary 3.7 *Let $\mathfrak{A}$ be an operator ideal on LCS's and Y a LCS. The following are all equivalent.*

(1) *Y is $\mathfrak{A}$–bornological.*

(2) $\mathfrak{L}^b(X,Y) \subset \mathfrak{A}^{\mathrm{bsur}}(X,Y)$ *for every LCS X.*

(3) $\mathfrak{L}(N,Y) = \mathfrak{A}^{\mathrm{bsur}}(N,Y)$ *for every normed space N.*

In case Y is infracomplete, they are all equivalent to

(3)$'$ $\mathfrak{L}(E,Y) = \mathfrak{A}^{\mathrm{bsur}}(E,Y)$ *for every Banach space E.*

If $\mathfrak{A}$ is surjective, we can replace $\mathfrak{A}^{\mathrm{bsur}}$ by $\mathfrak{A}$ in all above statements.

PROOF. We just mention that the last assertion follows from [18, Corollary 2.6]. $\square$

Example 3.8 When $\mathfrak{A}$ is the ideal $\mathfrak{N}$ of nuclear operators or the ideal $\mathfrak{P}$ of absolutely summing operators (resp. the ideal $\mathfrak{K}_p$ of precompact operators, the ideal $\mathfrak{W}$ of weakly compact operators), the corresponding $\mathfrak{A}$–bornological spaces are co–nuclear spaces (resp. semi–Montel spaces and semi–reflexive spaces). Corollary 3.7 serves as a prototype of a class of theorems concerning these spaces (see *e.g.* [3]).

Let $\mathfrak{C}$ be either the class of LCS's or the class of Banach spaces.

Theorem 3.9 *Let $\mathfrak{A}$ be an injective operator ideal on $\mathfrak{C}$ and $X, Y \in \mathfrak{C}$. If Y is a (topological) subspace of X then the $\mathfrak{A}$–bornology of Y is the subspace bornology inherited from the $\mathfrak{A}$–bornology of X.*

PROOF. Similar to Theorem 3.4. Note that we have $(\mathfrak{A}^{\mathrm{bsur}})^{\mathrm{inj}} = (\mathfrak{A}^{\mathrm{inj}})^{\mathrm{bsur}}$ by Propositions 2.1 and 2.2 in this case. $\square$

Corollary 3.10 *Let $\mathfrak{A}$ be an injective operator ideal on $\mathfrak{C}$. Then a subspace of an $\mathfrak{A}$–bornological space is again an $\mathfrak{A}$–bornological space.*

4 Grothendieck topologies and Grothendieck bornologies

4.1 Groth($\mathfrak{A}$)–topologies and Groth($\mathfrak{A}$)–spaces

Definition Let $\mathfrak{A}$ be an operator ideal on Banach spaces. We call a continuous seminorm p of a LCS X a Groth($\mathfrak{A}$)–*seminorm* if there is a continuous seminorm q of X such that $p \le q$ and $\widetilde{Q}_{pq} \in \mathfrak{A}(\widetilde{X}_q, \widetilde{X}_p)$.

Remark Two operator ideals $\mathfrak{A}$ and $\mathfrak{B}$ on Banach spaces are said to be *equivalent* if there are positive integers m and n such that $\mathfrak{A}^m \subseteq \mathfrak{B}$ and $\mathfrak{B}^n \subseteq \mathfrak{A}$. In this case, a continuous seminorm p of a LCS X is a Groth($\mathfrak{A}$)–seminorm if and only if p is a Groth($\mathfrak{B}$)–seminorm (cf. [10] or [7]). An operator ideal $\mathfrak{A}$ is said to be *quasi–injective* if $\mathfrak{A}$ is equivalent to an injective operator ideal. For example, the ideal $\mathfrak{N}$ of nuclear operators is quasi–injective since it is equivalent to the injective ideal $\mathfrak{P}$ of absolutely summing operators. In fact, $\mathfrak{P}^3 \subset \mathfrak{N} \subset \mathfrak{P}$ (cf. [20, p.145]).

Proposition 4.1 *Let $\mathfrak{A}$ be an operator ideal on Banach spaces and let $p, p_1, \ldots, p_n$ be Groth($\mathfrak{A}$)–seminorms of a LCS X.*

(a) λp is a Groth($\mathfrak{A}$)–seminorm of X for all $\lambda \geq 0$.

(b) If p_0 is a continuous seminorm of X such that $p_0 \leq p$ then p_0 is a Groth($\mathfrak{A}$)–seminorm.

(c) $p_1 + p_2 + \cdots + p_n$ is a Groth($\mathfrak{A}^{\text{inj}}$)–seminorm. In case $\mathfrak{A}$ is quasi–injective, $p_1 + p_2 + \cdots + p_n$ is a Groth($\mathfrak{A}$)–seminorm.

