JORDAN ISOMORPHISMS AND MAPS PRESERVING SPECTRA OF CERTAIN OPERATOR PRODUCTS

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Abstract. Let $\mathcal{A}_1, \mathcal{A}_2$ be (not necessarily unital or closed) standard operator algebras on locally convex spaces X_1, X_2, respectively. For $k \geq 2$, define different kinds of products $T_1 * \cdots * T_k$ on elements in $\mathcal{A}_i$, which covers the usual product $T_1 * \cdots * T_k = T_1 \cdots T_k$, and the Jordan triple product $T_1 * T_2 = T_2 T_1 T_2$. Let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a (not necessarily linear) map satisfying that $\sigma(\Phi(A_1) * \cdots * \Phi(A_k)) = \sigma(A_1 * \cdots * A_k)$ whenever any one of A_i's is of rank zero or one. It is shown that if the range of Φ contains all rank one and rank two operators then it must be a Jordan isomorphism multiplied by a root of unity. Similar results for self-adjoint operators acting on Hilbert spaces are obtained.

1. Introduction

Spectrum preserving linear maps between Banach algebras are extensively studied in connection to the Kaplansky's problem concerning the characterization of invertibility preserving linear maps; see [14]. A related question follows:

Is it true that between semi-simple Banach algebras every spectrum preserving unital surjective linear map is a Jordan homomorphism?

Jafarian and Sourour showed in [13] that the answer is positive for maps between $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$, the Banach algebras of bounded linear operators acting on complex Banach spaces X_1, X_2, respectively. There are many other papers concerning this type of linear preservers; for example, see [1, 2, 3, 10, 11, 19, 20, 21, 23]. We also mention [4, 6, 14] about invertibility preservers and spectrum compressers between semi-simple Banach algebras.

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Recently, there has been growing interests in the problem of characterizing spectrum preserving maps without the linearity assumption. Of course, nonlinear spectrum preserving transformations can be almost arbitrary. So, some mild additional assumptions are needed. In [17], Molnár considered surjective maps $\Phi : \mathcal{L}(X_1) \to \mathcal{L}(X_2)$ such that $\Phi(A)\Phi(B)$ and AB always have the same spectrum, and proved that such a map must be a Jordan isomorphism multiplied by ± 1. In [12], the authors considered surjective maps $\Phi : \mathcal{L}(X_1) \to \mathcal{L}(X_2)$ such that $\Phi(B)\Phi(A)\Phi(B)$ and BAB always have the same spectrum, and proved that such a map must be a Jordan isomorphism multiplied by a cubic root of unity. In fact, they obtained more general results concerning $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$, where $\mathcal{A}_1, \mathcal{A}_2$ are closed unital standard operator algebra on X_1, X_2, respectively. Moreover, in addition to the usual spectrum $\sigma(X)$, they also characterized preservers of the left spectrum, the right spectrum, the boundary of the spectrum, the full spectrum, the point spectrum, the compression spectrum, the approximate point spectrum, and the surjectivity spectrum of operators, etc.; see [7, 8]. Note that all these different types of spectra reduce to the usual spectrum for finite rank operators.

Instead of considering different types of products separately, the authors in [5] considered a general product $T_1 * \cdots * T_k$ on the algebra M_n of $n \times n$ complex matrices, which covers the usual product $T_1 * \cdots * T_k = T_1 \cdots T_k$ and the Jordan triple product $T_1 * T_2 = T_2 T_1 T_2$. They showed that a map $\Phi : M_n \to M_n$ satisfying

$$\sigma(\Phi(A_1) * \cdots * \Phi(A_k)) = \sigma(A_1 * \cdots * A_k) \tag{1.1}$$

for all $A_1, \ldots, A_k$ in M_n must be a Jordan isomorphism multiplied by a root of unity. Their results do not require that Φ is surjective.

The purposes of this paper are manifold. First, we develop new techniques in $\mathcal{L}(X)$ including various characterizations of rank one operators to extend the results in [5, 12, 17] to more general settings. These new techniques will be useful in other problems on $\mathcal{L}(X)$. Second, we refine the existing results by weakening the spectrum preserving properties. This will enhance the understanding of the analytic and algebraic properties of spectrum preserving maps on standard operator algebras.

Our results unify and generalize many known facts. In particular, two consequences of our general result (Theorem 3.2) are given below. Here, we suppose the range of a map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ between standard operator algebras contains all continuous rank one and rank two operators.

(a) If $\sigma(\Phi(A_1)\phi(A_2)) = \Phi(A_1A_2)$ whenever A_1, A_2 in $\mathcal{A}_1$ satisfy rank $(A_1A_2) \leq 1$, then Φ is a Jordan isomorphism multiplied by ± 1.

(b) If $\sigma(\Phi(A_1)\phi(A_2)\Phi(A_1)) = \Phi(A_1A_2A_1)$ whenever A_1, A_2 in $\mathcal{A}_1$ satisfy rank $(A_1A_2A_1) \leq 1$, then Φ is a Jordan isomorphism multiplied by a complex number μ with $\mu^3 = 1$.

We present a special but typical case of our results in Sections 2 and their most general forms in Section 3, and obtain analogous results for self-adjoint operators acting on Hilbert spaces in Section 4.

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2. Results for special operator products B^rAB^s

Let X be a (complex Hausdorff) locally convex (topological linear) space with the dual space X'. The $\sigma(X, X')$ topology is the weakest (Hausdorff) locally convex topology on X such that all f in X' defines a continuous linear functional $x \mapsto f(x)$ on X. Similarly, we can define the $\sigma(X', X)$ topology on X'. Denote by $\mathcal{L}(X)$ the algebra of all continuous linear operators on X, and by $\mathcal{F}(X)$ its subalgebra of all continuous finite rank operators $\sum_{i=1}^n f_i \otimes x_i$ on X. Here f_i belongs to X' and x_i belongs to X for each $i = 1, 2, \ldots, n$. The operator $f \otimes x$ on X is defined by sending y to $f(y)x$.

