FIXED POINT THEOREMS FOR NEW GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES AND APPLICATIONS

WATARU TAKAHASHI, NGAI-CHING WONG, AND JEN-CHIH YAO

Abstract. In this paper, we introduce a broad class of nonlinear mappings in a Hilbert space which contains the classes of nonexpansive mappings, nonspreading mappings, hybrid mappings and contractive mappings. Then we prove fixed point theorems for the class of such mappings. Using these results, we prove well-known and new fixed point theorems in a Hilbert space. We finally give an open problem which is related to nonspreading mappings and hybrid mappings.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive* [18], *nonspreading* [13], and *hybrid* [19] if

$$\|Tx - Ty\| \leq \|x - y\| \, ,$$

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

and

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$, respectively; see also [7], [8] and [20]. These mappings are independent and they are deduced from a firmly nonexpansive mapping in a Hilbert space; see [19]. A mapping $F : C \to H$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$; see, for instance, Browder [2], Goebel and Kirk [5], and Kohsaka and Takahashi [12]. A mapping $T : C \to H$ is said to be *contractive*, *contractively nonspreading*, and *contractively hybrid* if there exist $r \in [0, 1)$, $t \in [0, \frac{1}{2})$ and $s \in [0, \frac{1}{3})$ such that

$$\|Tx - Ty\| \leq r \, \|x - y\| \, ,$$

$$2\|Tx - Ty\|^2 \leq t\{\|Tx - y\|^2 + \|Ty - x\|^2\}$$

and

$$3\|Tx - Ty\|^2 \leq s\{\|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2\}$$

for all $x, y \in C$, respectively; see [6]. Recently Kawasaki and Takahashi [10] introduced the following nonlinear mapping in a Hilbert space. A mapping T from C

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into H is said to be *widely generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0\end{aligned}$$

for any $x, y \in C$; see also [11].

In this paper, motivated by these mappings, we introduce a broad class of nonlinear mappings in a Hilbert space which contains the classes of nonexpansive mappings, nonspreading mappings, hybrid mappings, contractive mappings, contractively nonspreading mappings and contractively hybrid mappings. Then we prove fixed point theorems for the class of such mappings. Using these results, we prove well-known and new fixed point theorems in a Hilbert space. We finally give an open problem which is related to nonspreading mappings and hybrid mappings.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [18]. Furthermore, in a Hilbert space, we have that

$$2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.2}$$

for all $x, y, z, w \in H$. Let C be a nonempty subset of H and let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T. A mapping T from C into H with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|Tx - u\| \leq \|x - u\|$ for any $x \in C$ and $u \in F(T)$. A nonexpansive mapping with a fixed point is quasi-nonexpansive. We also know that a nonspreading mapping and a hybrid mapping which have fixed points are quasi-nonexpansive; see [8] and [19]. It is well-known that if $T : C \to H$ is quasi-nonexpansive and C is closed and convex, then $F(T)$ is closed and convex; see Ito and Takahashi [9]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

we have that z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then we have from (2.1) that

$$\begin{aligned}\|z - Tz\|^2 &= \|\alpha x + (1 - \alpha)y - Tz\|^2 \\ &= \alpha\|x - Tz\|^2 + (1 - \alpha)\|y - Tz\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\leq \alpha\|x - z\|^2 + (1 - \alpha)\|y - z\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)^2\|x - y\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 \\ &= 0.\end{aligned}$$

This implies $Tz = z$. Thus $F(T)$ is convex. Let D be a nonempty closed convex subset of H and $x \in H$. We know that there exists a unique nearest point $z \in D$ such that $\|x - z\| = \inf_{y \in D} \|x - y\|$. We denote such a correspondence by $z = P_D x$. The mapping P_D is called the metric projection of H onto D. It is known that P_D is nonexpansive and

$$\langle x - P_D x, P_D x - u \rangle \geq 0$$

for all $x \in H$ and $u \in D$; see [18] for more details.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [17] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we can obtain the following result; see [14], [15], [16] and [17].

