ATTRACTIVE POINT AND WEAK CONVERGENCE THEOREMS FOR NEW GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper we introduce a broad class of nonlinear mappings which contains the class of contractive mappings and the class of generalized hybrid mappings in a Hilbert space. Then we prove an attractive point theorem for such mappings in a Hilbert space. Furthermore, we prove a mean convergence theorem of Baillon's type without convexity in a Hilbert space. Finally, we prove a weak convergence theorem of Mann's type [12] without closedness. These results generalize attractive point, mean convergence and weak convergence theorems proved by Takahashi and Takeuchi [18], and Kocourek, Takahashi and Yao [8].

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. Then we denote by $F(T)$ the set of fixed points of T and by $A(T)$ the set of attractive points [18] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

We know from [18] that $A(T)$ is closed and convex; see Lemma 2.3 in Section 2. This property is important. A mapping T of C into H is said to be contractive if there exists a real number α with $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|$$

for all $x, y \in C$. From Banach [2] it is known that any contractive mapping of a closed subset C of H into itself has a unique fixed point. Let C be a nonempty subset of H. A mapping T of C into H is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. From Baillon [1] we know the following mean convergence theorem in a Hilbert space.

Theorem 1.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into C with a fixed point. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

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is weakly convergent to a fixed point of T.

Kohsaka and Takahashi [10], and Takahashi [17] introduced the following nonlinear mappings. A mapping $T : C \to H$ is called nonspreading [10] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is called hybrid [17] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$; see also Iemoto and Takahashi [5] and Kohsaka and Takahashi [9]. Kocourek, Takahashi and Yao [8] introduced a wide class of nonlinear mappings which contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. A mapping $T : C \to H$ is called generalized hybrid [8] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\ \beta)$-generalized hybrid mapping. We know that (1,0), (2,1) and $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mappings are nonexpansive, nonspreading and hybrid mappings, respectively. Kocourek, Takahashi and Yao [8] proved a mean convergence theorem which generalizes the Baillon's theorem (Theorem 1.1); see also Takahashi and Yao [20]. Recently, Takahashi and Takeuchi [18] proved the Kocourek, Takahashi and Yao's mean convergence theorem without convexity.

In this paper, motivated by Kocourek, Takahashi and Yao [8], and Takahashi and Takeuchi [18], we introduce a broad class of nonlinear mappings of C into H which contains the class of contractive mappings and the class of generalized hybrid mappings. Then we prove an attractive point theorem for such mappings in a Hilbert space. Furthermore, we prove a mean convergence theorem of Baillon's type without convexity in a Hilbert space. Finally, we prove a weak convergence theorem of Mann's type [12] without closedness. These results generalize attractive point, mean convergence and weak convergence theorems proved by Takahashi and Takeuchi [18], and Kocourek, Takahashi and Yao [8].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot,\ \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [16]. Furthermore, in a Hilbert space, we have that

$$2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.2}$$

for all $x, y, z, w \in H$. Let C be a nonempty subset of H and let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T. A mapping T from C into H with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|Tx - u\| \leq \|x - u\|$ for any $x \in C$ and $u \in F(T)$. It is well-known that if $T : C \to H$ is quasi-nonexpansive and C is closed and convex, then $F(T)$ is closed and convex; see Ito and Takahashi [6]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

we have that z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1-\alpha)y$. Then we have from (2.1) that

$$\begin{aligned}
\|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\
&= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&\leq \alpha\|x - z\|^2 + (1-\alpha)\|y - z\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha(1-\alpha)^2\|x - y\|^2 + (1-\alpha)\alpha^2\|x - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha(1-\alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 \\
&= 0.
\end{aligned}$$

This implies $Tz = z$. Thus $F(T)$ is convex. Let D be a nonempty closed convex subset of H and $x \in H$. We know that there exists a unique nearest point $z \in D$ such that $\|x - z\| = \inf_{y \in D} \|x - y\|$. We denote such a correspondence by $z = P_D x$. The mapping P_D is called the metric projection of H onto D. It is known that P_D is nonexpansive and

$$\langle x - P_D x, P_D x - u \rangle \geq 0$$

for all $x \in H$ and $u \in D$; see [16] for more details. For proving main results in this paper, we also need the following lemma proved by Takahashi and Toyoda [19].

