Isometries between C*-algebras

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Abstract

Let A and B be C*-algebras and let T be a linear isometry from A *into* B. We show that there is a largest projection p in B^{**} such that $T(\cdot)p : A \longrightarrow B^{**}$ is a Jordan triple homomorphism and

$$T(ab^*c + cb^*a)p = T(a)T(b)^*T(c)p + T(c)T(b)^*T(a)p$$

for all a, b, c in A. When A is abelian, we have $\|T(a)p\| = \|a\|$ for all a in A. It follows that a (possibly non-surjective) linear isometry between any C*-algebras reduces *locally* to a Jordan triple isomorphism, by a projection.

1 Introduction

In his seminal paper [10], Kadison showed that a *surjective* linear isometry T between unital C*-algebras A and B is of the form $T(\cdot) = u\eta(\cdot)$ where u is a unitary element in B and η is a Jordan *-isomorphism. This result remains true in the non-unital case although the unitary element u generally comes from $B \oplus \mathbb{C}$ [13]. In both cases, T preserves the Jordan triple product:

$$T(ab^*c + cb^*a) = T(a)T(b)^*T(c) + T(c)T(b)^*T(a)$$

for all $a, b, c \in A$. In infinite-dimensional holomorphy, C*-algebras, and the larger class of JB*-triples, arise as tangent spaces to bounded symmetric domains and it has been shown in [11] that the geometry of these domains is completely determined by the Jordan triple structures of these spaces. Indeed, a bijective linear map T between two JB*-triples is an isometry if, and only if, it preserves the Jordan triple product:

$$T\{a, b, c\} = \{T(a), T(b), T(c)\}$$

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as shown in [11, Proposition 5.5] (see also [3, 4, 6, 16]). By polarization, T preserves the Jordan triple product if, and only if,

$$T\{a, a, a\} = \{T(a), T(a), T(a)\}.$$

The Jordan triple product in a C*-algebra is given by

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$$

and in particular, the above characterization of surjective linear isometries between JB*-triples extends Kadison's result as well as giving it a geometric perspective. It also highlights the importance of the Jordan triple product in the study of isometries of C*-algebras.

It is natural to ask to what extent the above triple-preserving property of a linear isometry persists if it is not surjective. We address this question in this paper. Let $T : A \longrightarrow B$ be a linear isometry, possibly non-surjective. We study T locally. Without surjectivity, the C^*-algebra and affine geometric techniques of [10, 4] can not be used directly to obtain conclusive results. Nevertheless, we show there is a largest projection $p \in B^{**}$, called the *structure projection* of T, such that $T(A)p$ is a Jordan subtriple of B^{**} and the map

$$T(\cdot)p : A \longrightarrow T(A)p$$

is a triple homomorphism with $T\{a, a, a\}p = \{T(a), T(a), T(a)\}p$ for all $a \in A$. The structure projection p is closed but the map $T(\cdot)p$ need not be injective. When A is abelian, we study the structure projection p in some detail, motivated by the question of the local behaviour of T, and show that the map $T(\cdot)p$ is isometric which also extends Holsztynski's result in [8] for non-surjective isometries between continuous function spaces (see also [9]). It follows that, for any A and B, the isometry T is reduced *locally* to a triple isomorphism by a projection in the sense that, for any $a \in A$, there is a closed projection $p_a \in B^{**}$ such that the map $T(\cdot)p_a$ is a triple isomorphism from the Jordan subtriple Z_a of A, generated by a, into B^{**} and

$$T\{x, y, z\}p_a = \{T(x), T(y), T(z)\}p_a$$

for all $x, y, z \in Z_a$. Although $T(A)p$ could be zero if A is nonabelian, we give conditions for $T(A)p$ to be non-zero in this case.

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2 Isometries of C*-algebras and their ranges

Throughout the paper, an isometry between Banach spaces is *not* assumed to be surjective. We first recall that a *JB*-triple* Z is a complex Banach space equipped with a Jordan triple product $\{\cdot,\cdot,\cdot\} : Z^3 \longrightarrow Z$ which is symmetric and linear in the outer variables, and conjugate linear in the middle variable such that for $a, b, c, x, y \in Z$, we have

(i) $\{a, b, \{c, x, y\}\} = \{\{a, b, c\}, x, y\} - \{c, \{b, a, x\}, y\} + \{c, x, \{a, b, y\}\}$;

(ii) the map $z \in Z \mapsto \{a, a, z\} \in Z$ is hermitian with nonnegative spectrum;

(iii) $\|\{a, a, a\}\| = \|a\|^3$.

A closed subspace of a JB*-triple is called a *subtriple* if it is closed with respect to the triple product. A linear map $T : Z \longrightarrow W$ between JB*-triples is called a *triple homomorphism* if it preserves the triple product in which case, the range $T(Z)$ is a subtriple of W and the kernel J of T is a *triple ideal* of Z, that is, $\{Z, Z, J\} + \{Z, J, Z\} \subset J$. We refer to [2, 17, 18, 20] for expositions as well as recent surveys of JB*-triples and symmetric Banach manifolds. In the sequel, we write $a^{(3)} = \{a, a, a\}$. We note that a norm-closed subspace Z of a C*-algebra is a JB*-triple if $a \in Z$ implies $aa^*a \in Z$, in which case Z is called a *JC*-triple* and the triple product is given by triple polarization

$$\begin{aligned} 2\{a, b, c\} &= ab^*c + cb^*a \\ &= \frac{1}{8} \sum_{\alpha^4=\beta^2=1} \alpha\beta(a + \alpha b + \beta c)(a + \alpha b + \beta c)^*(a + \alpha b + \beta c). \end{aligned}$$

In C*-algebras, the closed triple ideals are the closed algebra two-sided ideals [7, p.350].

We begin with a simple example of a linear isometry $T : A \longrightarrow B$ between abelian C*-algebras which is not a triple homomorphism.