Lemma 2.1. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n \mid n \in \mathbb{N}\}$ such that*

$$\mu_n \langle x_n, y \rangle = \langle z_0, y \rangle, \quad \forall y \in H.$$

3. Fixed point theorems

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is called *symmetric generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha \|Tx - Ty\|^2 + \beta(\|x - Ty\|^2 + \|Tx - y\|^2) + \gamma \|x - y\|^2 \\ + \delta(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0 \end{aligned} \tag{3.1}$$

for all $x, y \in C$. Such a mapping T is also called *$(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid.* If $\alpha = 1$, $\beta = \delta = 0$ and $\gamma = -1$ in (3.1), then the mapping T is nonexpansive. If $\alpha = 2$, $\beta = -1$ and $\gamma = \delta = 0$ in (3.1), then the mapping T is nonspreading. Furthermore, if $\alpha = 3$, $\beta = \gamma = -1$ and $\delta = 0$ in (3.1), then the mapping T is hybrid. Recently Kawasaki and Takahashi [10] introduced the following nonlinear mapping in a Hilbert space and they proved a fixed point theorem and a mean convergence theorem for the mappings. Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is said to be *widely generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \\ + \max\{\varepsilon \|x - Tx\|^2, \zeta \|y - Ty\|^2\} \leq 0 \end{aligned} \tag{3.2}$$

for any $x, y \in C$. Such a mapping T is called *$(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-widely generalized hybrid*. Replacing the variables x and y in (3.2), we have that

$$\alpha\|Ty-Tx\|^2+\beta\|y-Tx\|^2+\gamma\|Ty-x\|^2+\delta\|y-x\|^2 + \max\{\varepsilon\|y-Ty\|^2,\zeta\|x-Tx\|^2\}\le 0. \tag{3.3}$$

From (3.2) and (3.3) we have that

$$2\alpha\|Ty-Tx\|^2+(\beta+\gamma)(\|y-Tx\|^2+\|Ty-x\|^2)+2\delta\|y-x\|^2 + \max\{\varepsilon\|x-Tx\|^2,\zeta\|y-Ty\|^2\}+\max\{\varepsilon\|y-Ty\|^2,\zeta\|x-Tx\|^2\}\le 0. \tag{3.4}$$

From $\varepsilon\|x-Tx\|^2,\ \zeta\|y-Ty\|^2 \le \max\{\varepsilon\|x-Tx\|^2,\zeta\|y-Ty\|^2\}$, we have that

$$\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2\le 2\max\{\varepsilon\|x-Tx\|^2,\zeta\|y-Ty\|^2\}. \tag{3.5}$$

Similarly, we have that

$$\varepsilon\|y-Ty\|^2+\zeta\|x-Tx\|^2\le 2\max\{\varepsilon\|y-Ty\|^2,\zeta\|x-Tx\|^2\}. \tag{3.6}$$

Consequently, we have from (3.4), (3.5) and (3.6) that

$$2\alpha\|Ty-Tx\|^2+(\beta+\gamma)(\|y-Tx\|^2+\|Ty-x\|^2)+2\delta\|y-x\|^2 + \frac{1}{2}(\varepsilon+\zeta)(\|x-Tx\|^2+\|y-Ty\|^2)\le 0. \tag{3.7}$$

Such a mapping T is symmetric generalized hybrid. We first prove a fixed point theorem for symmetric generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha+2\beta+\gamma\ge 0$, (2) $\alpha+\beta+\delta>0$ *and* (3) $\delta\ge 0$ *hold. Then T has a fixed point if and only if there exists $z\in C$ such that $\{T^nz : n=0,1,\dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha+2\beta+\gamma>0$ on the condition* (1).

Proof. Suppose that T has a fixed point z. Then $\{T^nz : n=0,1,\dots\}=\{z\}$ and hence $\{T^nz : n=0,1,\dots\}$ is bounded. Conversely, suppose that there exists $z\in C$ such that $\{T^nz : n=0,1,\dots\}$ is bounded. Since T is an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping of C into itself, we have that

$$\alpha\|Tx-T^{n+1}z\|^2+\beta(\|x-T^{n+1}z\|^2+\|Tx-T^nz\|^2)+\gamma\|x-T^nz\|^2 + \delta(\|x-Tx\|^2+\|T^nz-T^{n+1}z\|^2)\le 0$$

for all $n\in\mathbb{N}\cup\{0\}$ and $x\in C$. Since $\{T^nz\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Since $\mu_n\|Tx-T^nz\|^2=\mu_n\|Tx-T^{n+1}z\|^2$ and $\mu_n\|x-T^nz\|^2=\mu_n\|x-T^nz\|^2$, we have that

$$(\alpha+\beta)\mu_n\|Tx-T^nz\|^2+(\beta+\gamma)\mu_n\|x-T^nz\|^2 + \delta(\|x-Tx\|^2+\mu_n\|T^nz-T^{n+1}z\|^2)\le 0.$$