Recall that a *standard operator algebra* $\mathcal{A}$ on a locally convex space X is a subalgebra of $\mathcal{L}(X)$ containing the algebra $\mathcal{F}(X)$ of all continuous finite rank operators on X. We do not assume $\mathcal{A}$ contains the identity operator I_X, or $\mathcal{A}$ is closed in any topology, however.

The following simple lemma is useful in this paper and was proved in [12]. For the sake of completeness, we give a short proof here. Recall that a rank one idempotent on X carries a form $f \otimes x$ with $f(x) = 1$.

Lemma 2.1. *Let X be a locally convex space, and let $A \in \mathcal{L}(X)$. Then $A = 0$ if and only if $f(Ax) = 0$ for all rank one idempotent $f \otimes x$ on X.*

Proof. We need only to check the "only if" part. For any f in X' and x in X, if $f(x) = \alpha \neq 0$ then $f(\frac{1}{\alpha}x) = 1$ implies $0 = f(A(\frac{1}{\alpha}x)) = \frac{1}{\alpha}f(Ax)$. Thus $f(Ax) = 0$. If $f(x) = 0$, then choose a g in X' such that $g(x) = 1$. Let $f_1 = g + f$ and $f_2 = g - f$. We have $f_1(x) = f_2(x) = 1$. Thus, by the assumption, $f_1(Ax) = f_2(Ax) = 0$. It follows that $f(Ax) = \frac{1}{2}(f_1 - f_2)(Ax) = 0$. Hence $A = 0$. ■

Lemma 2.2. *Let X be a locally convex space. Let (r, s) be a pair of non-negative integers such that $r + s \geq 1$. For any $A, \bar{A}$ in $\mathcal{L}(X)$, we have*

$$A = \bar{A} \quad \textit{if and only if} \quad \operatorname{tr}(B^rAB^s) = \operatorname{tr}(B^r\bar{A}B^s)$$

for every rank one idempotent operator B on X.

Proof. For any rank one idempotent $B = f\otimes x$ on X, we observe that

$$\operatorname{tr}(B^r AB^s) = \operatorname{tr}((f\otimes x)^r A(f\otimes x)^s) = f(Ax)f(x)^{r+s-1},$$

and

$$\operatorname{tr}(B^r \bar{A}B^s) = \operatorname{tr}((f\otimes x)^r \bar{A}(f\otimes x)^s) = f(\bar{A}x)f(x)^{r+s-1}.$$

So, $f((A-\bar{A})x) = 0$ whenever $f(x) = 1$. By Lemma 2.1, $A = \bar{A}$. ∎

Lemma 2.3. *Let A be any complex $n\times n$-matrix with rank at least k. Then there is a unitary $n\times n$ matrix U such that the leading k-by-k matrix of U^*AU is nonsingular.*

Proof. Suppose $A = PV$ such that P is positive semi-definite, and V is unitary. Let U be unitary such that $U^*VU = D$ is a diagonal unitary matrix. Then $U^*AU = (U^*PU)D$, and the sum of the absolute values of the k-by-k principal minors of U^*AU is the same as the sum of the k-by-k principal minors of U^*PU, which is the kth elementary symmetric function on the eigenvalues of P. This is positive because A has rank at least k, and therefore P has at least k positive eigenvalues, counting multiplicity. Now, applying a permutation similarity if needed, we see that the leading k-by-k matrix of U^*AU is invertible. ∎

Lemma 2.4. *Let X be a locally convex space. Let (r,s) be a pair of non-negative integers such that $r+s\geq 1$. Then a nonzero A in $\mathcal{L}(X)$ has rank one if and only if $\sigma(B^rAB^s)$ has at most 2 elements including 0 for all B in $\mathcal{L}(X)$ with rank 2.*

Proof. We check the sufficiency only. Suppose A has rank at least 2. Then there are x_1, x_2 in X such that $\{Ax_1, Ax_2\}$ is linearly independent. Let P be the projection of X onto $[x_1, x_2, Ax_1, Ax_2]$, which is the linear span of $\{x_1, x_2, Ax_1, Ax_2\}$ of dimension $n\leq 4$. Then we can think of $A_{11} := PAP$ as an $n\times n$ matrix with rank at least two. By Lemma 2.3, there is a unitary matrix U on $[x_1, x_2, Ax_1, Ax_2]$ such that $U^*A_{11}U$ has an invertible 2×2 leading submatrix. Thus, we may choose $\tilde{x}_1, \tilde{x}_2 \in [x_1, x_2, Ax_1, Ax_2]$ and a space decomposition of X into $[\tilde{x}_1, \tilde{x}_2]$ and its complement so that A has an operator matrix form

$$\begin{pmatrix} A_1 & * \\ * & * \end{pmatrix},$$

where A_1 is in upper triangular form with nonzero diagonal entries a_1, a_2. Now, use the same space decomposition to construct B so that B has operator matrix $\operatorname{diag}(1, b) \oplus 0$, where $a_1, a_2 b^{r+s}$ are two distinct nonzero numbers. Then $B^r A B^s$ has two distinct nonzero eigenvalues a_1 and $a_2 b^{r+s}$. ∎

The following is our main result. Here, we do not assume the map Φ is linear, multiplicative or continuous. But these comes out to be parts of the conclusion. Note also that Φ might not be surjective, and $\mathcal{A}_1, \mathcal{A}_2$ might not be closed in any topology or contain the identity operators.

Theorem 2.5. *Let $\mathcal{A}_1, \mathcal{A}_2$ be standard operator algebras on locally convex spaces X_1, X_2, respectively. Let (r, s) be a pair of non-negative integers such that $r + s \geq 1$. Suppose the range of a map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ contains all continuous operators of rank one and rank two, and*

$$\sigma(\Phi(B)^r \Phi(A) \Phi(B)^s) = \sigma(B^r A B^s) \tag{2.1}$$

whenever at least one of A, B in $\mathcal{A}_1$ is of rank zero or one. Then there is a scalar λ with $\lambda^{r+s+1} = 1$ and one of the following cases holds:

(1) there is a $\sigma(X_1, X_1') - \sigma(X_2, X_2')$ bi-continuous invertible linear operator $T : X_1 \to X_2$, such that

$$\Phi(A) = \lambda T A T^{-1}, \quad \forall A \in \mathcal{A}_1;$$

(2) there is a $\sigma(X_1', X_1) - \sigma(X_2, X_2')$ bi-continuous invertible linear operator $S : X_1' \to X_2$, such that

$$\Phi(A) = \lambda S A' S^{-1}, \quad \forall A \in \mathcal{A}_1.$$

Here $A' : X_1' \to X_1'$ is the dual map of $A : X_1 \to X_1$.