Proof. (a) and (b) are trivial. For (c), let $q_1, \ldots, q_n$ be continuous seminorms of X such that $p_i \leq q_i$ and $Q_i = \widetilde{Q}_{p_i q_i} \in \mathfrak{A}(\widetilde{X}_{q_i}, \widetilde{X}_{p_i})$, $i = 1, 2, \ldots, n$. Let $p_0 = p_1 + \cdots + p_n$ and $q_0 = q_1 + \cdots + q_n$. Let $J_p : \widetilde{X}_{p_0} \longrightarrow \oplus_{\ell_1} \widetilde{X}_{p_i}$ and $J_q : \widetilde{X}_{q_0} \longrightarrow \oplus_{\ell_1} \widetilde{X}_{q_i}$ be the canonical isometric embeddings. Let $j_k : \widetilde{X}_{p_k} \longrightarrow \oplus_{\ell_1} \widetilde{X}_{p_i}$ and $\pi_k : \oplus_{\ell_1} \widetilde{X}_{q_i} \longrightarrow \widetilde{X}_{q_k}$, $k = 1, \ldots, n$, be the canonical embeddings and projections, respectively. We want to prove that $Q_0 = \widetilde{Q}_{p_0 q_0}$ belongs to $\mathfrak{A}^{\text{inj}}(\widetilde{X}_{q_0}, \widetilde{X}_{p_0})$. Note that $J_p Q_0 = (j_1 Q_1 \pi_1 + j_2 Q_2 \pi_2 + \cdots + j_n Q_n \pi_n) J_q$. Since $Q_k \in \mathfrak{A}(\widetilde{X}_{q_k}, \widetilde{X}_{p_k})$, $k = 1, 2, \ldots, n$, $J_p Q_0 \in \mathfrak{A}(\widetilde{X}_{q_0}, \oplus_{\ell_1} \widetilde{X}_{p_1})$ and hence $Q_0 \in \mathfrak{A}^{\text{inj}}(\widetilde{X}_{q_0}, \widetilde{X}_{p_0})$. That is, p_0 is a Groth($\mathfrak{A}^{\text{inj}}$)–seminorm of X. □

Definition Let $\mathfrak{A}$ be a quasi–injective operator ideal on Banach spaces and X a LCS. The Groth($\mathfrak{A}$)–*topology* of X is defined to be the locally convex (Hausdorff) topology of X determined by all Groth($\mathfrak{A}$)–seminorms.

Recall that a LCS X is called a Groth($\mathfrak{A}$)–*space* for some operator ideal $\mathfrak{A}$ on Banach spaces if $id_X \in \mathfrak{A}^{\text{rup}}(X, X)$ (cf. [10]). It is easy to see that for a quasi–injective operator ideal $\mathfrak{A}$ on Banach spaces, a LCS X is a Groth($\mathfrak{A}$)–space if and only if the topology of X coincides with the Groth($\mathfrak{A}$)–topology. In this case, the completion $\widetilde{X}$ of X is a topological projective limit $\varprojlim \widetilde{Q}_{pq} \widetilde{X}_q$ of Banach spaces of type $\mathfrak{A}$ (cf. [7]).

4.2 Groth($\mathfrak{A}$)–bornologies and co–Groth($\mathfrak{A}$)–spaces

Definition Let $\mathfrak{A}$ be an operator ideal on Banach spaces. A bounded σ–disk B in a LCS X is said to be Groth($\mathfrak{A}$)–*bounded* in X if there is a bounded σ–disk A in X such that $B \subset A$ and the canonical map $J_{AB} \in \mathfrak{A}(X(B), X(A))$. Note that, in this case, both $X(A)$ and $X(B)$ are Banach spaces.

Remark If $\mathfrak{A}$ and $\mathfrak{B}$ are two equivalent operator ideals on Banach spaces then a bounded σ–disk B in a LCS X is Groth($\mathfrak{A}$)–bounded if and only if B is Groth($\mathfrak{B}$)–bounded (cf. [7]). An operator ideal $\mathfrak{A}$ is said to be *quasi–surjective* if $\mathfrak{A}$ is equivalent to a surjective operator ideal.

Proposition 4.2 *Let $\mathfrak{A}$ be an operator ideal on Banach spaces and let $B, B_1, \ldots, B_n$ be Groth($\mathfrak{A}$)–bounded σ–disks in a LCS X.*

(a) λB is Groth($\mathfrak{A}$)–bounded for all $\lambda \geq 0$.

(b) If B_0 is a bounded subset of X and $B_0 \subset B$ then the σ–disked hull $\Gamma_\sigma(B_0)$ of B_0 exists in X and is Groth($\mathfrak{A}$)–bounded in X.

(c) $\Gamma_\sigma(B_1 + \cdots + B_n)$ is Groth($\mathfrak{A}^{\mathrm{sur}}$)–bounded in X. In case $\mathfrak{A}$ is quasi–surjective, $\Gamma_\sigma(B_1 + \cdots + B_n)$ is Groth($\mathfrak{A}$)–bounded in X.

Proof. Similar to Proposition 4.1. □

Definition Let $\mathfrak{A}$ be a quasi–surjective operator ideal on Banach spaces. The Groth($\mathfrak{A}$)–*bornology* of a LCS X is defined to be the convex vector bornology of X determined by all Groth($\mathfrak{A}$)–bounded σ–disks in X.