Furthermore, since

$$\mu_n\|Tx-T^nz\|^2=\|Tx-x\|^2+2\mu_n\langle Tx-x,x-T^nz\rangle+\mu_n\|x-T^nz\|^2,$$

we have that

$$(\alpha+\beta+\delta)\|Tx-x\|^2+2(\alpha+\beta)\mu_n\langle Tx-x,x-T^nz\rangle + (\alpha+2\beta+\gamma)\mu_n\|x-T^nz\|^2+\delta\mu_n\|T^nz-T^{n+1}z\|^2\le 0.$$

From (1) $\alpha + 2\beta + \gamma \geq 0$ and (3) $\delta \geq 0$, we have that

$$(\alpha + \beta + \delta)\|Tx - x\|^2 + 2(\alpha + \beta)\mu_n\langle Tx - x, x - T^n z\rangle \leq 0. \tag{3.8}$$

Since there exists $p \in H$ from Lemma 2.1 such that

$$\mu_n\langle y, T^n z\rangle = \langle y, p\rangle$$

for all $y \in H$, we have from (3.8) that

$$(\alpha + \beta + \delta)\|Tx - x\|^2 + 2(\alpha + \beta)\langle Tx - x, x - p\rangle \leq 0. \tag{3.9}$$

Since C is closed and convex, we have that

$$p \in \overline{co}\{T^n x : n \in \mathbb{N}\} \subset C.$$

Putting $x = p$, we obtain from (3.9) that

$$(\alpha + \beta + \delta)\|Tp - p\|^2 \leq 0. \tag{3.10}$$

We have from (2) $\alpha + 2\beta + \delta > 0$ that $\|Tp - p\|^2 \leq 0$. This implies that p is a fixed point in T.

Next suppose that $\alpha + 2\beta + \gamma > 0$. Let p_1 and p_2 be fixed points of T. Then we have that

$$\begin{aligned}\alpha\|Tp_1 - Tp_2\|^2 + \beta(\|p_1 - Tp_2\|^2 + \|Tp_1 - p_2\|^2) + \gamma\|p_1 - p_2\|^2 \\ + \delta(\|p_1 - Tp_1\|^2 + \|p_2 - Tp_2\|^2) \leq 0\end{aligned}$$

and hence $(\alpha + 2\beta + \gamma)\|p_1 - p_2\|^2 \leq 0$. We have from $\alpha + 2\beta + \gamma > 0$ that $p_1 = p_2$. Therefore a fixed point of T is unique. This completes the proof. □

As a direct consequence of Theorem 3.1, we obtain the following theorem.

Theorem 3.2. *Let H be a Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta > 0$ *and* (3) $\delta \geq 0$ *hold. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition* (1).

Using Theorem 3.1, we also obtain the following theorem.

Theorem 3.3. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma > 0$, (2) $\beta \leq 0$, (3) $\beta + \gamma \leq 0$, *and* (4) $\beta + \delta \geq 0$ *hold. Then*

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, the sequence $\{T^n z\}$ converges to u.*

Proof. Let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping of C into itself satisfying four conditions (1), (2), (3) and (4) in the theorem. Take $x \in C$. Replacing x by $T^n x$ and y by $T^{n+1}x$ in (3.1), we have that

$$\begin{aligned}\alpha\|T^{n+1}x - T^{n+2}x\|^2 + \beta(\|T^n x - T^{n+2}x\|^2 + \|T^{n+1}x - T^{n+1}x\|^2) \\ + \gamma\|T^n x - T^{n+1}x\|^2 + \delta(\|T^n x - T^{n+1}x\|^2 + \|T^{n+1}x - T^{n+2}x\|^2) \leq 0\end{aligned} \tag{3.11}$$

for all $n \in \mathbb{N} \cup \{0\}$. From

$$\begin{aligned}\|T^n x - T^{n+2}x\|^2 \leq \|T^n x - T^{n+1}x\|^2 + \|T^{n+1}x - T^{n+2}x\|^2 \\ + 2\|T^n x - T^{n+1}x\|\|T^{n+1}x - T^{n+2}x\|\end{aligned}$$

and (2) $\beta \le 0$, we have that

$$\begin{aligned}\beta\|T^nx - T^{n+2}x\|^2 \ge \beta\|T^nx - T^{n+1}x\|^2 + \beta\|T^{n+1}x - T^{n+2}x\|^2 \\ + 2\beta\|T^nx - T^{n+1}x\|\|T^{n+1}x - T^{n+2}x\|.\end{aligned} \tag{3.12}$$