Proof. It suffices to consider the case when both X_1, X_2 have dimension at least 2. We divide the proof into several steps.

Assertion 1. Φ is injective, sends 0 to 0 and sends rank one operators to rank one operators.

The condition (2.1) implies that

$$\sigma(\Phi(B)^r \Phi(f \otimes x) \Phi(B)^s) = \sigma(B^r \cdot f \otimes x \cdot B^s) = \{0, f(B^{r+s}x)\}$$

for all B in $\mathcal{A}_1$. Since $\Phi(\mathcal{A}_1)$ contains all continuous operators of rank two, Lemma 2.4 implies that Φ sends each rank one operator $f \otimes x$ to an operator of rank at most one; the image has exactly rank one, by considering an B in $\mathcal{A}_1$ with $f(B^{r+s}x) = 1$ in the spectrum equality above.

It follows from Lemma 2.2 that $\Phi(0) = 0$.

Observe that if $\Phi(f \otimes x) = g \otimes y$, we will have

$$g(\Phi(B)^{r+s}y) = f(B^{r+s}x), \tag{2.2}$$

$$g(y)^{r+s-1}g(\Phi(B)y) = f(x)^{r+s-1}f(Bx), \quad \forall B \in \mathcal{A}_1. \tag{2.3}$$

Setting $B = f \otimes x$, we have

$$g(y)^{r+s+1} = f(x)^{r+s+1}. \tag{2.4}$$

It follows from Lemma 2.1 and (2.3) that Φ is injective.

Assertion 2. $\Phi(f \otimes x)$ is linear in f when x is fixed, and also linear in x when f is fixed.

Fix x in X. Suppose for f_1, f_2 in X_1' and a scalar α, we have g_1, g_2, g, g' in X_2' and y_1, y_2, y, y' in X_2 such that

$$\Phi(f_i \otimes x) = g_i \otimes y_i, \quad \text{for } i = 1, 2,$$

and

$$\Phi((f_1 + f_2) \otimes x) = g \otimes y, \quad \Phi(\alpha f_1 \otimes x) = g' \otimes y'.$$

Then (2.2) ensures that

$$g'(\Phi(B)^{r+s}y') = \alpha f_1(B^{r+s}x) = \alpha g_1(\Phi(B)^{r+s}y_1), \quad \forall B \in \mathcal{A}_1.$$

Since the range of Φ contains all continuous operators of rank one, by Lemma 2.1 for example, we have

$$g' \otimes y' = \alpha g_1 \otimes y_1, \quad \text{or} \quad \Phi(\alpha f_1 \otimes x) = \alpha \Phi(f_1 \otimes x).$$

On the other hand, (2.2) also ensures that

$$\begin{aligned} g(\Phi(B)^{r+s}y) &= f_1(B^{r+s}x) + f_2(B^{r+s}x) \\ &= g_1(\Phi(B)^{r+s}y_1) + g_2(\Phi(B)^{r+s}y_2), \quad \forall B \in \mathcal{A}_1. \end{aligned}$$

This gives rise to, by Lemma 2.1 again,

$$g \otimes y = g_1 \otimes y_1 + g_2 \otimes y_2. \tag{2.5}$$

In other words,

$$\Phi((f_1 + f_2) \otimes x) = \Phi(f_1 \otimes x) + \Phi(f_2 \otimes x), \quad \forall f_1, f_1 \in X', x \in X.$$

As a result, $\Phi(f \otimes x)$ is linear in f when x is fixed. Similarly, $\Phi(f \otimes x)$ is also linear in x when f is fixed.

By counting ranks, we note that (2.5) ensures either

$$y_1 = \alpha_1 y, \quad \text{and} \quad y_2 = \alpha_2 y, \tag{2.6}$$

or

$$g_1 = \beta_1 g, \quad \text{and} \quad g_2 = \beta_2 g, \tag{2.7}$$

for some scalars $\alpha_1, \alpha_2, \beta_1, \beta_2$.

From now on, we make the following

Assumption. The first case (2.6) happens for a linearly independent pair f_1, f_2 in X_1' and x in X_1.

Assertion 3. We can define an injective $\sigma(X_1', X_1) - \sigma(X_2', X_2)$ continuous linear operator $S_x' : X_1' \to X_2'$ such that

$$\Phi(f \otimes x) = S_x' f \otimes y, \quad \forall f \in X_1'.$$

To this end, let $f_3 \in X_1' \setminus \{0\}$, $g_3 \in X_2'$ and $y_3 \in X_2$ such that

$$\Phi(f_3 \otimes x) = g_3 \otimes y_3.$$

Suppose y_3 were linearly independent of y. By counting ranks in

$$\Phi((f_i + f_3) \otimes x) = g_i \otimes y_i + g_3 \otimes y_3 = \alpha_i g_i \otimes y + g_3 \otimes y_3, \quad i = 1, 2,$$

we see that g_1, g_2 are both scalar multiples of g_3. Hence $\Phi(f_1 \otimes x) = g_1 \otimes y_1 = \lambda g_2 \otimes y_2 = \lambda \Phi(f_2 \otimes x)$ for some scalar λ. This implies that $\Phi((f_1 - \lambda f_2) \otimes x) = 0$, and thus $f_1 = \lambda f_2$. This contradiction tells that y_3 depends on y, too.

At this stage, we show that for *this fixed* x in X_1, we have a fixed y in X_2 and a linear operator $S_x' : X_1' \to X_2'$ such that

$$\Phi(f \otimes x) = S_x' f \otimes y, \quad \forall f \in X_1'. \tag{2.8}$$

It follows from (2.2) that S_x' is injective and $\sigma(X_1', X_1) - \sigma(X_2', X_2)$ continuous.