Lemma 2.1. *Let D be a nonempty closed convex subset of H. Let P be the metric projection from H onto D. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for all $u \in D$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [15] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we can obtain the following result; see [11], [13] and [15].

Lemma 2.2. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n \mid n \in \mathbb{N}\}$ such that*

$$\mu_n\langle x_n, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

The following result obtained by Takahashi and Takeuchi [18] is important in this paper.

Lemma 2.3. *Let H be a Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H. Then $A(T)$ is a closed and convex subset of H.*

We also have the following result.

Lemma 2.4. *Let H be a Hilbert space, let C be a nonempty subset of H and let T be a quasi-nonexpansive mapping from C into H. Then $A(T) \cap C = F(T)$.*

Proof. Let $z \in A(T) \cap C$. From $z \in A(T)$ we have that

$$\|Tx - z\| \leq \|x - z\|, \quad \forall x \in C.$$

From $z \in C$ we have that $\|Tz - z\| \leq \|z - z\| = 0$ and hence $z \in F(T)$. Conversely, let $z \in F(T)$. Since $T : C \to H$ is quasi-nonexpansive, we have that

$$\|Tx - z\| \leq \|x - z\|, \quad \forall x \in C.$$

This implies $z \in A(T)$. It is obvious that $z \in C$. Thus $z \in A(T) \cap C$. This completes the proof. □

3. Attractive point theorems

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is called normally generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(1) $\alpha + \beta + \gamma + \delta \geq 0$;
(2) $\alpha + \gamma > 0$, or $\alpha + \beta > 0$;
(3) $\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0, \quad \forall x, y \in C.$

Such a mapping T is called $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid. If $\alpha + \beta = -\gamma - \delta = 1$, then an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping is a generalized hybrid mapping in the sense of Kocourek, Takahashi and Yao [8]. A normally generalized hybrid mapping $T : C \to H$ with a fixed point is quasi-nonexpansive. In fact, if y is a fixed point of T in (3), then we have that

$$\alpha\|Tx - y\|^2 + \beta\|x - y\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

and hence

$$(\alpha + \gamma)\|Tx - y\|^2 \leq (-\beta - \delta)\|x - y\|^2. \tag{3.1}$$

Since $\alpha + \gamma \geq -\beta - \delta$ and $\alpha + \gamma > 0$, we have that

$$\|Tx - y\|^2 \leq \frac{-\beta - \delta}{\alpha + \gamma}\|x - y\|^2 \leq \|x - y\|^2.$$

This implies that T is quasi-nonexpansive. Similarly, we have the desired result in the case of $\alpha + \beta > 0$. We first prove an attractive fixed point theorem for normally generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a nonempty subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping from C into itself. Then T has an attractive point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. Additionally, if C is closed and convex, then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition* (1).

Proof. Suppose that T has an attractive point z. Then $\|Tx - z\| \leq \|x - z\|$ for all $x \in C$. Therefore $\{T^n z \mid n = 0, 1, \dots\}$ is bounded. Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \dots\}$ is bounded. Since T is an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping of C into itself, we have that

$$\alpha\|Tx - T^{n+1}z\|^2 + \beta\|x - T^{n+1}z\|^2 + \gamma\|Tx - T^n z\|^2 + \delta\|x - T^n z\|^2 \leq 0$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x \in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Thus we have that

$$(\alpha + \gamma)\mu_n\|Tx - T^n z\|^2 + (\beta + \delta)\mu_n\|x - T^n z\|^2 \leq 0.$$