Example 2.1. Let $C(\Omega)$ and $C(\Omega \cup \{\beta\})$ be the C*-algebras of continuous functions on the closed unit disc $\Omega \subset \mathbb{C}$ and $\Omega \cup \{\beta\}$ respectively, where $\beta \in \mathbb{C}\backslash\Omega$. Define $T : C(\Omega) \longrightarrow C(\Omega \cup \{\beta\})$ by

$$(Tf)(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ \frac{1}{2}(f(1) + f(0)) & \text{if } x = \beta. \end{cases}$$

Then T is a linear isometry and $T(C(\Omega)) = \{h \in C(\Omega \cup \{\beta\}) : 2h(\beta) = h(1) + h(0)\}$ which is not a subtriple of $C(\Omega \cup \{\beta\})$. So T is not a triple isomorphism onto its range. Nevertheless, we have $T(f^{(3)}) = T(f)^{(3)}$ if $f(1) = f(0) = 0$.

Let $T : A \longrightarrow B$ be a linear isometry between C*-algebras. Although the range $T(A)$ need not be a subtriple of B, we show in Proposition 2.2 below that $T(A)$, cut down by a projection, is always a subtriple of B^{**}. This result will be used to study T locally later. In Example 2.1, such a projection is given by the characteristic function of Ω in $C(\Omega \cup \{\beta\})$.

We need some notation first. We denote by T^{**} the second dual map of T and for convenience, we often write Ta for $T(a)$. The identity of a unital C*-algebra will be denoted by $\mathbf{1}$. Given a C*-algebra A, we denote its closed unit ball by A_1, and by A_1^* the closed unit ball of the dual A^*. Let $Q(A) = \{\varphi \in A_1^* : \varphi \geq 0\}$ be the quasi-state space which is weak* compact and convex. Every weak* closed face of $Q(A)$ containing zero is of the form $F(p) = \{\varphi \in Q(A) : \varphi(\mathbf{1} - p) = 0\}$ for some closed projection $p \in A^{**}$, called the *support projection* of the face (cf. [5, 15] or [14, 3.11.10]). The polar decomposition of a functional $\psi \in A^*$ is denoted by $\psi(\cdot) = v^*|\psi|(\cdot) = |\psi|(v^*\cdot)$ where v^* is a partial isometry in A^{**}.

For each φ in $Q(A)$, we let $(\pi_\varphi, H_\varphi, \omega_\varphi)$ be the Gelfand-Naimark-Segal representation of A induced by φ. As usual, we also denote by π_φ the extended representation of A^{**} on the Hilbert space H_φ (see, for example, [14, p. 60]). For simplicity, we write $x\omega_\varphi$ for $\pi_\varphi(x)\omega_\varphi$ in H_φ whenever $x \in A^{**}$. Thus we have $x\omega_\varphi = 0$ if, and only if, $\varphi(x^*x) = 0$. Further, we have $\varphi(x^*x) = 0$ for all $\varphi \in F(p)$ if, and only if, $xp = 0$ (cf. [14, §3.10] and [1, Corollary 3.5]). We note that if φ is a pure state with support projection p, then $F(p) = [0,1]\varphi$.

Proposition 2.2. *Let A and B be C*-algebras and let $T : A \longrightarrow B$ be a linear isometry. Then there is a largest projection p in B^{**} such that*

(i) $T(\cdot)p : A \longrightarrow B^{**}$ is a triple homomorphism;

(ii) $T\{a,b,c\}p = \{Ta, Tb, Tc\}p$ for all a, b, c in A.

Further, p is a closed projection and $(Ta)^*(Tb)p = p(Ta)^*(Tb)$ for all a, b in A.

Proof. Let

$$\begin{aligned} F_1 &= \bigcap_{a \in A_1} \{\varphi \in Q(B) : (Ta^{(3)})\omega_\varphi = (Ta)^{(3)}\omega_\varphi\} \\ &= \bigcap_{a \in A_1} \left\{\varphi \in Q(B) : \varphi\left((Ta^{(3)} - (Ta)^{(3)})^*(Ta^{(3)} - (Ta)^{(3)})\right) = 0\right\}. \end{aligned}$$

Then F_1 is a weak* closed face of $Q(B)$ containing zero. For a in A_1, we define a weak* continuous affine map $\Phi_a : Q(B) \longrightarrow Q(B)$ by

$$\Phi_a(\varphi)(\cdot) = \varphi\left((Ta)^*(Ta) \cdot (Ta)^*(Ta)\right).$$

For $n = 1, 2, \ldots$, the sets

$$F_{n+1} = \{\varphi \in F_n : \Phi_a(\varphi) \in F_n, \forall a \in A_1\} = \bigcap_{a \in A_1} F_n \cap \Phi_a^{-1}(F_n)$$

form a decreasing sequence of weak* closed faces of $Q(B)$. The intersection $F = \bigcap_{n=1}^{\infty} F_n$ is a weak* closed face of $Q(B)$ containing zero. Let p be the closed projection in B^{**} supporting F:

$$F = F(p) = \{\varphi \in Q(B) : \varphi(\mathbf{1} - p) = 0\}.$$

For each a in A_1 and φ in F, we have

$$\Phi_a(\varphi)(\cdot) = \varphi\left((Ta)^*(Ta) \cdot (Ta)^*(Ta)\right) \in F,$$

and consequently,

$$\langle p(Ta)^*(Ta)\omega_\varphi, (Ta)^*(Ta)\omega_\varphi \rangle = \Phi_a(\varphi)(p) = \Phi_a(\varphi)(1) = \|(Ta)^*(Ta)\omega_\varphi\|^2.$$

Hence

$$p(Ta)^*(Ta)\omega_\varphi = (Ta)^*(Ta)\omega_\varphi, \quad \forall \varphi \in F = F(p)$$

and therefore

$$p(Ta)^*(Ta)p = (Ta)^*(Ta)p.$$

It follows that

$$p(Ta)^*(Ta) = (Ta)^*(Ta)p, \quad \forall a \in A.$$

By polarization, we have

$$p(Ta)^*(Tb) = (Ta)^*(Tb)p \tag{2.1}$$

for all $a, b \in A$. To verify (i), we note that

$$(Ta^{(3)})\omega_\varphi = (Ta)^{(3)}\omega_\varphi, \quad \forall \varphi \in F.$$

This gives

$$(Ta^{(3)})p = (Ta)^{(3)}p.$$

By triple polarization and (3.1), we get

$$T\{a, b, c\}p = \{Ta, Tb, Tc\}p = \{(Ta)p, (Tb)p, (Tc)p\}.$$

Finally, if q is a projection in B^{**} satisfying conditions (i) and (ii), then

$$F(q) = \{\varphi \in Q(B) : \varphi(\mathbf{1} - q) = 0\} \subseteq F_n, \quad n = 1, 2, \ldots$$

since $\Phi_a(F(q)) \subseteq F(q)$ for $a \in A_1$ and it is evident that $F(q) \subseteq F_1$. Therefore $F(q) \subseteq F(p)$ and $q \leq p$. The last assertion has been shown in (2.1). □

Remark 2.3. (a) Although the above result only requires T to be contractive, all subsequent applications of the result, including the next two remarks, requires T to be isometric.