Assertion 4. For any $\hat{x}$ in X_1, there is a $\hat{y}$ in X_2 and an injective $\sigma(X_1', X_1) - \sigma(X_2', X_2)$ continuous linear operator $S_{\hat{x}}'$ such that

$$\Phi(f \otimes \hat{x}) = S_{\hat{x}}' f \otimes \hat{y}, \quad \forall f \in X_1'.$$

It suffices to consider those $\hat{x}$ linearly independent of x. Assume for any linearly independent pair f_1, f_2 in X_1', we have $\hat{g}_1, \hat{g}_2$ in X_2' and y_1, y_2 in X_2 such that

$$\Phi(f_1 \otimes \hat{x}) = \hat{g}_1 \otimes y_1 \quad \text{and} \quad \Phi(f_2 \otimes \hat{x}) = \hat{g}_2 \otimes y_2.$$

Claim. y_1, y_2 are linearly dependent.

If not, by counting ranks in

$$\Phi((f_1 + f_2) \otimes \hat{x}) = \hat{g}_1 \otimes y_1 + \hat{g}_2 \otimes y_2,$$

we see that $\hat{g}_1, \hat{g}_2$ are linearly dependent. Suppose also

$$\Phi(f_1 \otimes x) = g_1 \otimes y \quad \text{and} \quad \Phi(f_2 \otimes x) = g_2 \otimes y.$$

Here, $g_i = S_x' f_i$ for $i = 1, 2$.

If y, y_1 are linearly dependent, as y_2 is linearly independent of y_1, we see that y, y_2 are linearly independent. By counting ranks in

$$\Phi(f_2 \otimes (x + \hat{x})) = g_2 \otimes y + \hat{g}_2 \otimes y_2,$$

we see that $g_2, \hat{g}_2$ are linearly dependent. Since $\hat{g}_1, \hat{g}_2$ are linearly dependent, $\hat{g}_1, g_2$ are linearly dependent, too. Be aware of that $f_1 \otimes \hat{x}, f_2 \otimes x$ are linearly independent. Write $y_1 = \alpha y$ and $\hat{g}_1 = \beta g_2$ for some nonzero scalars α, β. Then

$$\Phi(f_1 \otimes \hat{x}) = \hat{g}_1 \otimes y_1 = \alpha\beta g_2 \otimes y = \alpha\beta\Phi(f_2 \otimes x).$$

By (2.2),

$$f_1(B^{r+s}\hat{x}) = \alpha\beta f_2(B^{r+s}x), \quad \forall B \in \mathcal{A}_1.$$

By Lemma 2.1, $f_1 \otimes \hat{x} = \alpha\beta f_2 \otimes x$, a contradiction. Hence y, y_1 are linearly independent. Similarly, y, y_2 are linearly independent, too. Counting ranks again, we see that $g_i, \hat{g}_i$ are linearly dependent for $i = 1, 2$. This forces g_1, g_2 to be linearly dependent. It follows from the injectivity of Φ that f_1, f_2 are linearly dependent, a contradiction.

At this point, we can define an injective linear map $S'_{\hat{x}} : X'_1 \longrightarrow X'_2$ such that

$$\Phi(f \otimes \hat{x}) = S'_{\hat{x}} f \otimes \hat{y}, \quad \forall f \in X'_1.$$

Here $\hat{y}$ is a fixed element in X_2.

Assertion 5. $S'_x f, S'_{\hat{x}} f$ are linearly dependent for all f in X'_1.

Suppose not, and there is an f in X'_1 such that $g = S'_x f, \hat{g} = S'_{\hat{x}} f$ are linearly independent. By definition,

$$\Phi(f \otimes x) = g \otimes y \quad \text{and} \quad \Phi(f \otimes \hat{x}) = \hat{g} \otimes \hat{y}.$$

Counting ranks in

$$\Phi(f \otimes (x + \hat{x})) = g \otimes y + \hat{g} \otimes \hat{y},$$

we see that $y, \hat{y}$ are linearly dependent. Choose two f_1, f_2 from X'_1 such that

$$f_1(x) = f_1(\hat{x}) = f_2(x) = 1, \quad \text{and} \quad f_2(\hat{x}) = 0.$$

By (2.4),

$$|g_1(y)| = 1, \quad \text{and} \quad \hat{g}_2(\hat{y}) = 0.$$

Here $g_1 = S'_x f_1$ and $\hat{g}_2 = S'_{\hat{x}} f_2$. Now, (2.1) gives

$$\sigma((f_1 \otimes x)^r (f_2 \otimes \hat{x})(f_1 \otimes x)^s) = \sigma((g_1 \otimes y)^r (\hat{g}_2 \otimes \hat{y})(g_1 \otimes y)^s).$$

Since $\hat{y}$ linearly depends on y, we have $\hat{g}_2(y) = 0$, and thus

$$1 = f_1(x)^{r+s-1} f_1(\hat{x}) f_2(x) = g_1(y)^{r+s-1} g_1(\hat{y}) \hat{g}_2(y) = 0,$$

a contradiction.

At this moment, we can write

$$\Phi(f \otimes \hat{x}) = S'_x f \otimes \hat{y}.$$

Assertion 6. $\hat{y}$ is independent of f.

Suppose for any other $\bar{f}$ in X_1', which is linearly independent of f, we have $\Phi(\bar{f} \otimes \hat{x}) = S_x'\bar{f} \otimes \bar{y}$ for some $\bar{y}$ in X_2. By counting ranks in

$$\Phi((f + \bar{f}) \otimes \hat{x}) = S_x' f \otimes \hat{y} + S_x'\bar{f} \otimes \bar{y},$$

we see that $\hat{y}$ and $\bar{y}$ are linearly dependent, as $S_x' f$ and $S_x'\bar{f}$ are independent. Therefore, $\Phi(\bar{f} \otimes \hat{x}) = \alpha S_x'\bar{f} \otimes \hat{y}$ for some scalar α. On the other hand, $\Phi((f-\bar{f})\otimes\hat{x}) = \beta S_x'(f-\bar{f})\otimes\hat{y}$ for an other scalar β. This gives $\beta S_x'(f-\bar{f}) = S_x' f - \alpha S_x'\bar{f}$, and whence $\alpha = \beta = 1$ due to the linear independence of f and $\bar{f}$.