From $\|Tx - T^n z\|^2 = \|Tx - x\|^2 + 2\langle Tx - x, x - T^n z\rangle + \|x - T^n z\|^2$, We also have that

$$\begin{aligned}(\alpha + \gamma)\mu_n\|Tx - x\|^2 &+ 2(\alpha + \gamma)\mu_n\langle Tx - x, x - T^n z\rangle \\ &+ (\alpha + \gamma + \beta + \delta)\mu_n\|x - T^n z\|^2 \leq 0.\end{aligned}$$

From (1) $\alpha + \gamma + \beta + \delta \geq 0$, we have that

$$(\alpha + \gamma)\|Tx - x\|^2 + 2(\alpha + \gamma)\mu_n\langle Tx - x, x - T^n z\rangle \leq 0. \tag{3.2}$$

Since there exists $p \in C$ from Lemma 2.2 such that

$$\mu_n\langle y, T^n z\rangle = \langle y, p\rangle$$

for all $y \in H$, we have from (3.2) that

$$(\alpha + \gamma)\|Tx - x\|^2 + 2(\alpha + \gamma)\langle Tx - x, x - p\rangle \leq 0. \tag{3.3}$$

From (3.3) and (2.2) we obtain that

$$\begin{aligned}&(\alpha + \gamma)\|Tx - x\|^2 \\ &\quad + (\alpha + \gamma)(\|Tx - p\|^2 - \|Tx - x\|^2 - \|x - p\|^2) \leq 0\end{aligned}$$

and hence

$$(\alpha + \gamma)(\|Tx - p\|^2 - \|x - p\|^2) \leq 0.$$

Since $\alpha + \gamma > 0$, we have that

$$\|Tx - p\|^2 \leq \|x - p\|^2$$

for all $x \in C$. This implies $p \in A(T)$. In the case of $\alpha + \beta > 0$, we can obtain the result by replacing the variables x and y. Additionally, if C is closed and convex, then we have from $\{T^n x\} \subset C$ that

$$p \in \overline{co}\{T^n x : n \in \mathbb{N}\} \subset C.$$

Since $p \in A(T)$ and $p \subset C$, we have that

$$\|Tp - p\| \leq \|p - p\| = 0$$

and hence $p \in F(T)$. Conversely, if $z \in F(T)$, then it is obvious that $\{T^n z\} = \{z\}$ is bounded.

Next suppose that $\alpha + \beta + \gamma + \delta > 0$. Let p_1 and p_2 be fixed points of T. Then we have that

$$\begin{aligned}\alpha\|Tp_1 - Tp_2\|^2 &+ \beta\|p_1 - Tp_2\|^2 + \gamma\|Tp_1 - p_2\|^2 + \delta\|p_1 - p_2\|^2 \\ &= (\alpha + \beta + \gamma + \delta)\|p_1 - p_2\|^2 \leq 0\end{aligned}$$

and hence $p_1 = p_2$. Therefore a fixed point of T is unique. This completes the proof. □

Remark 3.1. We can also prove Theorem 3.1 by using the following condition instead of the condition (2):

(2)′ $\beta + \delta < 0$, or $\gamma + \delta < 0$.

In the case of the condition $\beta + \delta < 0$, we obtain from (1) that

$$\beta + \delta \geq -\alpha - \gamma.$$

Thus we obtain the desired result by Theorem 3.1. Similary, for the case of $\gamma+\delta < 0$, we can obtain the result by using the case of $\alpha + \beta > 0$.

As a direct consequence of Theorem 3.1, we obtain the following theorem.

Theorem 3.2. *Let H be a Hilbert space, let C be a nonempty bounded subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping from C into itself. Then T has an attractive point. Additionally, if C is closed and convex, then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition* (1)*.*

Note that an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping T above with $\alpha = 1$, $\beta = \gamma = 0$ and $-1 < \delta < 0$ is a contractive mapping. Using Theorem 3.1, we can show an attractive point theorem for contractive mappings in a Hilbert space.