(b) In the above proof, if T is surjective or $T(A)$ is a subtriple of B, then $F_1 = Q(B)$ and $p = \mathbf{1}$.

(c) For an arbitrary projection $p \in B^{**}$, conditions (i) and (ii) above are independent of each other in general and they need not imply (2.1). Consider, for instance, the identity map $T : A \longrightarrow A$, for which (ii) is satisfied by any projection, but only the central projections in A^{**} satisfy (i) and (2.1). Nevertheless, if $T^{**}(\mathbf{1})$ is unitary, then (i) implies (2.1) and hence (ii), for any projection $p \in B^{**}$. Indeed, if $T^{**}(\mathbf{1}) = \mathbf{1}$, then T commutes with involution and, by weak*-continuity of the triple product and (i), we have $T\{\mathbf{1}, \mathbf{1}, a\}p = \{\mathbf{1}p, \mathbf{1}p, T(a)p\}$ which gives $T(a)p = pT(a)p = pT(a)$ for $a = a^*$ and hence for all $a \in A$. For unitary $T^{**}(\mathbf{1})$, the map $T^{**}(\mathbf{1})^* T^{**}$ is unital and the preceding statement gives $pT(a)^*T(b) = p(T^{**}(\mathbf{1})^*T(a))^*(T^{**}(\mathbf{1})^*T(b)) = (T^{**}(\mathbf{1})^*T(a))^*(T^{**}(\mathbf{1})^*T(b))p = T(a)^*T(b)p$. If B is abelian, then of course (i) and (ii) are equivalent.

Definition 2.4. We denote by p_T the projection for the isometry T in Proposition 2.2 and call it the *structure projection* of T.

We give the following examples of structure projections p_T. Let M_n be the C*-algebra of $n \times n$ matrices.

Example 2.5. Let $T : M_2 \longrightarrow M_3$ be defined by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & a \end{pmatrix}.$$

Then T is a unital linear isometry and $T(M_2)$ is not a subtriple of M_3. The structure projection p_T is given by

$$p_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We note that Morita [12] has shown that a linear isometry $T : M_n \longrightarrow M_n$ is of the form $T(x) = uxv$ or $T(x) = ux^tv$ for some unitary $u, v \in M_n$ where x^t denotes the transpose of x.

Example 2.6. Let $A = C[0,1]$, $B = C([0,1] \cup \{2\})$ and define $T : A \longrightarrow B$ by

$$(Tf)(x) = \begin{cases} f(x) & \text{for } x \in [0,1] \\ \int_0^1 f(y)dy & \text{for } x = 2. \end{cases}$$

Then T is a unital linear isometry, $T(A) = \{h \in B : h(2) = \int_0^1 h(y)dy\}$ has codimension 1 in B and it is not a subtriple of B. We have $p_T = \chi_{[0,1]}$, the characteristic function of $[0,1]$, which is in B.

Example 2.7. Let $T : \mathbb{C} \longrightarrow M_2$ be defined by

$$T(a) = \begin{pmatrix} 0 & \frac{a}{2} \\ a & 0 \end{pmatrix}.$$

Then T is an isometry and $T(\mathbb{C})$ is not a subtriple of M_2. Also $T(1)$ is not unitary and $T(\mathbb{C})$ contains no nontrivial positive element. Its structure projection p_T is given by

$$p_T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which does not commute with $T(a)$ for $a \neq 0$. Also $T(a^{(3)}) \neq T(a)^{(3)}$ for all non-zero $a \in \mathbb{C}$.

Example 2.8. Let $K(H)$ be the C*-algebra of compact operators on a Hilbert space H with an orthonormal basis $\{e_1, e_2, \ldots\}$, and $B(H)$ the algebra of bounded operators on H. Define a linear isometry $T : c_0 \longrightarrow K(H)$ by

$$\begin{aligned} T(x) &= \frac{x_1}{2} e_1 \otimes e_1 + x_1 e_3 \otimes e_2 + \frac{x_2}{2} e_5 \otimes e_3 + x_2 e_7 \otimes e_4 + \cdots \\ &= \frac{1}{2}\sum_{n=1}^{\infty} x_n e_{4n-3} \otimes e_{2n-1} + \sum_{n=1}^{\infty} x_n e_{4n-1} \otimes e_{2n} \end{aligned}$$

where $x = (x_n) \in c_0$ and $(e_i \otimes e_k)(\cdot) = \langle \cdot, e_k \rangle e_i$. We have

$$x^{(3)} = (x_1^{(3)}, x_2^{(3)}, \ldots),$$

$$T(x^{(3)}) = \frac{1}{2}\sum_{n=1}^{\infty} x_n^{(3)} e_{4n-3} \otimes e_{2n-1} + \sum_{n=1}^{\infty} x_n^{(3)} e_{4n-1} \otimes e_{2n},$$

and

$$T(x)^{(3)} = \frac{1}{8}\sum_{n=1}^{\infty} x_n^{(3)} e_{4n-3} \otimes e_{2n-1} + \sum_{n=1}^{\infty} x_n^{(3)} e_{4n-1} \otimes e_{2n}$$

by orthogonality. Hence, for any projection q in $K(H)^{**} = B(H)$,

$$T(x^{(3)})q = T(x)^{(3)}q$$

if, and only if,

$$\Big(\sum_{n=1}^{\infty} x_n^{(3)} e_{4n-3} \otimes e_{2n-1}\Big)q = 0.$$

This happens for all x in c_0 exactly when $qe_{2n-1} = 0$ for $n = 1, 2, \ldots$. Therefore the structure projection p_T is the orthogonal projection onto $\operatorname{span}\{e_2, e_4, \ldots\}$ and we have

$$\|T(x)p_T\| = \|x\| \quad \text{and} \quad p_T(Tx) = 0$$

for all x in c_0.