Thus, we can obtain an injective $\sigma(X_1', X_1) - \sigma(X_2', X_2)$ continuous linear map $S' : X_1' \longrightarrow X_2'$ such that $S' = S_x'$ for all x in X_1. It then also follows that there is an injective $\sigma(X_1, X_1') - \sigma(X_2, X_2')$ continuous linear map $T : X_1 \longrightarrow X_2$ such that

$$\Phi(f \otimes x) = S' f \otimes Tx, \quad \forall f \in X_1', x \in X_1.$$

Assertion 7. $S' = \lambda T'^{-1}$, for some scalar λ with $\lambda^{r+s+1} = 1$. Here $T' : X_2' \longrightarrow X_1'$ is the dual map of T.

We check first that S' has denes range. Suppose that $S' X_1'$ is not $\sigma(X_2', X_2)$ dense in X_2'. Then there is a nonzero y in X_2 such that $S' f(y) = 0$. Since the range of Φ contains all continuous rank one operators on X_2, there is a B in $\mathcal{A}_1$ with $B \neq 0$ and $\Phi(B)X_2$ is spanned by y. Now by (2.3), we have

$$f(x)^{r+s-1} f(Bx) = S' f(Tx)^{r+s-1} S' f(\Phi(B)Tx) = 0, \quad \forall f \in X_1', \forall x \in X_1.$$

By Lemma 2.1, we have $B = 0$. This conflict tells us that S' does have dense range in X_2'. Similarly, we see that T have dense range in X_2. In particular, its dual map $T' : X_2' \longrightarrow X_1'$ is injective.

Applying (2.4), we have

$$(S' f(Tx))^{r+s+1} = (f(x))^{r+s+1}, \quad \forall f \in X_1', \forall x \in X_1.$$

By a connectedness argument, we can derive the existence of a scalar λ with $\lambda^{r+s+1} = 1$ such that

$$S' f(Tx) = \lambda f(x), \quad \forall f \in X_1', \forall x \in X_1.$$

It then follows

$$T' S' f = \lambda f, \quad \forall f \in X_1'.$$

Since T' is now known to be bijective, $S' = \lambda T'^{-1}$.

At this point, we have shown that

$$\Phi(A) = \lambda TAT^{-1}, \quad \forall A \in \mathcal{F}(X_1).$$

In general, for any A in $\mathcal{A}_1$, f in X_1' and x in X_1 with $f(x) = 1$, by putting $B = f \otimes x$ in (2.1) we have

$$f(Ax) = \lambda^{r+s}(T^{-1})'f(\Phi(A)Tx) = \lambda^{-1}f(T^{-1}\Phi(A)Tx).$$

By Lemma 2.1, we have

$$\Phi(A) = \lambda TAT^{-1}, \quad \forall A \in \mathcal{A}_1.$$

Finally, if the second case (2.7) happens for all pairs f_1, f_2 in X_1' and for all x in X_1, then arguing in a similar, and slightly easier, pattern we will arrive at the other possible conclusions. ∎

Recall that the *Mackey topology* of a locally convex space X is the (locally convex) topology $\tau(X, X')$ of uniform convergence on $\sigma(X', X)$ compact convex subsets of X'. A locally convex space X is called a *Mackey space* if its topology coincides with $\tau(X, X')$. Hilbert spaces, Banach spaces, Fréchet spaces, infrabarrelled spaces, bornological spaces, and Montel spaces are all Mackey spaces.

On the other hand, the *strong topology* of the dual space X' of X is the topology $\beta(X', X)$ of uniform convergence on bounded subsets of X. Equip X' with $\beta(X', X)$ and we get the *strong dual* X'_β of X. The strong dual $X''_{\beta\beta}$ of X'_β is called the *strong bidual* of X. X is *semi-reflexive* if $K_X X = X''_{\beta\beta}$, where K_X is the canonical embedding of X into $X''_{\beta\beta}$. If, in addition, the topology of X agrees with the strong topology then X is *reflexive*. The Mackey-Arens theorem implies that X is semi-reflexive if and only if $\beta(X', X) = \tau(X', X)$ (see, e.g., [22, Corollary 18.2]). Semi-reflexive metrizable locally convex spaces are reflexive (see, e.g., [22, Corollary 18.4]). Reflexive locally convex spaces are Mackey spaces.

Recall also that a locally convex space X is *barrelled* if every $\sigma(X', X)$ bounded set in X' is equicontinuous, and thus relatively $\sigma(X', X)$ compact by the Alaoglu-Bourbaki theorem (see, e.g., [22, Theorem 16.13]). In other words, X is barrelled if and only if its topology agrees with $\beta(X, X')$, where we observe $(X', \sigma(X', X))' = X$. Banach and Fréchet spaces are barrelled, and barrelled spaces are Mackey.

Theorem 2.6. *In the conclusion of Theorem 2.5, the continuity of T and S can be assumed in the Mackey topology.*

(1) *Assume Case 1 occurs. If X_1, X_2 are Banach or Fréchet spaces, then T is a linear homeomorphism in the metric topology.*

(2) *Assume Case 2 occurs. If X_1 (resp. X_2) is barrelled (in particular, Banach or Fréchet), then X_2 (resp. X_1) is semi-reflexive and $X_1 =$*

$(X_2')_\beta$ (resp. $X_2 = (X_1')_\beta$). In particular, if both X_1, X_2 are Banach or Fréchet spaces then they are reflexive and dual to each other.

