A Supplement to James' Theorem *

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A convex subset B of a real locally convex space X is said to have the *separation property* if it can be separated from any closed convex subset A of X, which is disjoint from B, by a closed hyperplane, *i.e.* there is a continuous linear functional f of X such that

$$\inf\{f(x) : x \in A\} > \sup\{f(x) : x \in B\}.$$

The famous R. James' Theorem (see [3–7] and [8]) asserts that a Banach space E is reflexive if and only if the closed unit ball U_E has the *James' property*, *i.e.* every continuous linear functional f of E attains its supremum in U_E. James' Theorem does not hold, however, for general normed spaces [6]. We prove in this talk that a normed space X is reflexive if and only if U_X has the separation property. Since the separation property is equivalent to, in Banach spaces, and implies, in general, James' property, our results can be viewed as a supplement to James'.

This talk sketches parts of our works in [1], in which details of proofs and other results on this subject can be found.

Theorem. *Let U be the closed unit ball of a real normed space X. U is weakly compact if and only if U has the separation property. In particular, a real normed space is reflexive if and only if its closed unit ball has the separation property.*

Proof. Only the implication from the separation property to weak compactness demands a proof, since the other direction follows clearly from the well-known strong separation theorem. Let $\tilde{U}$ be the closed unit ball of the completion $\tilde{X}$ of X. We want to show

*This research is partially supported by National Science Council of Taiwan, R.O.C.

that $\tilde{U} = U$. Suppose there were an element b in $\tilde{U}$ with $\| b \|= 1$ such that $b \notin U$, and b is contained in a supporting hyperplane H of $\tilde{U}$. W.O.L.G. $H = \{x \in \tilde{X} : f(x) = 1\}$ for some continuous linear functional f of $\tilde{X}$. In particular, $f(b) = 1$. Let $b_n = (1 + \frac{1}{n})b$, for each $n = 1, 2, 3, \cdots$. Let $B_{\tilde{X}}(a; \delta)$ denote the open ball $\{x \in \tilde{X} :\| x - a \|< \delta\}$. Since $B_{\tilde{X}}(b_n; \frac{1}{n}) \cap \{x \in \tilde{X} : f(x) > 1+\frac{1}{n}\}$ is non-empty and open in $\tilde{X}$ for each $n = 1, 2, \cdots$, and X is dense in $\tilde{X}$, we can choose a_n's from X so that $a_n \in B_{\tilde{X}}(b_n; \frac{1}{n}) \cap \{x \in \tilde{X} : f(x) > 1 + \frac{1}{n}\}$. Then $f(a_n) > 1 + \frac{1}{n}$, for $n = 1, 2, 3, \cdots$, and the sequence $\{a_n\}$ converges to b in norm.