Remark 2.9. Let $T : A \longrightarrow B$ be a linear isometry between C*-algebras. Let B be a C*-subalgebra of $\widetilde{B}$, with common approximate identity, and regard B^{**} as a subalgebra of $\widetilde{B}^{**}$. Then the structure projection $\widetilde{p_T}$ of the isometry $T : A \longrightarrow \widetilde{B}$ is the same as p_T. Evidently, we have $p_T \leq \widetilde{p_T}$. Suppose $p_T \neq \widetilde{p_T}$. Choose a state $\psi \in \widetilde{B}^*$ such that $\psi(p_T) < \psi(\widetilde{p_T})$. Then the state

$$\varphi(\cdot) = \frac{\psi(\widetilde{p_T} \cdot \widetilde{p_T})}{\psi(\widetilde{p_T})}$$

is in the closed face $F(\widetilde{p_T})$ of $Q(\widetilde{B})$ supported by $\widetilde{p_T}$. This means, by the proof of Proposition 2.2, that

$$\Phi_b^n(\varphi)((Ta^{(3)} - (Ta)^{(3)})^*((Ta^{(3)} - (Ta)^{(3)})) = 0 \quad (a, b \in A_1,\ n = 0, 1, 2, \ldots)$$

where $\Phi_b^0(\varphi) = \varphi$ and Φ_b^n is the nth iterate of Φ_b. The restriction $\varphi|_B$ is a state of B and clearly the above identity remains true when $\varphi|_B$ replaces φ, that is, $\varphi|_B \in F(p_T) \subseteq Q(B)$ which gives the contradiction

$$1 = \varphi(p_T) = \frac{\psi(\widetilde{p_T} p_T \widetilde{p_T})}{\psi(\widetilde{p_T})} = \frac{\psi(p_T)}{\psi(\widetilde{p_T})}.$$

So $p_T = \widetilde{p_T}$.

We note that, for a linear isometry $T : A \longrightarrow B$ between C*-algebras, the triple homomorphism $T(\cdot)p_T = 0$ if, and only if, $T^{**}(\mathbf{1})p_T = 0$. This follows from the weak* continuity of the triple product and the identity

$$T(a)p_T = T^{**}(a)p_T = T^{**}\{\mathbf{1}, \mathbf{1}, a\}p_T = \{T^{**}(\mathbf{1})p_T, T^{**}(\mathbf{1})p_T, T(a)p_T\}.$$

We study various necessary and sufficient conditions for $T(\cdot)p_T \neq 0$ in the next two sections. The above identity also shows that $T^{**}(\mathbf{1})p_T$ is a partial isometry in B^{**}.

3 Isometries from abelian C*-algebras

In this section, we study the structure projection of a linear isometry on an abelian C*-algebra. This is motivated by the intention to study a linear isometry locally, that is, to study its restriction on a subtriple generated by an element. We show in Theorem 3.10 below that when A is abelian, the structure projection p_T of an isometry T from A into any C*-algebra B is large enough to make the triple homomorphism $T(\cdot)p_T$ an isometry. Consequently, a linear isometry T on any C*-algebra reduces *locally* to a triple isomorphism via a projection, as shown in Corollary 3.12. We also give an alternative construction of p_T in Proposition 3.14 when the codomain B is a dual C*-algebra. We prove some lemmas first.

Definition 3.1. Let $T : A \longrightarrow B$ be a linear map between C*-algebras. For each φ in A^* with $\|\varphi\| = 1$, let

$$A_\varphi = \{a \in A : \varphi(a) = \|a\| = 1\}.$$

Similarly, for each ψ in B^* with $\|\psi\| = 1$, let

$$B_\psi = \{b \in B : \psi(b) = \|b\| = 1\}.$$

If $A_\varphi \neq \emptyset$, we define

$$Q_\varphi = \{\psi \in B^* : \|\psi\| = 1 \text{ and } T(A_\varphi) \subseteq B_\psi\}.$$

Lemma 3.2. *Let $T : A \longrightarrow B$ be a linear isometry between C*-algebras. For φ in A^* with $\|\varphi\| = 1$ and $A_\varphi \neq \emptyset$, the set Q_φ is a non-empty weak* closed face of B_1^*.*

Proof. We first note that Q_φ is an intersection of non-empty weak* closed faces of B_1^*:

$$Q_\varphi = \bigcap_{a \in A\varphi} \{\psi \in B_1^* : \psi(Ta) = 1\}.$$

We show these faces have finite intersection property. To this end, let $a_1, a_2, \ldots, a_n$ be in A_φ and let $a = \sum_{i=1}^n a_i$. Since $\varphi(a) = n$, we have $\|Ta\| = \|a\| = n$. Therefore, there is a norm one functional ψ in B^* such that $\psi(Ta) = n$. It follows that $\sum_{i=1}^n \psi(Ta_i) = n$ and so $\psi(Ta_i) = 1$ for $i = 1, 2, \ldots, n$. Consequently, we have $\psi \in \bigcap_{i=1}^n (Ta_i)^{-1}\{1\}$. □

Lemma 3.3. *Let $T : A \longrightarrow B$ be a linear isometry between C*-algebras, and let $\varphi \in A^*$ with $\|\varphi\| = 1$ and $A_\varphi \neq \emptyset$. Then for any $a \in A_\varphi$ and $\psi \in Q_\varphi \subseteq B_1^*$ with polar decomposition $\psi = v^*|\psi|$, we have*

(i) $\|(Ta)\omega_{|\psi|}\| = 1$;

(ii) $(Ta)\omega_{|\psi|} = v\omega_{|\psi|}$ *and* $(Ta)^* v\omega_{|\psi|} = \omega_{|\psi|}$ *in* $H_{|\psi|}$.

