SMOOTHLY EMBEDDED SUBSPACES OF A BANACH SPACE

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Abstract. We say that a Banach space Y embeds into a Banach space X smoothly if there is a linear isometry T from Y into X such that every subspace of TY is Hahn-Banach smooth in X (i.e., the ones with unique extension property). In this note, we show that they are exactly (isometric copies of) those subspaces of X having the half-space property.

1. Introduction

Recall that in a (real) Banach space X the (Gateaux) directional derivative of the norm is defined to be

$$G(y,z) = \lim_{t\to 0} \frac{\|y+tz\| - \|y\|}{t}, \quad \forall y, z \neq 0.$$

It is known in Banach's book [2] (see also [7, Section 5.4]) that $G(y,z)$ exists for all nonzero direction z in X if and only if y is a point of smoothness, i.e., there is a unique norm one linear functional f in the Banach dual space X^* of X such that $f(y) = \|y\|$. In fact, $f(z) = G(y,z)$ for all nonzero z in X in this case. We call X a smooth Banach space if every point in the unit sphere of X is a point of smoothness (see, e.g., [7]). Subspaces of a smooth space are obviously smooth.

A subspace Y of X is said to be Hahn-Banach smooth [6], or to have property U [10], if every norm one linear functional of Y has a unique norm one extension to X. In particular, every Banach space is Hahn-Banach smooth in itself. However, subspaces of a Hahn-Banach smooth subspace are not necessarily Hahn-Banach smooth. Moreover, a smooth subspace needs not be Hahn-Banach smooth, while a Hahn-Banach smooth subspace needs not be smooth either. Figure 1 below demonstrates two examples.

Definition 1. We say that a subspace Y of a Banach space X is a *smoothly embedded subspace* if every subspace of Y is Hahn-Banach smooth in X. We say that a Banach space Y *embeds into* X *smoothly* if Y is isometrically linear isomorphic to a smoothly embedded subspace of X; in this case, we will simply think Y is a smoothly embedded subspace of X and the embedding is the inclusion map.

Plainly, subspaces of a smoothly embedded subspace are again smoothly embedded. We also note that a smoothly embedded subspace is necessarily smooth itself. In fact,

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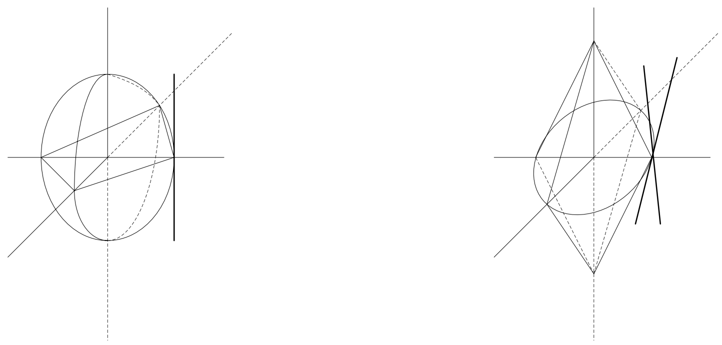

(a) $Y = \mathbb{R} \oplus_1 \mathbb{R}$
$X = (\mathbb{R} \oplus_1 \mathbb{R}) \oplus_2 \mathbb{R}$

(b) $Y = \mathbb{R} \oplus_2 \mathbb{R}$
$X = (\mathbb{R} \oplus_2 \mathbb{R}) \oplus_1 \mathbb{R}$

FIGURE 1. (a) Hahn-Banach smooth subspaces need not be smooth, and (b) smooth subspaces need not be Hahn-Banach smooth.

we know from [10] (see also [5, 12]) that a Banach space X embeds into itself smoothly if and only if X^* is strictly convex. Now, if Y embeds into X smoothly, then every subspace of Y is Hahn-Banach smooth in X, and thus also in Y. In other words, Y embeds into itself smoothly, or equivalently, Y^* is strictly convex. Consequently, Y is smooth (see, e.g., [7]).