Theorem 3.3. *Let H be a Hilbert space, let C be a nonempty subset of H and let T be a contractive mapping from C into C, that is, there exists a real number α with $0 < \alpha < 1$ such that*

$$\|Tx - Ty\| \leq \alpha\|x - y\|$$

for all $x, y \in C$. Then T has an attractive point.

Proof. Let $x \in C$. We have that

$$\begin{aligned}\|T^n x - x\| &\leq \|T^n x - T^{n-1}x\| + \|T^{n-1}x - T^{n-2}x\| + \cdots + \|Tx - x\| \\ &\leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + 1)\|Tx - x\| \\ &\leq \frac{1}{1-\alpha}\|Tx - x\|.\end{aligned}$$

Then $\{T^n x \mid n = 0, 1, \ldots\}$ is bounded. By Theorem 3.1 T has an attractive point. □

Using Theorem 3.1, we can show the following attractive point theorem for generalized hybrid mappings in a Hilbert space.

Theorem 3.4 (Takahashi and Takeuchi [18])**.** *Let H be a Hilbert space, let C be a nonempty subset of H and let T be a generalized hybrid mapping from C into C, that is, there exist real numbers α and β such that*

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Then T has an attractive point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded.

Proof. An (α, β)-generalized hybrid mapping T is an $(\alpha, 1 - \alpha, -\beta, -(1 - \beta))$-normally generalized hybrid mapping such that $\alpha + (1 - \alpha) - \beta - (1 - \beta) = 0 \geq 0$ and $\alpha + (1 - \alpha) = 1 > 0$. Then we have the desired result from Theorem 3.1. □

4. Mean convergence theorems

In this section, using the technique developed by Takahashi [13], we prove a mean convergence theorem of Baillon's type without convexity for normally generalized hybrid mappings in a Hilbert space.

Theorem 4.1. *Let H be a Hilbert space, let C be a nonempty subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping from C into C which has an attractive point. Let P be the metric projection from H onto $A(T)$. Then for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to an attractive point p of T, where $p = \lim_{n\to\infty} PT^n x$.

Proof. Since $A(T)$ is nonempty, we have that $\{T^n x\}$ is bounded for all $x \in C$. Since

$$\|S_n x - y\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x - y\| \leq \|x - y\|$$

for all $n \in \mathbb{N} \cup \{0\}$ and $y \in A(T)$, we have that $\{S_n x \mid n = 0, 1, \dots\}$ is bounded. Then there exists a strictly increasing sequence $\{n_i\}$ and $p \in H$ such that $\{S_{n_i} x \mid i = 0, 1, \dots\}$ is weakly convergent to p. We first show that $p \in A(T)$. Indeed, since T is an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping of C into itself, we have that

$$\alpha\|Tz - T^{k+1}x\|^2 + \beta\|z - T^{k+1}x\|^2 + \gamma\|Tz - T^k x\|^2 + \delta\|z - T^k x\|^2 \leq 0$$

for all $k \in \mathbb{N} \cup \{0\}$ and $z \in C$. We also have that

$$\begin{aligned}\gamma\|Tz - T^k x\|^2 = (\alpha + \gamma)(&\|Tz - z\|^2 + \|z - T^k x\|^2 + 2\langle Tz - z, z - T^k x\rangle) \\ &- \alpha\|Tz - T^k x\|^2.\end{aligned}$$

Since $-\alpha - \beta - \gamma \leq \delta$ from (1), we obtain that

$$\begin{aligned}\alpha(\|Tz - T^{k+1}x\|^2 - \|Tz - T^k x\|^2) + \beta(\|z - T^{k+1}x\|^2 - \|z - T^k x\|^2) \\ + 2(\alpha + \gamma)\langle Tz - z, z - T^k x\rangle + (\alpha + \gamma)\|z - Tz\|^2 \leq 0.\end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \dots, n-1$ and dividing by n, we obtain that

$$\begin{aligned}\frac{\alpha}{n}(\|Tz - T^n x\|^2 - \|Tz - x\|^2) + \frac{\beta}{n}(\|z - T^n x\|^2 - \|z - x\|^2) \\ + 2(\alpha + \gamma)\langle Tz - z, z - S_n x\rangle + (\alpha + \gamma)\|z - Tz\|^2 \leq 0.\end{aligned}$$