Proof. Given $a \in A_\varphi$ and $\psi \in Q_\varphi$, we have $Ta \in B_\psi$ and therefore,

$$\begin{aligned} 1 &= \psi(Ta) = |\psi|(v^*(Ta)) \\ &= \left\langle v^*(Ta)\omega_{|\psi|}, \omega_{|\psi|} \right\rangle = \left\langle (Ta)\omega_{|\psi|}, v\omega_{|\psi\|} \right\rangle = \left\langle \omega_{|\psi|}, (Ta)^* v\omega_{|\psi\|} \right\rangle . \end{aligned}$$

Since $\|v\omega_{|\psi|}\| = 1$ and $\|(Ta)\omega_{|\psi|}\| \leq \|Ta\| = 1$, we have $\|(Ta)\omega_{|\psi|}\| = 1$ and $(Ta)\omega_{|\psi|} = v\omega_{|\psi|}$. Similarly, we have $(Ta)^* v\omega_{|\psi|} = \omega_{|\psi|}$. □

In the *remaining lemmas of this section*, we assume that A is an *abelian* C*-algebra and is identified with the algebra $C_0(X)$ of continuous functions on a locally compact Hausdorff space X, vanishing at infinity. Fix a linear isometry $T : C_0(X) \longrightarrow B$, where B is any C*-algebra. We write

$$A_x = A_{\delta_x} = \{f \in C_0(X) : f(x) = \|f\| = 1\};$$

$$Q_x = Q_{\delta_x} = \{\psi \in B^* : \|\psi\| = 1 \text{ and } T(A_x) \subseteq B_\psi\}$$

where δ_x is the point mass at x. Note that $A_x \neq \emptyset$ for all x in X.

We let $Q = \bigcup_{x\in X} Q_x$ and define $|Q_x| = \{|\psi| : \psi \in Q_x\}$, $|Q| = \bigcup_{x\in X} |Q_x|$.

Lemma 3.4. *Given $x \neq x'$ in X, we have $|Q_x| \cap |Q_{x'}| = \emptyset$.*

Proof. We first show that $Q_x \cap Q_{x'} = \emptyset$. Suppose, otherwise, that there exists $\psi \in Q_x \cap Q_{x'}$. Then $TA_x \subseteq B_\psi$ and $TA_{x'} \subseteq B_\psi$. Let $f \in A_x$ and $f' \in A_{x'}$ with $ff' = 0$. Since T is an isometry and $\|f + f'\| = 1$, we have $\|Tf + Tf'\| = 1$. But $\psi(Tf) = \psi(Tf') = 1$ implies $\|Tf + Tf'\| \geq 1 + 1 = 2$ which is a contradiction.

Now suppose there exists $\psi \in |Q_x| \cap |Q_{x'}|$ with $\psi = |\varphi| = |\varphi'|$ and $\varphi \in Q_x$, $\varphi' \in Q_{x'}$. Let $\varphi = v^*|\varphi|$ and $\varphi' = v'^*|\varphi'|$ be the polar decompositions. By Lemma 3.3, given f in $C_0(X)$, we have

$$\begin{aligned} f \in A_x &\implies (Tf)\omega_\psi = v\omega_\psi; \\ f \in A_{x'} &\implies (Tf)\omega_\psi = v'\omega_\psi. \end{aligned}$$

We can choose an f in $A_x \cap A_{x'}$ which then gives $v\omega_\psi = v'\omega_\psi$. Consequently, for every a in A we have

$$\varphi(a) = \psi(v^*a) = \langle a\omega_\psi, v\omega_\psi \rangle_\psi = \langle a\omega_\psi, v'\omega_\psi \rangle_\psi = \psi(v'^*a) = \varphi'(a).$$

Hence $\varphi = \varphi' \in Q_x \cap Q_{x'}$ which is impossible. □

Definition 3.5. Define $\sigma : |Q| \longrightarrow X$ by

$$\sigma(|\psi|) = x \quad \text{for } \psi \in Q_x.$$

Let $P(B)$ be the set of all pure states of B. The following lemma shows that $|Q| \cap P(B) \neq \emptyset$.

Lemma 3.6. $\sigma(|Q| \cap P(B)) = X$.

Proof. Consider the isometry T from $A = C_0(X)$ onto $T(A)$. The adjoint map T^* sends the set $\partial T(A)_1^*$ of extreme points in the closed unit ball of $T(A)^*$ onto the extreme points of the closed unit ball of $C_0(X)^*$. In particular, for each x in X, there is a ψ in $\partial T(A)_1^*$ with $T^*\psi = \delta_x$. Let $\widetilde{\psi}$ be an extreme point in B_1^* extending ψ. Let $\widetilde{\psi} = v^*|\widetilde{\psi}|$ be the polar decomposition of $\widetilde{\psi}$. Then $\widetilde{\psi}(Tf) = T^*\psi(f) = f(x)$ for all f in $C_0(X)$ which implies that $\widetilde{\psi} \in Q_x$ and $|\widetilde{\psi}| \in |Q_x| \cap P(B)$. Hence $\sigma(|\widetilde{\psi}|) = x$. □

Let $q = \bigvee\{p_\varphi : \varphi \in |Q| \cap P(B)\}$ be the atomic projection in B^{**} supporting all pure states in $|Q|$ where p_φ is the minimal projection in B^{**} supporting the pure state φ. Note that q depends on T.

Lemma 3.7. *For all f in $C_0(X)$, we have $\|(Tf)q\| = \|Tf\|$.*

Proof. Let $\|f\| = |f(x)| > 0$ for some x in X. Then $\frac{f}{f(x)} \in A_x$ and $\frac{Tf}{f(x)} \in B_\psi$ for some $\psi \in Q_x$ with $|\psi| \in |Q| \cap P(B)$ by Lemma 3.6. It follows from Lemma 3.3 that $\|(Tf)\omega_{|\psi|}\| = \|f\| = \|Tf\|$. So $\|Tf\| \geq \|(Tf)q\| \geq \|(Tf)p_{|\psi|}\| \geq \|(Tf)\omega_{|\psi|}\| = \|Tf\|$. □

Lemma 3.8. *Let $\varphi = |\rho|$ for some ρ in Q with polar decomposition $\rho = v^*\varphi$. Let $f \in C_0(X)$. If $f(\sigma(\varphi)) = 0$, then $(Tf)\omega_\varphi = (Tf)^*v\omega_\varphi = 0$.*