Proof. It is well-known that $\sigma(X, X') - \sigma(Y, Y')$ continuous linear operators between X and Y are exactly those $\tau(X, X') - \tau(Y, Y')$ continuous linear operators. Moreover, if A is a continuous linear operator from a locally convex space X into another Y then A is $\sigma(X, X') - \sigma(Y, Y')$ continuous, and its dual map $A' : Y' \longrightarrow X'$ is $\beta(Y', Y) - \beta(X', X)$ continuous (see, e.g., [22, Proposition 17.14]). Thus T and S in Theorem 2.5 are both continuous in the Mackey topologies. Rather than noting that Banach and Fréchet spaces are Mackey, we can also prove (1) directly by using the closed graph theorem.

For (2), we note that a locally convex space X is semi-reflexive if and only if $(X, \sigma(X, X'))$ is quasi-complete, i.e. all bounded Cauchy nets converges (see, e.g., [22, Theorem 18.2]). Now, $S : (X_1', \sigma(X_1', X_1)) \longrightarrow (X_2, \sigma(X_2, X_2'))$ is a linear homeomorphism. If X_1 is barrelled, then $(X_1', \sigma(X_1', X_1))$ is quasi-complete, and thus X_2 is semi-reflexive. Because S' induces a linear homeomorphism from $(X_2', \beta(X_2', X_2))$ onto $(X_1, \beta(X_1, X_1'))$, we see that $(X_2')_\beta = X_1$.

On the other hand, the range $\Phi(\mathcal{A}_1)$ is again a standard operator algebra on X_2. The inverse map $\Phi^{-1} : \Phi(\mathcal{A}_1) \longrightarrow \mathcal{A}_1$ satisfies a condition similar to (2.1), and clearly the range of Φ^{-1} contains $\mathcal{F}_2(X_1)$. Hence, one can conclude Case 2 again. In case X_2 is barrelled, we can conclude that X_1 is semi-reflexive and $(X_1')_\beta = X_2$ in a similar manner. ■

We remark that the barrelledness condition in Theorem 2.6(2) is sharp, as it is known that a Mackey space X has a quasi-complete dual space $(X', \sigma(X', X))$ if and only if X is barelled (see, e.g., [15, 23.6(4)]).

One reason why we are interested in such a generality of Theorem 2.5 is that the whole theory depends on the dual pairs $\langle X_1, X_1' \rangle, \langle X_2, X_2' \rangle$ rather than the particular topologies of the underlying spaces X_1, X_2. The following example provides us an other reason. We think this is a very important case we should not forget.

Example 2.7. Let X be a Banach space. Consider the map $\Phi : \mathcal{L}(X) \longrightarrow \mathcal{L}(X')$ defined by

$$\Phi(A) = A', \quad \forall A \in \mathcal{L}(X).$$

Here $A' : X' \longrightarrow X'$ is the dual map of A. Note that the range of Φ might not contain all norm continuous rank one operators on X'. However, if we equip X' with the $\sigma(X', X)$ topology, then the range of Φ does contain all

$\sigma(X', X)$ continuous finite rank operators. Thus we can apply Theorem 2.5. Note also that X need not be reflexive. Anyway, we have

$$(X', \sigma(X', X))'_\beta = X.$$

Remark 2.8. We do not have $r = s$ in general, even if Case (2) in Theorem 2.5 holds. Indeed, we always have

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\},$$

and thus

$$\begin{aligned}\sigma(B^r AB^s) \cup \{0\} &= \sigma(AB^{r+s}) \cup \{0\} = \sigma(B^s AB^r) \cup \{0\} \\ &= \sigma(B'^r A' B'^s) \cup \{0\} = \sigma(\Phi(B)^r \Phi(A)\Phi(B)^s) \cup \{0\},\end{aligned}$$

for all A, B in $\mathcal{A}_1$. We can drop 0 from the above equalities, since Φ sends invertible elements to invertible elements.

3. Applications to generalized operator products

Definition 3.1. Fix a positive integer $k \geq 2$ and a finite sequence $(i_1, i_2, \ldots, i_m)$ such that $\{i_1, i_2, \ldots, i_m\} = \{1, 2, \ldots, k\}$ and there is an i_p not equal to i_q for all other q. Define a product for operators $T_1, \ldots, T_k$ by

$$T_1 * \cdots * T_k = T_{i_1} \cdots T_{i_m}.$$

Clearly, this general product covers the usual product $T_1 * \cdots * T_k = T_1 \cdots T_k$ and the Jordan triple product $T_1 * T_2 = T_2 T_1 T_2$.

Theorem 3.2. *Let $\mathcal{A}_i$ be a standard operator algebra on a complex locally convex space X_i for $i = 1, 2$. Consider the product of operators $T_1 * \cdots * T_k$ defined in Definition 3.1. Suppose a map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ satisfies*

$$\sigma(\Phi(A_1) * \cdots * \Phi(A_k)) = \sigma(A_1 * \cdots * A_k) \tag{3.1}$$

whenever any one of $A_1, \ldots, A_k$ in $\mathcal{A}_1$ has rank zero or one. Suppose also that the range of Φ contains all continuous linear operators on X_2 of rank one and rank two. Then there exist a scalar λ with $\lambda^m = 1$ and one of the following cases holds.

(1) There exists an invertible operator T in $\mathcal{L}(X_1, X_2)$ such that

$$\Phi(A) = \lambda TAT^{-1} \quad \textit{for all } A \in \mathcal{A}_1.$$

(2) There exists an invertible operator S in $\mathcal{L}(X_1', X_2)$ such that

$$\Phi(A) = \lambda SA'S^{-1} \quad \textit{for all } A \in \mathcal{A}_1.$$

In this case, the ordered indices

$$(i_{p+1}, \ldots, i_m, i_1, \ldots, i_{p-1}) = (i_{p-1}, \cdots, i_1, i_m, \ldots, i_{p+1}).$$

The continuity of T and S above can also be assumed in the Mackey topologies.

Suppose further that X_1 and X_2 are Banach or Fréchet spaces. Then T and S are continuous in the metric topologies. If the second case happens, then both X_1 and X_2 are reflexive and dual to each other.

