INVERTIBILITY AND FREDHOLMNESS OF LINEAR COMBINATIONS OF QUADRATIC, k-POTENT AND NILPOTENT OPERATORS

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Abstract. Recently, the invertibility of linear combinations of two idempotents has been studied by several authors. Let P and Q be idempotents in a Banach algebra. It was shown that the invertibility of $P+Q$ is equivalent to that of $aP+bQ$ for nonzero a,b with $a+b\neq 0$. In this note, we obtain a similar result for square zero operators and those operators satisfying $x^2=dx$ for some scalar d. More generally, we show that if P,Q satisfy a quadratic polynomial $(x-c)(x-d)$ then the linear combination $aP+bQ-c(a+b)$ being invertible or Fredholm (and the index) is independent of the choice of the nonzero scalars a,b. However, this is not the case when P and Q are involutions, unitaries, partial isometries, k-potents ($k\geq 3$) and other nilpotents, as counterexamples are provided.

1. Introduction

The importance of idempotents and square zero elements cannot be overemphasized. For example, it is shown in [9] that every bounded linear operator on a complex infinite dimensional Hilbert space is a sum of at most five idempotents, and also a sum of at most five operators having square zero. See also [5, 11, 10].

Recently, there have been several papers devoted to the invertibility of a linear combination of two idempotents. In 2004, Baksalary and Baksalary [1] discuss the invertibility of linear combinations of idempotent matrices P and Q. They show that the invertibility of $P+Q$ is equivalent to that of $aP+bQ$ for nonzero coefficients a,b with $a+b\neq 0$. In 2006, Du, Yao and Deng [3] show that the same result also holds for idempotent operators on an infinite dimensional Hilbert space. About the

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same time, Koliha and Rakočević state an open problem in [7]: Suppose P and Q are idempotent operators on a Hilbert space. Is it true that $P+Q$ is Fredholm if and only if $aP+bQ$ is Fredholm where the nonzero coefficients a, b satisfy $a+b \neq 0$? Gau and Wu [6] find an affirmative answer for this question and they also show that the index of $aP+bQ$ is independent of the choice of the coefficients. Most recently, we are informed by Professor H. K. Du that Koliha and Rakočević [8] have already obtained independently the same result for idempotent operators on a Banach space. Du *et. al.* [4] have also extended these results further for other spectral properties of the linear combination $aP+bQ$.

In this note, we discuss the invertibility of a linear combination $aP+bQ$, satisfying the quadratic equation $(x-c)(x-d)=0$ for some scalars c, d. When $c=0, d=1$, it is the idempotent case and already done. We show that in the case P, Q are square zero operators on a Banach space, i.e. when $c=d=0$, the invertibility, as well as the Fredholmness and the index, of a linear combination $aP+bQ$ is independent of the choice of the nonzero coefficients a, b. This is also true for those quadratic operators P, Q satisfying $P^2=dP$ and $Q^2=dQ$ for some scalar $d \neq 0$, provided that $ab \neq 0$ and $a+b \neq 0$.

We finally obtain a complete solution involving operators satisfying a quadratic equation. Our new result states that the invertibility and the Fredholmness, and the index, of $aP+bQ-c(a+b)$ do not depend on the choice of the nonzero coefficients a, b provided that $a+b \neq 0$. However, in all other cases, including involutions, unitaries, partial isometries, other k-potents and other nilpotents, even in a finite dimensional setting, it fails to retain any such properties, and there are counterexamples provided by us.

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2. Invertibility of linear combinations of quadratic operators

The original proof of the following result was quite lengthy. We thank C. K. Li and Y. T. Poon for showing us the current arguments. Unlike the idempotent case, we do not need to exclude the case $a + b = 0$ below.