Replacing n by n_i, we have that

$$\begin{aligned}\frac{\alpha}{n_i}(\|Tz - T^{n_i} x\|^2 - \|Tz - x\|^2) + \frac{\beta}{n_i}(\|z - T^{n_i} x\|^2 - \|z - x\|^2) \\ + 2(\alpha + \gamma)\langle Tz - z, z - S_{n_i} x\rangle + (\alpha + \gamma)\|z - Tz\|^2 \leq 0.\end{aligned}$$

Letting $i \to \infty$, we obtain that

$$2(\alpha + \gamma)\langle Tz - z, z - p\rangle + (\alpha + \gamma)\|z - Tz\|^2 \leq 0.$$

As in the proof of Theorem 3.1, we obtain that

$$(\alpha + \gamma)(\|Tz - p\|^2 - \|z - Tz\|^2 + \|z - p\|^2) + (\alpha + \gamma)\|z - Tz\|^2 \leq 0.$$

From $\alpha+\gamma>0$ we have that for all $z \in C$,

$$\|p - Tz\|^2 \leq \|p - z\|^2.$$

This implies $p \in A(T)$. Similarly, we can obtain the desired result for the case of $\alpha+\beta>0$.

Since $A(T)$ is nonempty, closed and convex from Lemma 2.3, the metric projection P from H onto $A(T)$ is well-defined. We also obtain that

$$\|T^{n+1}x - y\| \leq \|T^n x - y\|$$

for all $n \in \mathbb{N} \cup \{0\}$ and $y \in A(T)$. By Lemma 2.1, there exists $q \in A(T)$ such that $\{PT^n x \mid n = 0, 1, \ldots\}$ is convergent to q. To complete the proof, we show that $q = p$. Note that the metric projection P satisfies

$$\langle z - Pz, Pz - u\rangle \geq 0$$

for all $z \in H$ and $u \in A(T)$; see [15]. Therefore

$$\langle T^k x - PT^k x, PT^k x - y\rangle \geq 0$$

for all $k \in \mathbb{N} \cup \{0\}$ and $y \in A(T)$. Since P is the metric projection from H onto $A(T)$, we obtain that

$$\begin{aligned}\|T^n x - PT^n x\| &\leq \|T^n x - PT^{n-1}x\| \\ &\leq \|T^{n-1}x - PT^{n-1}x\|,\end{aligned}$$

that is, $\{\|T^n x - PT^n x\| \mid n = 0, 1, \ldots\}$ is non-increasing. Therefore we obtain

$$\begin{aligned}\langle T^k x - PT^k x, y - q\rangle &\leq \langle T^k x - PT^k x, PT^k x - q\rangle \\ &\leq \|T^k x - PT^k x\| \cdot \|PT^k x - q\| \\ &\leq \|x - Px\| \cdot \|PT^k x - q\|.\end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \ldots, n-1$ and dividing by n, we obtain

$$\left\langle S_n x - \frac{1}{n}\sum_{k=0}^{n-1} PT^k x, y - q \right\rangle \leq \frac{\|x - Px\|}{n} \sum_{k=0}^{n-1} \|PT^k x - q\|.$$

Since $\{S_{n_i}x \mid i = 0, 1, \ldots\}$ is weakly convergent to p and $\{PT^n x \mid n = 0, 1, \ldots\}$ is convergent to q, we obtain that

$$\langle p - q, y - q\rangle \leq 0.$$

Putting $y = p$, we obtain

$$\|p - q\|^2 \leq 0$$

and hence $q = p$. This completes the proof. □

As in the proof of Theorem 3.4, from Theorem 4.1 we can prove the following mean convergence theorem for generalized hybrid mappings in a Hilbert space.