We have already noted that Y embeds into X smoothly if and only if Y is Hahn-Banach smooth in X and Y^* is strictly convex. Note that when Y is reflexive, Y^* is strictly convex if and only if Y is smooth (see, e.g., [7]). So in a reflexive Banach space X, every smooth and Hahn-Banach smooth subspace embeds into X smoothly. In [13], however, a nonreflexive smooth Banach space X is given, whose dual space X^* is not strictly convex. Thus, there is a smooth Banach space X containing a Hahn-Banach smooth subspace, i.e., X itself, which does not embeds into X smoothly. In particular, smooth Banach spaces do not necessarily embed into itself smoothly.

There are a number of geometric conditions to describe smoothness of a Banach space X. See, e.g., [1, 4, 3, 6, 8, 9, 11]. One of them is the half-space property. A *nested sequence* $\{B_n = B(x_n, r_n)\}$ *of balls* in a Banach space X is a sequence of (open) balls centered at $x_n \in X$ and of radius $r_n \to \infty$ such that $B_n \subseteq B_{n+1}$ for all $n \geq 1$.

Definition 2. We say that a subspace Y of a Banach space X has the *half-space property in* X if for every nested sequence of balls $B(y_n, r_n)$ in X with all centers y_n from Y, the union $B = \bigcup_{n=1}^{\infty} B(y_n, r_n)$ is either the whole space X or an open half-space.

It is shown in [14] (see also [4, 8]) that a Banach space X has the half-space property in itself if and only if X^* is strictly convex, and thus if and only if X embeds into itself smoothly. We will show a local version in Theorem 3 that a subspace Y has the half-space property in X if and only if Y is smoothly embedded into X.

We hope our results be helpful in the study of the Banach space geometry and the approximation theory as those about Hahn-Banach smoothness and the half-space property demonstrated in, e.g., [10, 14, 15].

2. Results

Theorem 3. *Let Y be a subspace of a Banach space X. Then Y embeds into X smoothly if and only if Y has the half-space property in X.*

Proof. Suppose Y has the half-space property in X. Let Y_0 be a subspace of Y, and let f_0 be a norm one linear functional in Y_0^*. Let $y_n \in Y_0$ with $\|y_n\| = 1$ such that

$$1 - \frac{1}{2^{n+1}} < f_0(y_n) \leq 1,$$

and let

$$B_n = B(y_1 + \cdots + y_n, \frac{2n-1}{2}), \quad n = 1, 2, \ldots.$$

Then $\{B_n\}$ is a nested sequence of balls in X with centers from $Y_0 \subseteq Y$. By the half-space property of Y in X, there is a norm one linear functional g in X^* such that

$$\bigcup_{n=1}^{\infty} B_n = \{x \in X : g(x) > \alpha\}$$

for some real number α.

Let f be any norm one extension of f_0 in X^*. Notice that for any z in X with $\|z\| \leq 1$, we have

$$f(y_1 + \cdots + y_n + \frac{2n-1}{2} z) > n - (\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}}) - \frac{2n-1}{2} > 0.$$

Therefore, $f(B_n) \subseteq (0, +\infty)$ for all $n = 1, 2, \ldots$. In other words,

$$g(x) > \alpha \quad \Longrightarrow \quad f(x) > 0, \quad \forall x \in X.$$

Hence, $f = \lambda g$ for some real number λ. Indeed, $f = g$, and thus every norm one linear functional of Y_0 extends to a unique norm one linear functional of X. So every subspace Y_0 of Y is Hahn-Banach smooth in X.

Conversely, suppose Y embeds into X smoothly. In particular, Y^* is strictly convex. By [14] (see also [4, 8]), Y has the half-space property in itself. Let $\{B(y_n, r_n)\}$ be a nested sequence of balls in X with centers y_n from Y and radius $r_n \to +\infty$, whose union B is not the whole of X. Intersecting with Y, they give rise to a nested sequence of balls in Y. By translation we can assume that $0 \in B_1$. With the half-space property of Y in itself, we have a norm one linear functional f_0 in Y^* such that

$$B \cap Y = \bigcup_{n=1}^{\infty} B(y_n, r_n) \cap Y = \{y \in Y : f_0(y) > \beta\}$$

for some real number β. Since Y is Hahn-Banach smooth in X, there is a unique norm one extension f_1 of f_0 to X.