Theorem 2.1. *Let P, Q be a pair of square zero elements of a unital algebra. Then the following are equivalent.*

(1) *There are nonzero scalars a, b such that $aP + bQ$ is invertible.*
(2) *$aP + bQ$ is invertible for any nonzero scalars a, b.*
(3) *The Jordan product $PQ + QP$ is invertible.*

Proof. Observe that

$$(aP + bQ)^2 = ab(PQ + QP).$$

Hence the invertibility of $aP + bQ$ is equivalent to that of $PQ + QP$, which is independent of the choice of the nonzero scalars a, b. □

Recall that a bounded linear operator T from a Banach space E into a Banach space F is said to be *Fredholm* if $\dim(\ker T) < \infty$, and $\dim(F/TE) < \infty$. In this case, T has closed range (see, e.g., [2, 28A]). The *index* of a Fredholm operator T is defined by $\mathrm{Ind}(T) = \dim(\ker T) - \dim(F/TE)$.

Theorem 2.2. *Let P, Q be a pair of square zero operators on a real or complex Banach space E. Then $P + Q$ is Fredholm if and only if $aP + bQ$ is Fredholm for any, and thus all, nonzero scalars a, b. Moreover, we have*

$$\mathrm{Ind}(P + Q) = \mathrm{Ind}(aP + bQ)$$

in this case.

Proof. Note that T in the Banach algebra $\mathcal{B}(E)$ of all bounded linear operators on E is Fredholm if and only if the coset $T + \mathcal{K}(E)$ is invertible in the quotient Banach algebra $\mathcal{B}(E)/\mathcal{K}(E)$ (see, e.g., [2, 28J and 28K]). Here, $\mathcal{K}(E)$ is the closed ideal of all compact linear operators on E. The stability of the Fredholmness of aP +bQ follows from Theorem 2.1.

Now assume both $P+Q$ and $aP+bQ$ are Fredholm. Recall the index formula (see, e.g., [2, 28N])

$$\mathrm{Ind}(ST)=\mathrm{Ind}(S)+\mathrm{Ind}(T).$$

Since

$$\mathrm{Ind}(aP+bQ)^2=\mathrm{Ind}(ab(PQ+QP))=\mathrm{Ind}(PQ+QP)=\mathrm{Ind}(P+Q)^2,$$

we have $\mathrm{Ind}(aP+bQ)=\mathrm{Ind}(P+Q)$. □

Since there are positive results in both idempotent and square zero element cases, one might expect we could possibly do something for a pair of operators satisfying a fixed quadratic polynomial. We can assume the general form of the quadratic polynomial is $f(x)=x^2+\alpha x+\beta=(x-c)(x-d)$. The question states: Let P,Q be two operators such that $f(P)=f(Q)=0$. Does the invertibility of $aP+bQ$ depend on the choice of the coefficients a,b, provided that $ab\neq 0$ and/or $a+b\neq 0$.

When $\alpha=-1$ and $\beta=0$, it reduces to the idempotent case. When $\alpha=\beta=0$, it reduces to the square zero element case. However, it is not valid when $\beta\neq 0$. Indeed, we can construct counterexamples easily. For instance, consider $f(x)=(x-1)^2=x^2-2x+1$. Let

$$P=\begin{bmatrix}1&4\\0&1\end{bmatrix}\quad\text{and}\quad Q=\begin{bmatrix}1&0\\1&1\end{bmatrix}.$$

Clearly, $f(P)=f(Q)=0$. However, $P+Q$ is not invertible while $P+2Q$ is.

In the following example, $f(P)=f(Q)=0$ where $f(x)=x^2-1$. In other words, P,Q are involutions. Let $P=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$ and $Q=\begin{bmatrix}1&0\\0&1\end{bmatrix}$. Then $P+Q$ is not invertible, while $P+2Q$ is.