From (3.11) and (3.12) we have that

$$\begin{aligned}(\alpha+\beta)\|T^{n+1}x - T^{n+2}x\|^2 &+ (\beta+\gamma)\|T^nx - T^{n+1}x\|^2 \\ &+ 2\beta\|T^nx - T^{n+1}x\|\|T^{n+1}x - T^{n+2}x\| \\ &+ \delta(\|T^nx - T^{n+1}x\|^2 + \|T^{n+1}x - T^{n+2}x\|^2) \le 0.\end{aligned}$$

From (4) $\beta + \delta \ge 0$ we have that

$$\begin{aligned}(\alpha+\beta)\|T^{n+1}x - T^{n+2}x\|^2 &+ (\beta+\gamma)\|T^nx - T^{n+1}x\|^2 \\ &- 2\delta\|T^nx - T^{n+1}x\|\|T^{n+1}x - T^{n+2}x\| \\ &+ \delta(\|T^nx - T^{n+1}x\|^2 + \|T^{n+1}x - T^{n+2}x\|^2) \le 0\end{aligned}$$

and hence

$$\begin{aligned}(\alpha+\beta)\|T^{n+1}x - T^{n+2}x\|^2 &+ (\beta+\gamma)\|T^nx - T^{n+1}x\|^2 \\ &+ \delta(\|T^nx - T^{n+1}x\| - \|T^{n+1}x - T^{n+2}x\|)^2 \le 0.\end{aligned}$$

Since $\delta \ge 0$ from (2) and (4), we obtain that

$$(\alpha+\beta)\|T^{n+1}x - T^{n+2}x\|^2 + (\beta+\gamma)\|T^nx - T^{n+1}x\|^2 \le 0. \tag{3.13}$$

Using (1) $\alpha + 2\beta + \gamma > 0$ and (3) $\beta + \gamma \le 0$, we obtain that $\alpha + \beta > -(\beta+\gamma) \ge 0$. Then we have from (3.13) that

$$\|T^{n+1}x - T^{n+2}x\|^2 \le \frac{-(\beta+\gamma)}{\alpha+\beta}\|T^nx - T^{n+1}x\|^2 \tag{3.14}$$

and

$$0 \le \frac{-(\beta+\gamma)}{\alpha+\beta} < 1. \tag{3.15}$$

Putting $\lambda = (\frac{-(\beta+\gamma)}{\alpha+\beta})^{\frac{1}{2}}$, we have that for any $n \in \mathbb{N}$,

$$\begin{aligned}\|x - T^nx\| &\le \|x - Tx\| + \|Tx - T^2x\| + \cdots + \|T^{n-1}x - T^nx\| \\ &\le \|x - Tx\| + \lambda\|x - Tx\| + \cdots + \lambda^{n-1}\|x - Tx\| \\ &\le \|x - Tx\| + \lambda\|x - Tx\| + \cdots + \lambda^{n-1}\|x - Tx\| + \dots \\ &= \|x - Tx\|(1 + \lambda + \cdots + \lambda^{n-1} + \dots) \\ &= \|x - Tx\|\frac{1}{1-\lambda}.\end{aligned}$$

Thus the sequence $\{T^nx\}$ is bounded. On the other hand, from $\alpha + 2\beta + \gamma > 0$ and $\beta + \gamma \le 0$, we have $\alpha + \beta > 0$. Furthermore from $\delta \ge 0$, we have $\alpha + \beta + \delta > 0$. Thus we have from Theorem 3.1 that T has a unique fixed point u in X.