Proof. Without loss of generality, we may assume that $\|f\| = 1$. By Urysohn's Lemma, it suffices to show that if f vanishes in a neighborhood of $\sigma(\varphi)$ in X, then $(Tf)\omega_\varphi = (Tf)^*v\omega_\varphi = 0$. For this, we choose g in $A_{\sigma(\varphi)}$ such that $fg = 0$. Then

$$\|g\| = 1 = g(\sigma(\varphi))$$

and

$$\|f + g\| = 1 = (f + g)(\sigma(\varphi)).$$

By Lemma 3.3, we have

$$(Tg)\omega_\varphi = v\omega_\varphi = T(f + g)\omega_\varphi$$

and

$$(Tg)^*v\omega_\varphi = \omega_\varphi = (T(f + g))^*v\omega_\varphi.$$

Consequently $(Tf)\omega_\varphi = (Tf)^*v\omega_\varphi = 0$. □

Lemma 3.9. *Let $\psi \in Q$ have polar decomposition $\psi = v^*\varphi$ where $\varphi = |\psi|$. Then for all f in $C_0(X)$, we have $(Tf)\omega_\varphi = f(\sigma(\varphi))v\omega_\varphi$ and $(Tf)^*v\omega_\varphi = \overline{f(\sigma(\varphi))}\omega_\varphi$.*

Proof. Recall that $\sigma(\varphi) = x$ if $\psi \in Q_x$. Pick $h \in C_0(X)$ such that $h(\sigma(\varphi)) = 1 = \|h\|$, that is, $h \in A_{\sigma(\varphi)}$. Since

$$(f - f(\sigma(\varphi))h)(\sigma(\varphi)) = 0,$$

Lemma 3.8 gives

$$T(f - f(\sigma(\varphi))h)\omega_\varphi = (T(f - f(\sigma(\varphi))h))^*v\omega_\varphi = 0.$$

Therefore

$$(Tf)\omega_\varphi = f(\sigma(\varphi))(Th)\omega_\varphi = f(\sigma(\varphi))v\omega_\varphi$$

since $(Th)\omega_\varphi = v\omega_\varphi$ by Lemma 3.3. Similarly, we have, by Lemma 3.3 again,

$$(Tf)^*v\omega_\varphi = \overline{f(\sigma(\varphi))}(Th)^*v\omega_\varphi = \overline{f(\sigma(\varphi))}\omega_\varphi.$$

□

We are now ready to prove that $T(\cdot)p_T$ is an isometry if A is abelian.

Theorem 3.10. *Let $T : A \longrightarrow B$ be a linear isometry between C^*-algebras and let A be abelian. Let $p_T \in B^{**}$ be the structure projection of T. Then we have*

$$\|(Ta)p_T\| = \|a\| \quad (a \in A).$$

Proof. Let $q \in B^{**}$ be the atomic projection, determined by T, in Lemma 3.7. We show that $T(\cdot)q$ is a triple homomorphism from $A = C_0(X)$ onto $T(A)q$. Let $\varphi \in |Q| \cap P(B)$ with $\varphi = |\psi|$ for some $\psi \in Q$. Let $\psi = v^*\varphi$ be the polar decomposition. By Lemma 3.9, we have

$$(Tf^{(3)})\omega_\varphi = f^{(3)}(\sigma(\varphi))v\omega_\varphi = f(\sigma(\varphi))\overline{f(\sigma(\varphi))}f(\sigma(\varphi))v\omega_\varphi = (Tf)^{(3)}\omega_\varphi.$$

Hence, by the definition of q, we have

$$(Tf^{(3)})q = (Tf)^{(3)}q$$

for every f in $C_0(X)$, and hence the map $T(\cdot)q$ is a triple homomorphism. On the other hand, using Lemma 3.9 again, we get

$$(Tg)^*(Tf)\omega_\varphi = \overline{g(\sigma(\varphi))}f(\sigma(\varphi))\omega_\varphi$$

which gives $q(Tg)^*(Tf)\omega_\varphi = (Tg)^*(Tf)\omega_\varphi$ since $q\omega_\varphi = \omega_\varphi$. Therefore $q(Tg)^*(Tf)q = (Tg)^*(Tf)q$ and q commutes with $(Tg)^*(Tf)$ for all f, g in $C_0(X)$. It follows that q satisfies condition (ii) in Proposition 2.2 and so $q \le p_T$ by maximality of p_T. By Lemma 3.7, $T(\cdot)q$ is an isometry which implies that $T(\cdot)p_T$ is such also. □

Remark 3.11. When B is abelian, Theorem 3.10 gives a result of Holsztynski [8, 9] as a special case.

Given any element a in a C*-algebra or, more generally, a JB*-triple A, the (closed) subtriple Z_a of A generated by a is linearly isometric (and hence triple isomorphic) to an abelian C*-algebra [11, Corollary 1.15]. Applying the above theorem to the restriction of a linear isometry to Z_a, we obtain the following local result on linear isometries between C*-algebras.

Corollary 3.12. *Let $T : A \longrightarrow B$ be a linear isometry, where A is a JB*-triple and B is a C*-algebra. Then for every $a \in A$, there is a largest projection $p_a \in B^{**}$, which is closed, such that $T(\cdot)p_a : Z_a \longrightarrow B^{**}$ is an isometry and a triple homomorphism satisfying*

$$T\{x, y, z\}p_a = \{Tx, Ty, Tz\}p_a$$

for all $x, y, z \in Z_a$.

Remark 3.13. (a) Clearly, $p_T \leq p_a$, but it can happen that $p_T \neq p_a = \mathbf{1}$. In Example 2.1, we have $p_T \neq \mathbf{1}$ and if $a \in C(\Omega)$ satisfies $a(0) = a(1) = 0$, then every $b \in Z_a$ also satisfies $b(0) = b(1) = 0$ since $\{f \in C(\Omega) : f(0) = f(1) = 0\}$ is a (closed) subtriple of $C(\Omega)$ containing a. Therefore T restricts to a triple isomorphism on Z_a, in other words, $p_a = \mathbf{1}$.

(b) The condition $T\{a, a, a\} = \{Ta, Ta, Ta\}$ alone need not imply that $p_a = \mathbf{1}$. This amounts to saying that the condition $T(a^{(3)}) = T(a)^{(3)}$ need not imply $T(a^{(2n+1)}) = (Ta)^{(2n+1)}$ for all n. Consider the unital isometry T in Example 2.6 and the function

$$f(x) = \frac{25}{4} - \frac{63}{4}x^2$$

in $C[0, 1]$. A simple calculation gives

$$(Tf)(2) = \int_0^1 f(x)dx = 1$$

and

$$T(f^{(3)})(2) = \int_0^1 f^{(3)}(x)dx = \int_0^1 \left(\frac{25}{4} - \frac{63}{4}x^2\right)^3 dx = 1.$$

Therefore, we have $T(f^{(3)}) = (Tf)^{(3)}$, but $T(f^{(5)}) \neq (Tf)^{(5)}$ since

$$T(f^{(5)})(2) = \int_0^1 f^{(5)}(x)dx = -\frac{20959168}{11264} \neq 1 = (Tf)^{(5)}(2).$$

In the proof of Theorem 3.10, the two maps $T(\cdot)q$ and $T(\cdot)p_T$ are actually equal if B is a dual C*-algebra. We show this in the next proposition as well as giving an exact formula relating q and p_T.