Definition A LCS is called a *co*–Groth($\mathfrak{A}$)–*space* if all bounded σ–disks in X are Groth($\mathfrak{A}$)–bounded. It is equivalent to say that $id_X \in \mathfrak{A}^{\mathrm{lup}}(X, X)$.

It is easy to see that for a quasi–surjective operator ideal $\mathfrak{A}$ on Banach spaces, an infracomplete LCS X is a co–Groth($\mathfrak{A}$)–space if and only if the von Neumann bornology $\mathfrak{M}_{\mathrm{von}}(X)$ of X coincides with the Groth($\mathfrak{A}$)–bornology. In this case, the complete convex bornological space X is a bornological inductive limit $\varinjlim J_{AB}X(B)$ of Banach spaces of type $\mathfrak{A}$.

5 Coincidence of ideal topologies (bornologies) and Grothendieck topologies (bornologies)

Theorem 5.1 *Let $\mathfrak{A}$ be an operator ideal on Banach spaces. The Groth$(\mathfrak{A}^{\rm inj})$–topology coincides with the $\mathfrak{A}^{\rm rup}$–topology on every LCS and the Groth$(\mathfrak{A}^{\rm sur})$–bornology coincides with the $\mathfrak{A}^{\rm lup}$–bornology on every infracomplete LCS. In particular, we have*

(*a*) *A LCS X is a Groth$(\mathfrak{A}^{\rm inj})$–space if and only if X is an $\mathfrak{A}^{\rm rup}$–topological space.*

(*b*) *An infracomplete LCS X is a co–Groth$(\mathfrak{A}^{\rm sur})$–space if and only if X is an $\mathfrak{A}^{\rm lup}$–bornological space.*

(*c*) *The $\mathfrak{A}$–topology (resp. $\mathfrak{A}$–bornology) coincides with the Groth$(\mathfrak{A}^{\rm inj})$–topology (resp. Groth$(\mathfrak{A}^{\rm sur})$–bornology) on Banach spaces.*

Proof. Let p be an $\mathfrak{A}^{\rm rup}$–continuous seminorm of X. Then $\widetilde{Q}_p \in (\mathfrak{A}^{\rm rup})^{\rm inj}(X, \widetilde{X}_p) = (\mathfrak{A}^{\rm inj})^{\rm rup}(X, \widetilde{X}_p)$, by [18, Proposition 3.5]. Consequently, a factorization of $\widetilde{Q}_p = ST$ exists, where $S \in \mathfrak{A}^{\rm inj}(E, \widetilde{X}_p)$ and $T \in \mathfrak{L}(X, E)$ for some Banach space E. Define

$$q(x) = \|S\| \, \|Tx\|, \forall x \in X.$$

Then q is a continuous seminorm of X such that

$$p(x) = \|\widetilde{Q}_p(x)\| = \|STx\| \leq \|S\| \, \|Tx\| = q(x), \forall x \in X.$$

Note that T induces an R in $\mathfrak{L}(\widetilde{X}_q, E)$ such that $T = R\widetilde{Q}_q$. Now, $\widetilde{Q}_{pq} = SR \in \mathfrak{A}^{\rm inj}(\widetilde{X}_q, \widetilde{X}_p)$. Therefore, p is a Groth$(\mathfrak{A}^{\rm inj})$–seminorm of X.

Conversely, if p is a Groth$(\mathfrak{A}^{\rm inj})$–seminorm of X then there exists a continuous seminorm q of X such that $p \leq q$ and $\widetilde{Q}_{pq} \in \mathfrak{A}^{\rm inj}(\widetilde{X}_q, \widetilde{X}_p)$. As a result, $\widetilde{Q}_p = \widetilde{Q}_{pq}\widetilde{Q}_q \in \mathfrak{A}^{\rm inj}(X, \widetilde{X}_p)$ and thus p is $\mathfrak{A}$–continuous.

We leave the bornological case to the readers, and comment that the assumption on infracompleteness is merely to give us a chance to utilize the extension condition. □

Remark Let $\mathfrak{A}$ be an operator ideal on Banach spaces and $\mathfrak{A}_0$ be an extension of $\mathfrak{A}$ to LCS's. It is plain that if $\mathfrak{A}_0 \subset \mathfrak{A}^{\rm rup}$ then $\mathfrak{A}_0$–topology $=$ $\mathfrak{A}^{\rm rup}$–topology; and if $\mathfrak{A}_0 \subset \mathfrak{A}^{\rm lup}$ then $\mathfrak{A}_0$–bornology $=$ $\mathfrak{A}^{\rm lup}$–bornology at least on infracomplete LCS's. For instance, $\mathfrak{N} = \mathfrak{N}_{\mathbb{B}}^{\rm inf}$ [20, p.144], where $\mathfrak{N}_{\mathbb{B}}$ is the quasi–injective ideal of nuclear operators between *Banach spaces*. Consequently, $\mathfrak{N}$–topology $=$ Groth$(\mathfrak{N}_{\mathbb{B}})$–topology on every

LCS. This explains why the constructions of Grothendieck and Randkte match in the case of nuclear spaces. The discussion is similar for Schwartz and infra–Schwartz spaces and their “co–spaces”.

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