Let us prove (ii). We have from (3.1) that for every $x, y \in C$,

$$\begin{aligned}\alpha\|T^{n+1}x - Ty\|^2 &+ \beta(\|T^nx - Ty\|^2 + \|T^{n+1}x - y\|^2) \\ &+ \gamma\|T^nx - y\|^2 + \delta(\|T^nx - T^{n+1}x\|^2 + \|y - Ty\|^2) \le 0\end{aligned} \tag{3.16}$$

for all $n \in \mathbb{N} \cup \{0\}$. From $\delta \geq 0$ that

$$\alpha\|T^{n+1}x - Ty\|^2 + \beta(\|T^n x - Ty\|^2 + \|T^{n+1}x - y\|^2) + \gamma\|T^n x - y\|^2 \leq 0. \tag{3.17}$$

Since $\{T^n x\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Thus we have that

$$(\alpha + \beta)\mu_n\|T^n x - Ty\|^2 + (\beta + \gamma)\mu_n\|T^n x - y\|^2 \leq 0$$

and hence

$$\mu_n\|T^n x - Ty\|^2 \leq \frac{-(\beta + \gamma)}{\alpha + \beta}\mu_n\|T^n x - y\|^2. \tag{3.18}$$

We define a function $g : C \to \mathbb{R}$ as follows:

$$g(y) = \mu_n\|T^n x - y\|, \quad \forall y \in C.$$

We know from [17] that $g : C \to \mathbb{R}$ is a continuous function. For any $z \in C$, we have from (3.14) that $\{T^m z\}_{m=1}^{\infty}$ is a Cauchy sequence in C. Since C is complete, $\{T^m z\}$ converges. Let $T^m z \to u$. We also have from (3.18) that

$$g(T^{m+1}z) = \mu_n\|T^n x - T^{m+1}z\|^2 \leq r\mu_n\|T^n x - T^m z\|^2 = rg(T^m z),$$

where $r = \frac{-(\beta+\gamma)}{\alpha+\beta}$. Since g is continuous, we obtain that $g(u) \leq rg(u)$. Thus we have that

$$\mu_n\|T^n x - u\|^2 = g(u) \leq rg(u) = r\mu_n\|T^n x - u\|^2.$$

From $0 \leq r < 1$, we have $\mu_n\|T^n x - u\|^2 = 0$. Since

$$\begin{aligned}\|Tu - u\|^2 &= \|Tu - T^n x\|^2 + \|T^n x - u\|^2 + 2\langle Tu - T^n x, T^n x - u\rangle \\ &\leq 2\|Tu - T^n x\|^2 + 2\|T^n x - u\|^2\end{aligned}$$

for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\|Tu - u\|^2 &\leq 2\mu_n\|T^n x - Tu\|^2 + 2\mu_n\|T^n x - u\|^2 \\ &\leq 2r\mu_n\|T^n x - u\|^2 + 2\mu_n\|T^n x - u\|^2 \\ &= 0.\end{aligned}$$

Then $Tu = u$. We know already that $T^m z \to u$ and a fixed point u of T is unique. This completes the proof. $\square$

Using Theorems 3.1 and 3.3, we prove the following fixed point theorems. Before proving it, we introduce a more broad class of nonlinear mappings which contains the class of symmetric generalized hybrid mappings. A mapping T from C into H is called *symmetric more generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta(\|x - Ty\|^2 + \|Tx - y\|^2) + \gamma\|x - y\|^2 + \delta(\|x - Tx\|^2 + \|y - Ty\|^2) + \zeta\|x - y - (Tx - Ty)\|^2 \leq 0 \tag{3.19}$$

for all $x, y \in C$. Such a mapping T is called *$(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid*.

Theorem 3.4. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) $\delta + \zeta \geq 0$ *hold. Then T has a fixed point if and only if there exists $z \in C$*

such that $\{T^n z : n = 0, 1, \dots\}$ *is bounded. In particular, a fixed point of* T *is unique in the case of* $\alpha + 2\beta + \gamma > 0$ *on the condition* (1).