A C*-algebra B is called a *dual C*-algebra* if $I^{\perp\perp} = I$ for all closed one-sided ideals I of B, where for any closed left (resp. right) ideal I (resp. J) of B, we define $I^{\perp} = \{b \in B : Ib = \{0\}\}$ (resp. $J^{\perp} = \{b \in B : bJ = \{0\}\}$). It is known that a C*-algebra B is dual if and only if every maximal abelian subalgebra of B is generated by minimal projections, or equivalently, B is a c_0-sum of algebras of compact operators on Hilbert spaces (cf. [19, p.157]). Therefore, a unital dual C*-algebra is finite-dimensional. Given a dual C*-algebra B, the minimal projections in B are also minimal in B^{**}, and every singular state of B^{**} vanishes on B.

Given b in B^{**}, we denote by $r(b)$ the right support projection of b which is the smallest projection in B^{**} satisfying $br(b) = b$. If T is a linear isometry from a C*-algebra A into B, then for the partial isometry $T^{**}(\mathbf{1})p_T$, we have $r(T^{**}(\mathbf{1})p_T) = p_T T^{**}(\mathbf{1})^* T^{**}(\mathbf{1})p_T$.

Proposition 3.14. *Let p_T be the structure projection of $T : A \longrightarrow B$ in Theorem 3.10 and q the projection in its proof. Let B be a dual C*-algebra. Then we have*

(i) $T(\cdot)p_T = T(\cdot)q$;

(ii) q *is the right support projection of* $T^{**}(\mathbf{1})p_T$;

(iii) $p_T = q + \mathbf{1} - r(TA)$ *where* $r(TA) = \bigvee\{r(T(a)) : a \in A\}$.

Proof. (i) We note that $q \leq p_T$ from the proof of Theorem 3.10. Let $z = p_T - q$. We show that $T(\cdot)z = 0$. Suppose otherwise. Then $T(\cdot)z : A \longrightarrow T(A)z$ is a non-zero triple homomorphism as $T(a^{(3)})z = T(a^{(3)})p_T z = (Ta)^{(3)} p_T z = (Ta)^{(3)} z$, and z commutes with $T(a)^*T(a)$ because p_T and q do. Hence the quotient $A/\ker T(\cdot)z$ is isometrically triple isomorphic to $T(A)z$. If we identify A with $C_0(X)$, then $A/\ker T(\cdot)z$ identifies with $C_0(Y)$, where Y is a nonempty closed subset of X and the quotient map is just the restriction map. Pick $y \in Y$. Applying Lemma 3.2 to the isometry $C_0(Y) \longrightarrow T(A)z \subseteq B^{**}$, we find an extreme point ψ in $(B^{**})^*_1$ such that $\psi((Tf)z) = 1$ whenever $f \in C_0(X)$ satisfies $f(y) = \|f\| = 1$. Let $\psi = v^*|\psi|$ be the polar decomposition with $v \in B^{****}$. Then $|\psi|$ is a pure state of B^{**} and $|\psi|(z) = 1$ by Schwarz inequality. Hence

$$|\psi|(q) = |\psi|(qz) = 0.$$

We note that $|\psi|((Tf)^*Tf) = 1$ since $1 = |\psi|(v^*(Tf)z) = |\psi|(v^*Tf) \leq |\psi|((Tf)^*Tf) \leq 1$. It follows that $|\psi|$ is a pure normal state of B^{**} as it does not vanish on B and a pure state is normal or singular. Therefore ψ is normal on B^{**} since $B^* = B^{***}z_0$ for some central projection z_0 in B^{****} (cf. [19, p. 126]) and we have $\psi z_0 = v^*|\psi|z_0 = v^*|\psi| = \psi$. Therefore $|\psi| \in |Q_y| \cap P(B)$ because $\psi((Tf)(\mathbf{1}-z)) = |\psi|(v^*(Tf)(\mathbf{1}-z)) = 0$ yields $\psi(Tf) = \psi((Tf)z) = 1$ for $f \in A_y$. It follows that $|\psi|(q) = 1$, by the definition of q, which gives a contradiction.

(ii) By weak* continuity and Lemma 3.9, we have

$$T^{**}(\mathbf{1})^*T^{**}(\mathbf{1})\omega_\varphi = \omega_\varphi, \quad \forall \varphi \in |Q|.$$

Therefore

$$T^{**}(\mathbf{1})^*T^{**}(\mathbf{1})q = q$$

and

$$p_T T^{**}(\mathbf{1})^*T^{**}(\mathbf{1})p_T = (T^{**}(\mathbf{1})p_T)^*(T^{**}(\mathbf{1})p_T) = (T^{**}(\mathbf{1})q)^*(T^{**}(\mathbf{1})q) = q.$$

(iii) Since $T(A)z = 0$, we have

$$p_T - q = z \le \mathbf{1} - r(TA).$$

On the other hand, since $T(\cdot)(\mathbf{1} - r(TA)) = 0$, we have

$$\mathbf{1} - r(TA) \le p_T \quad \text{and} \quad q(\mathbf{1} - r(TA)) = 0$$

which gives

$$p_T = q + \mathbf{1} - r(TA).$$

□

The use of dual C*-algebras in Proposition 3.14 hints at the atomic property of B^{**} and a general formulation of the result, without any assumption on B, should relate the atomic part of p_T to q, as the following example shows.

Example 3.15. Let $A = C_0(0,1]$ and $T : A \longrightarrow C[-1,1]$ be the natural embedding, namely, Tf agrees with f on $(0,1]$ and is zero elsewhere. Then we have $p_T = \mathbf{1}$, $r(TA) = \bigvee_{f\in A} T(f) = \chi_{(0,1]} \in C[-1,1]^{**}$ and $q = z_{\text{at}}\chi_{(0,1]}$ is in the atomic part of $C[-1,1]^{**}$, where z_{at} is the maximal atomic projection in $C[-1,1]^{**}$. We see, in this case, $T(\cdot)p_T z_{\text{at}} = T(\cdot)q$ and $p_T z_{\text{at}} = q + (\mathbf{1} - r(TA))z_{\text{at}}$.