In the case that $\beta=0$, nevertheless, we do have an affirmative result like that for idempotents. This happens when P,Q are zeroes of $f(x)=x^2-dx$ for some scalar d. Of course, we only need to discuss the case $d\neq 0$ here. Notice that $P/d,Q/d$ are idempotents. The assertion now follows from the special case $d=1$ as shown in [8]. More precisely, the invertibility and the Fredholmness of $aP+bQ=ad(P/d)+bd(Q/d)$ is equivalent to that of $(P/d+Q/d)$, or equivalently, $P+Q$, provided $ab\neq 0$ and $a+b\neq 0$.

The above affine transformation technique can be generalized to give the following complete solution for quadratic operators.

Theorem 2.3. *Let P, Q be bounded linear operators on a real or complex Banach space E. Suppose P, Q satisfy the quadratic equation $(x-c)(x-d)=0$. Then the invertibility and the Fredholmness of a linear combination $aP+bQ-c(a+b)$ is independent of the choice of the scalars a, b, provided that $ab \neq 0$ and $a+b \neq 0$. In this case,*

$$\operatorname{Ind}(P+Q-2c) = \operatorname{Ind}(aP+bQ-c(a+b)),$$

where c and d can be interchanged. When $c=d$, we can also include the case $a+b=0$.

Proof. We make use of the affine transformation, and notice that $(P-c)/(d-c)$ and $(Q-c)/(d-c)$ are idempotents if $c \neq d$. The assertions now follow from the special case $c=0, d=1$ as shown in [8]. Indeed, for any coefficients a, b with $ab \neq 0$ and $a+b \neq 0$,

$$\begin{aligned} aP+bQ-c(a+b) &= aP-ac+bQ-bc \\ &= (d-c)\left\{a\frac{P-c}{d-c}+b\frac{Q-c}{d-c}\right\}. \end{aligned}$$

Hence the invertibility and the Fredholmness of $aP+bQ-c(a+b)$ are equivalent to those of the sum of idempotents $(P-c)/(d-c)+(Q-c)/(d-c)$, and thus those of $(P-c)+(Q-c)=P+Q-2c$. They also have the same indices if they are Fredholm.

Finally, if $c=d$, we have the same results by applying the square-zero case shown in Theorems 2.1 and 2.2. □

One might notice that in the proof of the stability of the invertibility of a linear combinations $aP+bQ-c(a+b)$ in Theorem 2.3 and also in [8], one does not use any property of the underlying Banach space, except for the open mapping theorem. Hence we can state an extension of a result about sums of idempotents given in [8].

Theorem 2.4. *Let A be a real or complex unital Fréchet algebra. Let P, Q in A be zeroes of a quadratic polynomial $f(x)=(x-c)(x-d)$. Then the sum $P+Q-2c$ is invertible in A if and only if $aP+bQ-c(a+b)$ is invertible in A for any, and thus all, nonzero scalars a, b. Here we assume, in addition, $a+b \neq 0$ in the case that $c \neq d$. Moreover, the roles of c and d can be interchanged.*

Proof. Let $\pi : A \to \mathcal{L}(A)$ be the left regular representation of the algebra A into the algebra of all continuous linear operators on the Fréchet space A. It is well known that a is invertible in A if and only if $\pi(a)$ is invertible in $\mathcal{L}(A)$. In fact, $a^{-1} = \pi(a)^{-1}(1)$. Hence, Theorem 2.3 applies. $\square$

3. Counterexamples for other cases

The counterexample involving two involutions in Section 2 can also serve as one of two unitary matrices for which the invertibility of a linear combination of them depends on the coefficients. Furthermore, recall that a bounded linear operator P on a complex Hilbert space is called a *partial isometry* if $PP^*P = P$. We will give a counterexample in the following to show the invertibility of a linear combination $aP + bQ$ of two partial isometry matrices depends on the choice of the coefficients a and b with $ab \neq 0$ and $a + b \neq 0$.

Example 3.1. Consider the partial isometries

$$P = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Note that $P + Q = \begin{bmatrix} 1 & -\sqrt{3} \\ 0 & 0 \end{bmatrix}$ is not invertible, while $2P + 4Q = \begin{bmatrix} 3 & -3\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ is invertible.