Proof. Since $T : C \to C$ is an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping, there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ satisfying (3.19). We also have that

$$\begin{aligned} \|x - y - (Tx - Ty)\|^2 &= \|x - Tx\|^2 + \|y - Ty\|^2 \\ &\quad - \|x - Ty\|^2 - \|y - Tx\|^2 + \|x - y\|^2 + \|Tx - Ty\|^2 \end{aligned} \tag{3.20}$$

for all $x, y \in C$. Thus we obtain from (3.19) that

$$\begin{aligned} &(\alpha + \zeta)\|Tx - Ty\|^2 + (\beta - \zeta)(\|x - Ty\|^2 + \|Tx - y\|^2) \\ &\quad + (\gamma + \zeta)\|x - y\|^2 + (\delta + \zeta)(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0. \end{aligned} \tag{3.21}$$

The conditions (1) $\alpha + 2\beta + \gamma \geq 0$ and (2) $\alpha + \beta + \delta + \zeta > 0$ are equivalent to $(\alpha + \zeta) + 2(\beta - \zeta) + (\gamma + \zeta) \geq 0$ and $(\alpha + \zeta) + (\beta - \zeta) + (\delta + \zeta) > 0$, respectively. Furthermore, since (3) $\delta + \zeta \geq 0$ holds, we have the desired result from Theorem 3.1. □

Theorem 3.5. *Let* H *be a real Hilbert space, let* C *be a nonempty closed convex subset of* H *and let* T *be an* $(\alpha, \beta, \gamma, \delta, \zeta)$*-symmetric more generalized hybrid mapping from* C *into itself such that the conditions* (1) $\alpha + 2\beta + \gamma > 0$*,* (2) $\beta \leq \zeta$*,* (3) $\beta + \gamma \leq 0$*, and* (4) $\beta + \delta \geq 0$ *hold. Then*

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, the sequence $\{T^n z\}$ converges to u.*

Proof. As in the proof of Theorem 3.4, we have that

$$\begin{aligned} &(\alpha + \zeta)\|Tx - Ty\|^2 + (\beta - \zeta)(\|x - Ty\|^2 + \|Tx - y\|^2) \\ &\quad + (\gamma + \zeta)\|x - y\|^2 + (\delta + \zeta)(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0. \end{aligned} \tag{3.22}$$

The conditions (1) $\alpha + 2\beta + \gamma > 0$ and (2) $\beta \leq \zeta$ are equivalent to $(\alpha + \zeta) + 2(\beta - \zeta) + (\gamma + \zeta) > 0$ and $\beta - \zeta \leq 0$, respectively. Furthermore, since (3) $\beta + \gamma \leq 0$ and (4) $\beta + \delta \geq 0$ are equivalent to $(\beta - \zeta) + (\gamma + \zeta) \leq 0$ and $(\beta - \zeta) + (\delta + \zeta) \geq 0$ respectively, we have the desired result from Theorem 3.3. □

4. Applications

In this section, we prove well-known and new fixed point theorems in a Hilbert space by using fixed point theorems obtained in Section 3.

Let H be a Hilbert space and let C be a nonempty subset of H. Then $U : C \to H$ is called a *widely strict pseudo-contraction* if there exists $r \in \mathbb{R}$ with $r < 1$ such that

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + r\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

We call such U a *widely r-strict pseudo-contraction.* If $0 \leq r < 1$, then U is a *strict pseudo-contraction*; see [4]. Furthermore, if $r = 0$, then U is nonexpansive. Conversely, let $T : C \to H$ be a nonexpansive mapping and define $U : C \to H$ by $U = \frac{1}{1+n}T + \frac{n}{1+n}I$ for all $x \in C$ and $n \in \mathbb{N}$. Then U is a widely $(-n)$-strict pseudo-contraction. In fact, from the definition of U, it follows that $T = (1 + n)U - nI$. Since T is nonexpansive, we have that for any $x, y \in C$,

$$\|(1 + n)Ux - nx - ((1 + n)Uy - ny)\|^2 \leq \|x - y\|^2$$

and hence

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 - n\|(I - U)x - (I - U)y\|^2.$$

Using Theorem 3.4, we first prove the following fixed point theorem.

Theorem 4.1. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let U be a widely strict pseudo-contraction from C into itself, i.e., there exists $r \in \mathbb{R}$ with $r < 1$ such that*

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + r\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C. \tag{4.1}$$

Then U has a fixed point in C.