Proof. Let i_p be a fixed index differing from all other indices i_q as in Definition 3.1. We consider only a special class of products $A_1 * \cdots * A_k$ in which $A_{i_p} = A$ and all other $A_{i_q} = B$ such that one of A, B is of rank zero or one. The condition (3.1) now reduces to the condition (2.1) in Theorem 2.5. Applying Theorems 2.5 and 2.6, we have the desired forms of Φ.

Finally, assuming $\Phi(A) = \lambda S A' S^{-1}$ as in Case (2). It follows from an argument similar to the one in Remark 2.8 that

$$\sigma(A_{i_p} A_{i_{p+1}} \cdots A_{i_m} A_{i_1} \cdots A_{i_{p-1}}) = \sigma(A_{i_p} A_{i_{p-1}} \cdots A_{i_1} A_{i_m} \cdots A_{i_{p+1}}),$$

whenever $A_1, A_2, \ldots, A_k \in \mathcal{A}_1$ and $A_{i_p} = f \otimes x$ has rank one. This amounts to

$$f(A_{i_{p+1}} \cdots A_{i_m} A_{i_1} \cdots A_{i_{p-1}} x) = f(A_{i_{p-1}} \cdots A_{i_1} A_{i_m} \cdots A_{i_{p+1}} x),$$

for all f in X_1' and x in X_1. Therefore,

$$A_{i_{p+1}} \cdots A_{i_m} A_{i_1} \cdots A_{i_{p-1}} = A_{i_{p-1}} \cdots A_{i_1} A_{i_m} \cdots A_{i_{p+1}}.$$

Suppose $i_{p+1} \neq i_{p-1}$. Then we can choose two linearly independent vectors x_1, x_2 in X_1 and an f in X_1' such that $f(x_1) = f(x_2) = 1$. Assign $A_{i_{p+1}} = f \otimes x_1$, $A_{i_{p-1}} = f \otimes x_2$ and all other A_k to be the two dimension projection of X_1 onto the linear span $[x_1, x_2]$. In this way, we shall arrive a contradiction

$$x_1 = A_{i_{p+1}} \cdots A_{i_m} A_{i_1} \cdots A_{i_{p-1}} x_1 = A_{i_{p-1}} \cdots A_{i_1} A_{i_m} \cdots A_{i_{p+1}} x_1 = x_2.$$

Therefore, $i_{p+1} = i_{p-1}$. Inductively, we will have the equalities of other indices. ∎

Remark 3.3. (a) As mentioned in [5] the assumption that there exists an i_p such that $i_q \neq i_p$ for all other i_q in Definition 3.1 is necessary for the conclusion of Theorem 3.2. For example, if $A * B = AABB$, then one can pick an involution A_0 different from I, and consider Φ such that $\Phi(A_0) = I$, $\Phi(I) = A_0$ and $\Phi(A) = A$ for all other A. Then Φ is surjective such that $\Phi(A) * \Phi(B)$ and $A * B$ always have the same spectrum, but Φ is not of the form (1) or (2) in Theorem 3.2.

(b) The assumption that the range of Φ contains all rank two continuous linear operators is necessary for the infinite dimensional case even if we assume that Φ is linear and preserves rank one idempotents. For example, let H be an infinite dimensional complex Hilbert space and V an isometry on H that is not unitary. Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a linear map defined by $\Phi(A) = VAV^*$ for every A. Then

$$\Phi(A_1) * \cdots * \Phi(A_k) = VA_1V^* * \cdots * VA_kV^* = V(A_1 * \cdots * A_k)V^*$$

and

$$\begin{aligned} &\sigma(\Phi(A_1) * \cdots * \Phi(A_k)) \cup \{0\} = \sigma(V(A_1 * \cdots * A_k)V^*) \cup \{0\} \\ &= \sigma((A_1 * \cdots * A_k)V^*V) \cup \{0\} = \sigma(A_1 * \cdots * A_k) \cup \{0\}. \end{aligned}$$

As a result, $\sigma(\Phi(A_1)*\cdots*\Phi(A_k)) = \sigma(A_1*\cdots*A_k)$ whenever $A_1*\cdots*A_k$ is not invertible. In particular, the equality holds whenever $A_1 * \cdots * A_k$ has finite rank.

Note that, however, Theorems 2.5 and 3.2 indeed apply if we think of Φ as a map from $\mathcal{L}(H)$ onto $\mathcal{L}(K)$ with $K = VH$.

(c) If Case (1) in Theorem 3.2 holds, then clearly equation (3.1) holds for any operators $A_1, \dots, A_k$ in $\mathcal{A}_1$. In fact, if the conclusion (1) holds, then $A_1 * \cdots * A_k$ and $\Phi(A_1) * \cdots * \Phi(A_k)$ will always have the same left spectrum, the right spectrum, the boundary of the spectrum, the full spectrum, the point spectrum, the compression spectrum, the approximate point spectrum and the surjectivity spectrum, etc.

(d) Assume Case (2) in Theorem 3.2 holds. In the finite dimensional case, equation (3.1) will also hold for any matrices $A_1, \dots, A_k$. In the infinite dimensional case, one may consider different types of spectra and the same conclusion holds in some occasions, but not always; see [12]. For example, let X be a reflexive infinite dimensional complex Banach space on which there exists a left invertible operator A_0 that is not invertible. Thus, $\sigma_l(A_0) \neq \sigma_l(A_0')$, where $\sigma_l(T)$ denotes the left spectrum. Let $\Phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective map. Then Φ satisfies $\sigma_l(\Phi(B)\Phi(A)\Phi(B)) = \sigma_l(BAB)$ for all A, B in $\mathcal{L}(X)$ if and only if there exists an invertible T in $\mathcal{L}(X)$ such that $\Phi(A) = \mu TAT^{-1}$ for all A, where μ is a cubic root of unity. In fact, Φ satisfies equation (3.1). So, by Theorem 3.2 with $A_1 * A_2 = A_2A_1A_2$, Φ has either the form (1) $\Phi(A) = \mu TAT^{-1}$ for all A or the form (2) $\Phi(A) = \mu SA^*S^{-1}$ for all A, where $\mu^3 = 1$. However, (2) cannot occur in this situation; otherwise, $\sigma_l(A_0) = \sigma_l(\Phi(I)\Phi(A_0)\Phi(I)) = \sigma_l(A_0')$.