Theorem 4.2 (Takahashi and Takeuchi [18]). *Let H be a Hilbert space, let C be a nonempty subset of H and let T be a generalized hybrid mapping from C into C, that is, there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Suppose $A(T) \neq \emptyset$ and let P be the metric projection from H onto $A(T)$. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to an attractive point p of T, where $p = \lim_{n\to\infty} PT^n x$.

5. Weak convergence theorems of Mann's type

In this section, we prove a weak convergence theorem of Mann's type [12] for normally generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following lemma.

Lemma 5.1. *Let H be a Hilbert space and let C be a nonempty subset of H. Let $T : C \to H$ be a normally generalized hybrid mapping. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in A(T)$.*

Proof. Since $T : C \to H$ is a normally generalized hybrid mapping, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that (1) $\alpha + \beta + \gamma + \delta \geq 0$, (2) $\alpha + \gamma > 0$, or $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0 \tag{5.1}$$

for all $x, y \in C$. Suppose $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Replacing x by x_n in (5.1), we have that

$$\alpha\|Tx_n - Ty\|^2 + \beta\|x_n - Ty\|^2 + \gamma\|Tx_n - y\|^2 + \delta\|x_n - y\|^2 \leq 0. \tag{5.2}$$

From this inequality, we have that

$$\begin{aligned} &\alpha(\|Tx_n - x_n\|^2 + \|x_n - Ty\|^2 + 2\langle Tx_n - x_n, x_n - Ty\rangle) + \beta\|x_n - Ty\|^2 \\ &\quad + \gamma(\|Tx_n - x_n\|^2 + \|x_n - y\|^2 + 2\langle Tx_n - x_n, x_n - y\rangle) + \delta\|x_n - y\|^2 \leq 0. \end{aligned}$$

We apply a Banach limit μ to both sides of this inequality. We have that

$$\begin{aligned} &\alpha\mu_n(\|Tx_n - x_n\|^2 + \|x_n - Ty\|^2 + 2\langle Tx_n - x_n, x_n - Ty\rangle) + \beta\mu_n\|x_n - Ty\|^2 \\ &\quad + \gamma\mu_n(\|Tx_n - x_n\|^2 + \|x_n - y\|^2 + 2\langle Tx_n - x_n, x_n - y\rangle) + \delta\mu_n\|x_n - y\|^2 \leq 0 \end{aligned}$$

and hence

$$\begin{aligned} &\alpha\mu_n\|x_n - Ty\|^2 + \beta\mu_n\|x_n - Ty\|^2 \\ &\qquad + \gamma\mu_n\|x_n - y\|^2 + \delta\mu_n\|x_n - y\|^2 \leq 0. \end{aligned}$$

Thus we have

$$(\alpha + \beta)\mu_n\|x_n - Ty\|^2 + (\gamma + \delta)\mu_n\|x_n - y\|^2 \leq 0.$$

From $\|x_n - Ty\|^2 = \|x_n - y\|^2 + \|y - Ty\|^2 + 2\langle x_n - y, y - Ty\rangle$, we also have

$$(\alpha + \beta)(\mu_n\|x_n - y\|^2 + \|y - Ty\|^2 + 2\mu_n\langle x_n - y, y - Ty\rangle) + (\gamma + \delta)\mu_n\|x_n - y\|^2 \leq 0.$$

From $\alpha + \beta + \gamma + \delta \geq 0$ we obtain that

$$(\alpha + \beta)(\|y - Ty\|^2 + 2\mu_n\langle x_n - y, y - Ty\rangle) \leq 0.$$

Since $x_n \rightharpoonup z$, we have that

$$(\alpha + \beta)(\|y - Ty\|^2 + 2\langle z - y, y - Ty\rangle) \leq 0.$$

Using (2.2), we have that

$$(\alpha + \beta)(\|y - Ty\|^2 + \|z - Ty\|^2 - \|z - y\|^2 - \|y - Ty\|^2) \leq 0.$$

Since $\alpha + \beta > 0$, we have that

$$\|z - Ty\|^2 - \|z - y\|^2 \le 0$$

for all $y \in C$. This implies $z \in A(T)$. Similarly, we can obtain the desired result for the case of $\alpha + \gamma > 0$. This completes the proof. □

We can prove the following theorem by using Lemma 5.1 and the technique developed by Ibaraki and Takahashi [3, 4].