Proof. We first assume that $r \leq 0$. We have from (4.1) that for any $x, y \in C$,

$$\|Ux - Uy\|^2 - \|x - y\|^2 - r\|(I - U)x - (I - U)y\|^2 \leq 0. \tag{4.2}$$

Then U is a $(1, 0, -1, 0, -r)$-symmetric more generalized hybrid mapping. Furthermore, (1) $\alpha+2\beta+\gamma = 1-1 \geq 0$, (2) $\alpha+\beta+\delta+\zeta = 1-r > 0$ and (3) $\delta+\zeta = -r \geq 0$ in Theorem 3.4 are satisfied. Thus U has a fixed point from Theorem 3.4. Assume that $0 \leq r < 1$ and define a mapping T as follows:

$$Tx = \lambda x + (1 - \lambda)Ux, \quad \forall x \in C,$$

where $r \leq \lambda < 1$. Then T is a mapping from C into itself and $F(T) = F(U)$. From $Tx = \lambda x + (1 - \lambda)Ux$, we also have that

$$Ux = \frac{1}{1-\lambda}Tx - \frac{\lambda}{1-\lambda}x.$$

Thus we obtain from (4.1) and (2.1) that

$$\begin{aligned}
0 &\geq \|\frac{1}{1-\lambda}Tx - \frac{\lambda}{1-\lambda}x - (\frac{1}{1-\lambda}Ty - \frac{\lambda}{1-\lambda}y)\|^2 \\
&\quad - \|x - y\|^2 - r\|x - y - \{\frac{1}{1-\lambda}Tx - \frac{\lambda}{1-\lambda}x - (\frac{1}{1-\lambda}Ty - \frac{\lambda}{1-\lambda}y)\}\|^2 \\
&= \|\frac{1}{1-\lambda}(Tx - Ty) - \frac{\lambda}{1-\lambda}(x - y)\|^2 \\
&\quad - \|x - y\|^2 - r\|\frac{1}{1-\lambda}(x - y) - \frac{1}{1-\lambda}(Tx - Ty)\|^2 \\
&= \frac{1}{1-\lambda}\|Tx - Ty\|^2 - \frac{\lambda}{1-\lambda}\|x - y\|^2 \\
&\quad + \frac{1}{1-\lambda} \cdot \frac{\lambda}{1-\lambda}\|x - y - (Tx - Ty)\|^2 - \|x - y\|^2 \\
&\quad - \frac{r}{(1-\lambda)^2}\|x - y - (Tx - Ty)\|^2 \\
&= \frac{1}{1-\lambda}\|Tx - Ty\|^2 - \frac{1}{1-\lambda}\|x - y\|^2 + \frac{\lambda - r}{(1-\lambda)^2}\|x - y - (Tx - Ty)\|^2.
\end{aligned}$$

Then T is $(\frac{1}{1-\lambda}, 0, -\frac{1}{1-\lambda}, 0, \frac{\lambda-r}{(1-\lambda)^2})$-symmetric more generalized hybrid. From

$$\frac{1}{1-\lambda} - \frac{1}{1-\lambda} = 0, \quad \frac{1}{1-\lambda} + \frac{\lambda - r}{(1-\lambda)^2} > 0 \quad \text{and} \quad \frac{\lambda - \mathrm{r}}{(1-\lambda)^2} \geq 0,$$

(1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ and (3) $\delta + \zeta \geq 0$ in Theorem 3.4 are satisfied. Thus T has a fixed point in C from Theorem 3.4 and hence U has a fixed point. This completes the proof. $\square$

Using Theorem 3.3, we can also prove the following fixed point theorem.

Theorem 4.2. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let $T : C \to C$ be a contractive mapping, i.e., there exists a real number r with $0 \le r < 1$ such that*

$$\|Tx - Ty\| \le r\|x - y\| \tag{4.3}$$

for all $x, y \in C$. Then the following hold:

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, the sequence $\{T^n z\}$ converges to u.*

Proof. We have from (4.3) that

$$\|Tx - Ty\|^2 - r^2\|x - y\|^2 \le 0$$

for all $x, y \in C$. This implies that T is $(1, 0, -r^2, 0)$-symmetric generalized hybrid. For $\alpha, \beta, \gamma, \delta$ in Theorem 3.3, we also have that

$$\alpha + 2\beta + \gamma = 1 - r^2 > 0,\ \beta = 0 \le 0,\ \beta + \gamma = -r^2 \le 0 \text{ and } \beta + \delta = 0 \ge 0.$$

From Theorem 3.3, we have the desired result. □

Using Theorem 3.1, we can prove the following fixed point theorems.