4. Results on self-adjoint operators

Let H be a complex Hilbert space and $\mathcal{S}(H)$ be the real linear space of all self-adjoint operators in $\mathcal{L}(H)$. Note that, $\mathcal{S}(H)$ is a Jordan ring. In this section we solve the problems discussed in Sections 2 and 3 for maps on $\mathcal{S}(H)$. Our results refine those in [5] under the assumption that the range of Φ contains all self-adjoint operators of rank one and rank two.

It suffices to consider the case when both X_1, X_2 have dimension at least 2. We begin with an observation.

Lemma 4.1. *Let $T_1 * \cdots * T_k = T_{i_1} \cdots T_{i_p} \cdots T_{i_m}$ be a general product on $\mathcal{S}(H)$ defined as in Definition 3.1. Then there exists a positive integer n with*

$$m = 2n - 1, \quad i_p = n, \quad \text{and} \quad i_k = i_{2n-k} \quad \text{for all } k = 1, \ldots, n.$$

Proof. Since the products are all self-adjoint, we have

$$T_{i_1} \cdots T_{i_p} \cdots T_{i_m} = T_{i_m} \cdots T_{i_p} \cdots T_{i_m}.$$

If $i_1 \neq i_m$, we put $T_{i_1} = P$, $T_{i_m} = Q$, and all other $T_{i_j} = I_H$, the identity operator on H. Here P, Q are any pair of projections on H. Then we get

$$PQP \cdots Q = QPQ \cdots P.$$

This cannot happen when, e.g., $P = e_1 \times e_1$ and $Q = \frac{1}{\sqrt{2}}(e_1 + e_2) \times (e_1 + e_2)$. Here e_1, e_2 are two orthogonal elements of norm one, and the operator $e \times e$ is defined by $x \mapsto \langle x, e \rangle \, e$. This contradiction shows that $i_1 = i_m$. Inductively, we have $i_k = i_{m-k+1}$ for $k = 1, 2, \ldots, m$. Since i_p is distinct from any other index, we must have $m = 2n - 1$ and $i_p = n$ for some positive integer n. ■

Theorem 4.2. *Let $\mathcal{S}(H_i)$ be the set of all self-adjoint operators on a complex Hilbert space H_i for $i = 1, 2$. Consider the product $T_1 * \cdots * T_k$ defined in Definition 3.1. Suppose $\Phi : \mathcal{S}(H_1) \to \mathcal{S}(H_2)$ satisfies*

$$\sigma(\Phi(A_1) * \Phi(A_2) * \cdots * \Phi(A_k)) = \sigma(A_1 * A_2 * \cdots * A_k), \tag{4.1}$$

whenever any one of $A_1, A_2, \ldots, A_k$ in $\mathcal{S}(H_1)$ has rank zero or one. Suppose also that the range of Φ contains all rank one and rank two self-adjoint operators on H_2. Then there exist a scalar ξ in $\{-1, 1\}$ with $\xi^m = 1$ and a linear or conjugate linear surjective isometry $U : H_1 \to H_2$ such that

$$\Phi(A) = \xi U A U^*, \quad \forall A \in \mathcal{S}(H_1).$$

To prove our result, it is important to characterize rank one operators in terms of the general products of self-adjoint operators. We have the following lemmas.

Lemma 4.3. *Let (r, s) be a pair of non-negative integers such that $r + s \geq 1$. Let H be a complex Hilbert space, and let $0 \neq A \in \mathcal{S}(H)$. Then the following statements are equivalent.*

(a) A has rank one.
(b) For any B in $\mathcal{S}(H)$, the spectrum $\sigma(B^rAB^s)$ contains at most one nonzero element.
(c) There does not exist B in $\mathcal{S}(H)$ of rank two such that $\sigma(B^rAB^s)$ contains two distinct nonzero elements.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear. The proof of (c) $\Rightarrow$ (a) is similar to that of Lemma 2.4. ∎

Proof of Theorem 4.2. Observing that Lemmas 2.1 and 2.2 in Section 2 hold when we consider only rank one self-adjoint idempotents $x \times x$ in $\mathcal{L}(H)$, where $x \times x(y) := \langle y, x\rangle\, x$ and $\|x\| = 1$. Together with Lemma 4.3, Assertion 1 in the proof of Theorem 2.5 is valid. In other words, Φ is injective, sends 0 to 0 and sends rank one self-adjoint operators to rank one self-adjoint operators. More precisely, for all x in H_1 there is a u in H_2, unique up to a complex modular one multiple, such that

$$\Phi(x \times x) = \xi\, u \times u,$$

where $\xi \in \{-1, 1\}$. Write $u = Tx$. It follows from the spectrum equality that $\|Tx\| = \|x\|$ and $\xi^m = 1$. As in (2.3), we have

$$\langle Bx, x\rangle = \xi\, \langle \Phi(B)Tx, Tx\rangle\,, \quad \forall B \in \mathcal{S}(H_1), \forall x \in H_1. \tag{4.2}$$

By a connectedness argument, we see that the choice of $\xi = \pm 1$ is uniform for all x in H_1.

Putting $B = y \times y$ in (4.2), we see that

$$|\,\langle x, y\rangle\,| = |\,\langle Tx, Ty\rangle\,|, \quad \forall x, y \in H_1.$$

By Wigner's theorem (cf. [9]; see also [16, 18]), we can assume T is either a linear or a conjugate-linear isometry from H_1 into H_2. If the range of T were not dense in H_2 then we can choose a self-adjoint operator B on H_1 such that $B \neq 0$, $\Phi(B)$ has rank one and $\Phi(B)Tx = 0$ for all x in H_1. But then (4.2) gives rise to the contradiction that $B = 0$. Consequently, T is a linear or a conjugate linear isometry from H_1 onto H_2. Finally, (4.2) gives

$$\phi(B) = \xi TBT^*, \quad \forall B \in \mathcal{S}(H_1).$$

∎

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