Observe that B is open and convex in X. By the separation theorem, there is a norm one linear functional f of X supporting B. In other words, there is a real scalar α such that

$$\sup\{f(b) : b \in B\} = \alpha.$$

If B were not a half space, there were an z' not in B such that $f(z') < \alpha$. By the separation theorem again, there is a norm one linear functional g of X such that

$$\sup\{g(x) : x \in B\} < g(z').$$

In particular, $g \neq f$. As $0 \in B_n$, we have $\|\frac{y_n}{r_n}\| < 1$ for all $n = 1, 2, \ldots$. Now

$$f(y_n) + r_n = \sup\{f(x) : x \in B_n\} = \alpha.$$

This implies

$$f(\frac{y_n}{r_n}) + 1 = \frac{\alpha}{r_n} \to 0 \text{ as } n \to \infty.$$

Hence the restriction $f|_Y$ of f to Y has norm one. Similarly, $g|_Y$ also has norm one. This rules out the possibility that $f = -g$, as $B \cap Y$ is an half-space in Y and thus both f and g assume unbounded values there. Since both $f|_Y$ and $g|_Y$ support $B \cap Y$, we see that they are $\pm f_0$. Thus, both f and g are $\pm f_1$ by the uniqueness. This gives a contradiction. So Y has the half-space property in X. □

Example 4. Let $Y = l_1$ and $X = l_1 \oplus_2 \mathbb{R}$. In this case, Y is Hahn-Banach smooth in X but not smoothly embedded into X. We demonstrate that Y does not have the half-space property in X directly. Let $\{e_n\}$ be the canonical basis of ℓ_1 and $\mathbb{R} = \text{span}\{e_0\}$ with $\|e_0\| = 1$. Let $y_n = e_1 + e_2 + \cdots + e_n$, $r_n = n$, and $B_n = B(y_n, r_n)$. If $x \in B_n$, then for $n < m$ we have

$$\begin{aligned}\|x - (e_1 + \cdots + e_m)\| &\leq \|x - (e_1 + \cdots + e_n)\| + \|e_{n+1} + \cdots + e_m\| \\ &< n + (m - n) = m.\end{aligned}$$

Hence $x \in B_m$, and thus $\{B_n\}$ is a nested sequence of balls with centers $y_n \in Y$.

We claim that $B = \bigcup_{n=1}^{\infty} B_n$ is not a half-space. Suppose, on contrary, there existed an $f \in X^*$ such that $B = \{x \in X : f(x) > 0\}$, as 0 belongs to the closure of B. For every $\alpha \neq 0$, we have

$$\begin{aligned}\|\alpha e_0 - y_n\| &= \|\alpha e_0 - (e_1 + \cdots e_n)\| \\ &= \sqrt{\alpha^2 + n^2} > n, \quad \forall n = 1, 2, \ldots.\end{aligned}$$

It means $\alpha e_0 \notin B$, and thus $\alpha f(e_0) \leq 0, \forall \alpha \neq 0$. Consequently, we have $f(e_0) = 0$. Moreover,

$$\begin{aligned}\|2e_1 + e_0 - y_n\| &= \|2e_1 + e_0 - (e_1 + \cdots + e_n)\| \\ &= \|(e_1 - e_2 - e_3 - \cdots - e_n) + e_0\| \\ &= \sqrt{n^2 + 1} > n, \quad \forall n = 1, 2, \ldots.\end{aligned}$$

This implies $2e_1 + e_0 \notin B$, and thus $2f(e_1) + f(e_0) \leq 0$. Consequently, $f(e_1) \leq 0$. However, this conflicts with the fact that $e_1 \in B$ which ensures $f(e_1) > 0$.

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