Theorem 4.3. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let $T : C \to C$ be contractively nonspreading, i.e., there exists a real number s with $0 \le s < \frac{1}{2}$ such that*

$$\|Tx - Ty\|^2 \le s\{\|Tx - y\|^2 + \|Ty - x\|^2\}$$

for all $x, y \in C$. Then T has a unique fixed point u in C.

Proof. Setting $r = \frac{s}{1-s}$, we have $r - rs = s$ and hence $s = \frac{r}{1+r}$. From $0 \le s < \frac{1}{2}$, we have $0 \le r$ and

$$r < 1 \Leftrightarrow \frac{r}{1+r} = s < \frac{1}{2}.$$

Thus we have $0 \le r < 1$. Furthermore, we have

$$(1 + r)\|Tx - Ty\|^2 \le r\{\|Tx - y\|^2 + \|Ty - x\|^2\}$$

for all $x, y \in C$. This implies that

$$(1 + r)\|Tx - Ty\|^2 - r(\|x - Ty\|^2 + \|Tx - y\|^2) \le 0$$

for all $x, y \in C$. That is, T is a $(1 + r, -r, 0, 0)$-symmetric generalized hybrid mapping. For $\alpha, \beta, \gamma, \delta$ in Theorem 3.1, we also have that

$$\alpha + 2\beta + \gamma = 1 - r > 0\ , \alpha + \beta + \delta = 1 - r > 0 \text{ and } \delta = 0 \ge 0.$$

From Theorem 3.1, we have the desired result. □

Theorem 4.4. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let $T : C \to C$ be contractively hybrid, i.e., there exists a real number s with $0 \le s < \frac{1}{3}$ and*

$$\|Tx - Ty\|^2 \le s\{\|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2\}$$

for all $x, y \in C$. Then T has a unique fixed point u in C.

Proof. Setting $r = \frac{2s}{1-s}$, we have $r - rs = 2s$ and hence $s = \frac{r}{2+r}$. From $0 \le s < \frac{1}{3}$, we have $0 \le r$ and

$$r < 1 \Leftrightarrow \frac{r}{2+r} = s < \frac{1}{3}.$$

Thus we have $0 \le r < 1$. Furthermore, we have

$$(2+r)\|Tx - Ty\|^2 \le r\{\|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2\}$$

for all $x, y \in C$. This implies that

$$(2+r)\|Tx - Ty\|^2 - r(\|x - Ty\|^2 + \|Tx - y\|^2) - r\|x - y\|^2 \le 0$$

for all $x, y \in C$. Thus T is a $(2+r, -r, -r, 0)$-symmetric generalized hybrid mapping. For $\alpha, \beta, \gamma, \delta$ in Theorem 3.1, we also have that

$$\alpha + 2\beta + \gamma = 2 - 2r > 0 \ , \alpha + \beta + \delta = 2 > 0 \text{ and } \delta = 0 \ge 0.$$

From Theorem 3.1, we have the desired result. □

5. An open problem

In 1967, Browder [3] proved the famous strong convergence theorem with implicit iteration for nonexpansive mappings in a Hilbert space.

Theorem 5.1 ([3])**.** *Let H be a Hilbert space, let C be a bounded closed convex subset of H and let T be a nonexpansive mapping of C into C. Fixed $u \in C$ and define a net $\{y_\alpha\}$ in C by*

$$y_\alpha = \alpha u + (1-\alpha)Ty_\alpha, \quad \forall \alpha \in (0,1).$$

Then $\{y_\alpha\}$ converges strongly to Pu as $\alpha \to +0$, where P is the metric projection of H onto $F(T)$.

We have not known whether such a theorem for nonspreading mappings and hybrid mappings holds or not. Putting

$$T_\alpha x = \alpha u + (1-\alpha)Tx, \quad \forall x \in C$$

in Browder's theorem, we can show easily that T_α is a contractive mapping of C into itself and T_α has a unique fixed point y_α in C by Banach [1]. However, it is difficult to use the above methods for nonspreading mappings and hybrid mappings.

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(Wataru Takahashi) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp

(Ngai-Ching Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan
E-mail address: wong@math.nsysu.edu.tw

(Jen-Chih Yao) Center for General Education, Kaohsiung Medical University, Kaohsiung 80702, Taiwan
E-mail address: yaojc@kmu.edu.tw