Let A be the closed convex hull of a_n's in X. We want to show that $A \cap U = \emptyset$. Suppose an element y in X exists such that $y \in A \cap U$. Let N be a positive integer such that $B_{\tilde{X}}(b; \frac{2}{N}) \cap B_{\tilde{X}}(y; \frac{2}{N}) = \emptyset$. Note that $f(y) = 1$. Since $y \in A$, there exists a sequence y_n's of convex combinations of a_n's converges to y in norm. For each n, write $y_n = \sum_{i=1}^{k_n} \alpha_i^n a_i$, where $\alpha_i^n \geq 0$ for $i = 1, 2, \cdots, k_n$, $\sum_{i=1}^{k_n} \alpha_i^n = 1$ and k_n is a positive integer depending on n. Since $y_n \to y$ in norm. There exists a positive integer M_1 such that $f(y_n) < 1 + \frac{1}{N}$ for all $n \geq M_1$. For each $n \geq M_1$, $1 + \frac{1}{N} > f(\sum_{i=1}^{k_n} \alpha_i^n a_i) = \sum_{i=1}^{k_n} \alpha_i^n f(a_i) > \sum_{i=1}^{k_n} \alpha_i^n (1 + \frac{1}{i}) = 1 + \sum_{i=1}^{k_n} \frac{\alpha_i^n}{i}$. This implies $\sum_{i=1}^{k_n} \frac{\alpha_i^n}{i} < \frac{1}{N}$. On the other hand, there exists a positive integer M_2 such that $y_n \in B_{\tilde{X}}(y; \frac{2}{N}), \forall n \geq M_2$. Let M =max$\{M_1, M_2\}$. For $n \geq M$, $\| y_n - b \|=\| \sum_{i=1}^{k_n} \alpha_i^n a_i - b \|=\| \sum_{i=1}^{k_n} \alpha_i^n (a_i - b) \|\leq \sum_{i=1}^{k_n} \alpha_i^n \| a_i - b \|< 2 \sum_{i=1}^{k_n} \frac{\alpha_i^n}{i} < \frac{2}{N}$. This implies $y_n \in B_{\tilde{X}}(b; \frac{2}{N}), \forall n \geq M$. This contradicts to the fact that $B_{\tilde{X}}(b; \frac{2}{N}) \cap B_{\tilde{X}}(y; \frac{2}{N}) = \emptyset$. Hence $A \cap U = \emptyset$.

By the separation property of U, there is a continuous linear functional g of X such that

$$\sup\{g(u) : u \in U\} < \inf\{g(a) : a \in A\}.$$

Let g' be the continuous extension of g to $\tilde{X}$. Since $a_n \to b$ as $n \to \infty$ in $\tilde{X}$,

$$g'(b) = \lim_{n \to \infty} g(a_n) \geq \inf\{g(a) : a \in A\} > \sup\{g(u) : u \in U\} \geq g'(b).$$

This is a contradiction! Therefore $U = \tilde{U}$, and thus X is a Banach space. Since U has the separation property, by James' Theorem, U is weakly compact in X. $\square$

Let us recall the classical theorem that a Banach space is reflexive if and only if its unit ball is weakly sequentially compact [2]. The following extends some James' results from Banach spaces to normed spaces, cf. [5]. A proof can be found in [1].

Corollary. *Let B be the closed unit ball of a real normed space N. Then the following* *are equivalent:*

(1) *B is weakly compact.*

(2) *B is weakly countably compact.*

(3) *For each sequence x_n's in B there is an x in B such that for all continuous linear functionals f,*

$$\underline{\lim} f(x_n) \leq f(x) \leq \overline{\lim} f(x_n).$$

(4) *If K_n's is a decreasing sequence of closed convex sets in X and $B \cap K_n$ is non-empty for each n, then $B \cap (\cap_{n \geq 1} K_n)$ is non-empty.*

(5) *B is weakly sequentially compact.*

(6) *If S is a weakly closed set and $B \cap S$ is empty, then $d(B, S) = \inf\{\|b - s\| : b \in B, s \in S\} > 0$.*

(7) *B has the separation property.*

References

[1] Hwa-long Gau and Ngai-ching Wong, *Some converses of the strong separartion theorem*, preprint.

[2] W. F. Eberlein, *Weak compactness in Banach spaces,* Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 51-53.

[3] R. C. James, *Weakly compact sets,* Trans. Amer. Math. Soc. **113** (1964), 129-140.

[4] R. C. James, *Weak compactness and separation*, Canad. J. Math. **16** (1964), 204-206.

[5] R. C. James, *Weakly compactness and reflexivity*, Israel J. Math. **2** (1964), 101-119.

[6] R. C. James, *A counterexample for a sup theorem in normed spaces*, Israel J. Math. **9** (1971), 511-512.

[7] R. C. James, *Reflexivity and the sup of linear functionals*, Israel J. Math. **13** (1972), 289-300.

[8] J. D. Pryce, *Weak compactness in locally convex spaces,* Proc. Amer. Math. Soc. **17** (1966), 148-155.