Theorem 5.1. *Let H be a Hilbert space and let C be a convex subset of H. Let $T : C \to C$ be a normally generalized hybrid mapping with $A(T) \neq \emptyset$ and let P be the mertic projection of H onto $A(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to an element $v \in A(T)$, where $v = \lim_{n\to\infty} Px_n$.

Proof. Let $z \in A(T)$. Then we have that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)Tx_n - z\|^2 \\ &\le \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|Tx_n - z\|^2 \\ &\le \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 \\ &= \|x_n - z\|^2\end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \|x_n - z\|^2$ exists. Then $\{x_n\}$ is bounded. We also have from (2.1) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)Tx_n - z\|^2 \\ &= \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|Tx_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2 \\ &\le \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2 \\ &= \|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2.\end{aligned}$$

Thus we have

$$\alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Since $\lim_{n\to\infty} \|x_n - z\|^2$ exists and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, we have that

$$\|Tx_n - x_n\| \to 0. \tag{5.3}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. By Lemma 5.1 and (5.3), we obtain that $v \in A(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. To complete the proof, we show $v_1 = v_2$. We know $v_1, v_2 \in A(T)$ and hence $\lim_{n\to\infty} \|x_n - v_1\|^2$ and $\lim_{n\to\infty} \|x_n - v_2\|^2$ exist. Put

$$a = \lim_{n\to\infty}(\|x_n - v_1\|^2 - \|x_n - v_2\|^2).$$

Note that for $n = 1, 2, \dots,$

$$\|x_n - v_1\|^2 - \|x_n - v_2\|^2 = 2\langle x_n, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2.$$

From $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$, we have

$$a = 2\langle v_1, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2 \tag{5.4}$$

and

$$a = 2\langle v_2, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2. \tag{5.5}$$

Combining (5.4) and (5.5), we obtain $0 = 2\langle v_2 - v_1, v_2 - v_1\rangle$. Thus we obtain $v_2 = v_1$. This implies that $\{x_n\}$ converges weakly to an element $v \in A(T)$. Since $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $z \in A(T)$ and $n \in \mathbb{N}$, we obtain from Lemma 2.1 that $\{Px_n\}$ converges strongly to an element $p \in A(T)$. On the other hand, we have from the property of P that

$$\langle x_n - Px_n, Px_n - u\rangle \geq 0$$

for all $u \in A(T)$ and $n \in \mathbb{N}$. Since $x_n \rightharpoonup v$ and $Px_n \to p$, we obtain

$$\langle v - p, p - u\rangle \geq 0$$

for all $u \in A(T)$. Putting $u = v$, we obtain $p = v$. This means $v = \lim_{n\to\infty} Px_n$. This completes the proof. □

Using Theorem 5.1, we can show the following weak convergence theorem of Mann's type for generalized hybrid mappings in a Hilbert space.

Theorem 5.2 (Kocourek, Takahashi and Yao [8]). *Let H be a Hilbert space and let C be a closed convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an element $v \in F(T)$.

Proof. As in the proof of Theorem 3.4, a generalized hybrid mapping is a normally generalized hybrid mapping. Since $\{x_n\} \subset C$ and C is closed and convex, we have from Theorem 5.1 that $v \in A(T) \cap C$. A normally generalized hybrid mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive, we have from Lemma 2.4 that $A(T) \cap C = F(T)$. Thus $\{x_n\}$ converges weakly to an element $v \in F(T)$. □

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