On the other hand, $P = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ are partial isometries. In this case, $P + Q$ is invertible while $\frac{4+2\sqrt{3}+\sqrt[4]{3}}{2}P + Q$ is not invertible.

Recall that P is a *k-potent* in an algebra if $P^k = P$, and P is an *involution* if $P^2 = I$. If P is an involution then $P^k = P$ for all odd positive integers k. The following example shows that there always exist a k-potent P for any odd integer $k \geq 3$ and an idempotent Q, the invertibility of $aP + bQ$ depends on the choice of the coefficients with $ab \neq 0$ and $a + b \neq 0$.

Example 3.2. Consider the involution $P = \begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix}$ and the idempotent $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We see that $P + Q = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$ is invertible, while $2P + Q = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}$ is not invertible.

On the other hand, $P = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ is an involution and $Q = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ is an idempotent. In this case, $P + Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible, while $2P + Q = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$ is invertible.

The assertion in Example 3.2 can be extended to the case P and Q are k-potents for any $k \geq 3$.

Example 3.3. Let $k \geq 4$ be a positive integer, and $\theta = \dfrac{2\pi}{k-1}$. Notice that $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a k-potents, and $Q = \begin{bmatrix} -\cos\theta & \dfrac{\cos^2\theta + \cos\theta}{\sin\theta} \\ -\sin\theta & 1 + \cos\theta \end{bmatrix}$ is an idempotent. Clearly,

$$P + Q = \begin{bmatrix} 0 & \dfrac{\cos^2\theta + \cos\theta - \sin^2\theta}{\sin\theta} \\ 0 & 1 + 2\cos\theta \end{bmatrix}$$

is not invertible. However, $\det(aP + bQ) = a(a - b)$. By choosing $b = 2a$, we get $\det(aP + bQ) = -a^2 \neq 0$, and thus $aP + bQ$ is invertible in this situation. For $k = 3$, we refer to Example 3.2.

Conversely, we consider now a positive integer $k \geq 6$, and $\theta = \dfrac{2\pi}{k-1}$. The matrix $P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is a k-potent and $Q = \begin{bmatrix} -\cos\theta & \dfrac{\cos^2\theta + \cos\theta}{\sin\theta} \\ -\sin\theta & 1 + \cos\theta \end{bmatrix}$ is an idempotent. Note that $\det(P + Q) = 2(1 + \cos\theta) \neq 0$, and thus $P + Q$ is invertible. We then need to find some nonzero scalars a and b satisfying $a + b \neq 0$, and

$\det(aP+bQ) = a^2+ab+2ab\cos\theta = 0$. This amounts to

$$a = -(1+2\cos\theta)b.$$

For the case $k = 3, 5$, see Example 3.2. For the case $k = 4$, choose

$$P = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Both of them are 4-potents. Now $P+Q = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$ is invertible, but $P+\frac{1}{2}Q = \begin{bmatrix} -2 & 1 \\ -3 & \frac{3}{2} \end{bmatrix}$ is not invertible.

As a conclusion, the invertibility of the sum $P+Q$ of k-potents is not always equivalent to the invertibility of $aP+bQ$ for any nonzero a and b with $a+b \neq 0$, when $k \geq 3$.

In view of Theorem 2.1, we are also interested in the cases of other nilpotents, $P^n = Q^n = 0$ with $n \geq 3$. We give a counterexample below to show that the statement of Theorem 2.1 is, however, false in this generality.

Example 3.4. Let

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then both P and Q are cube zero matrices, namely, $P^3 = Q^3 = 0$. Notice that $P+Q = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible, but $2P-Q = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ is not invertible.

On the other hand, let

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $P^3 = Q^3 = 0$. In this case, $P+Q = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is not invertible but $2P+Q = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible.

As a conclusion, the invertibility of a linear combination $aP+bQ$ of two cube zero matrices depends on the choice of the nonzero coefficients a, b.

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