4 Isometries into abelian C*-algebras

Every C*-algebra can be embedded into an abelian C*-algebra by a linear isometry. It is therefore natural to consider isometries into abelian C*-algebras. We begin with a description of the structure projection.

Proposition 4.1. *Let $T : A \longrightarrow B$ be a linear isometry between C*-algebras and let B be abelian. Then $p_T = \bigwedge_{a\in A} p_a$ where p_a is the projection in Corollary 3.12.*

Proof. Let $p = \bigwedge_{a\in A} p_a$. We only need to prove $p_T \geq p$. For every $a \in A$, we have

$$T\{a,a,a\}p = T\{a,a,a\}p_a p = \{Ta,Ta,Ta\}p_a p = \{Ta,Ta,Ta\}p.$$

Since B is abelian, $T(\cdot)p : A \longrightarrow B^{**}$ is a triple homomorphism. Hence $p_T \geq p$ by the maximality of p_T in Proposition 2.2. □

By a *character* ρ of a C*-algebra A, we mean an algebra homomorphism $\rho : A \longrightarrow \mathbb{C}\backslash\{0\}$. It is clear that the algebra M_2 does not have a character. Also, a C*-algebra is abelian if, and only if, its pure states are all characters.

Lemma 4.2. *Let N be a von Neumann algebra. Then N has a weak* continuous character if, and only if, N contains an abelian summand.*

Proof. The sufficiency is obvious. Suppose N has a weak* continuous character ρ. Then N must contain a type I summand N_I for otherwise, the 'Halving Lemma' implies that N is of the form $D\otimes M_2$ (cf. [19, Proposition V.1.22]) and the restriction of ρ to $\mathbf{1}\otimes M_2$ is a character which is impossible. Since N_I is of the form $\sum_k N_k \otimes B(H_{n_k})$ where N_k is abelian and $B(H_{n_k})$ is a type I_{n_k}-factor, N_I must contain an abelian summand because the contrary would imply $\rho|_{N_I} = 0$ and $\rho = 0$. □

The above lemma implies that a C*-algebra A has a character if, and only if, A^{**} contains an abelian summand. We show below that this condition is equivalent to the non-triviality of the map $T(\cdot)p_T$ if T is a linear isometry from A into an abelian C*-algebra B.

Proposition 4.3. *Let $T : A \longrightarrow B$ be a linear isometry between C*-algebras where B is abelian. Let $p_T \in B^{**}$ be the structure projection of T. Then*

(i) $T(\cdot)p_T$ is an isometry if, and only if, A is abelian.

(ii) $T(\cdot)p_T \neq 0$ if, and only if, A admits a character.

Proof. (i) The necessity is obvious since $T(A)p_T$ is an abelian JB*-triple. The sufficiency follows from Theorem 3.10.

For (ii), we first assume that $T(\cdot)p_T \neq 0$. Then there exists a character ρ of B^{**} which does not vanish on $T(A)p_T$, and hence the composite $\rho\circ(T(\cdot)p_T) : A \longrightarrow \mathbb{C}$ is a non-zero triple homomorphism. Since the closed triple ideals of C*-algebras are algebra ideals, it follows that $A/\ker\rho\circ(T(\cdot)p_T)$ is a one-dimensional C*-algebra and the natural quotient map $\tilde{\rho}: A \longrightarrow A/\ker\rho\circ(T(\cdot)p_T)$ is a character of A.

Conversely, let η be a character of A and let $B = C_0(Y)$ for some locally compact Hausdorff space Y. Then η is a pure state of A. Since the extreme points in the closed unit ball of $T(A)^*$ can be extended to the extreme points in the closed unit ball of $C_0(Y)^*$, we have $\eta = T^*(\lambda\delta_y|_{T(A)})$ for some y in Y and $|\lambda| = 1$ where $T^*: T(A)^* \longrightarrow A^*$ is an isometry. The support projection $p_{\delta_y} \in C_0(Y)^{**}$ of δ_y is a minimal projection and we have $\lambda T(a^{(3)})p_{\delta_y} = \lambda T(a^{(3)})(y)p_{\delta_y} = \eta(a^{(3)})p_{\delta_y} = \eta(a)^{(3)}p_{\delta_y} = \lambda T(a)^{(3)}p_{\delta_y}$ for all a in A. Therefore $p_{\delta_y} \leq p_T$ by maximality of p_T, and thus $T(\cdot)p_T \neq 0$. □

Remark 4.4. Let A, B and T be as in Proposition 4.3. If A has a character, then we actually have

$$\|T(a)p_T\| = \sup\{|\eta(a)| : \eta \text{ is a character of } A\},$$

which gives an alternative proof of the sufficiency in (i). The identity follows from

$$\begin{aligned}\|T(a)p_T\| &= \sup\{|\rho(T(a)p_T)| : \rho \text{ is a character of } B^{**}\}\\ &= \sup\{|\tilde{\rho}(a)| : \ \rho \text{ is a character of } B^{**}\}\\ &\leq \sup\{|\eta(a)| : \eta \text{ is a character of } A\},\end{aligned}$$

where $\tilde{\rho}$ is the quotient map $A \longrightarrow A/\ker\rho\circ(T(\cdot)p_T)$ and the last term is at most $\|T(a)p_T\|$ from the proof of (ii).

The result of Proposition 4.3 does not hold if B is nonabelian. In Example 2.5, we have $T(\cdot)p_T \neq 0$ for some linear isometry $T: M_2 \longrightarrow M_3$. We conclude with the following example.

Example 4.5. There is a linear isometry $T: M_2 \longrightarrow B(H)$, where $B(H)$ is the algebra of bounded operators on an infinite dimensional separable Hilbert space H, such that $T(\cdot)p_T = 0$.

To see this, let Y be the closed unit ball of M_2^* and j be the canonical linear embedding of M_2 into $C(Y)$. Take a faithful nondegenerate representation π of $C(Y)$ on a separable Hilbert space H. Then $T = \pi\circ j$ is a linear isometry from M_2 into $B(H)$. By Remark 2.9 and Proposition 4.3, we have $T(\cdot)p_T = T(\cdot)p_j = 0$.

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