DISJOINTNESS PRESERVING n-SHIFTS ON $C_0(X)$

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Abstract. We study disjointness preserving (quasi-)n-shift operators on $C_0(X)$, where X is locally compact and Hausdorff. When $C_0(X)$ admits a quasi-n-shift T, there is a countable subset of $X_\infty = X \cup \{\infty\}$ equipped with a tree-like structure, called φ-tree, with exactly n joints such that the action of T on $C_0(X)$ can be implemented as a shift on the φ-tree. If T is even an n-shift, then the φ-tree is dense in X and thus X is separable. By analyzing the structure of the φ-tree, we show that every (quasi-)n-shift on c_0 can always be written as a product of n (quasi-)shifts. Although it is not the case for general $C_0(X)$ as shown by our counter examples, we may do so after dilation.

1. Introduction

A linear operator S from a Banach space E into itself is called an *n-shift* if

(a) S is injective and has closed range;
(b) S has corank n;
(c) The intersection $\bigcap_{m=1}^{\infty} S^m E$ of the range spaces of all powers S^m of S is zero.

S is called a *quasi-n-shift* if S satisfies conditions (a) and (b). When $n = 1$, we will simply call S a *shift* or a *quasi-shift* accordingly. Crownover [3] showed that S is a shift on a Banach space if and only if it is similar to the unilateral shift on a sequence space. In fact, every n-shift on a Banach space is similar to an operator on a sequence space shifting the first n coordinates of a vector to the right (Proposition 4.3).

Let X and Y be locally compact Hausdorff spaces. Let $C_0(X)$ and $C_0(Y)$ be Banach spaces of continuous (real- or complex-valued) functions defined on X and Y vanishing at infinity, respectively. In the papers of Gutek *et. al.* [6] and Farid and Varadarajan [4], isometric shifts and quasi-shifts on $C(X)$ $(= C_0(X))$ are studied for compact Hausdorff spaces X. When the underlying scalar field is the complex $\mathbb{C}$, Haydon [7] provided examples to demonstrate such shifts do exist in some compact connected Hausdorff space as well as in the Cantor set. This is an interesting complement to the fact found by Holub [9] that the *real* Banach space $C(X, \mathbb{R})$ of continuous real-valued functions defined on X admits no shift at all if X is compact and connected. More recently, Rajagopalan [14] and Araujo and Font [2] discussed related questions in this direction.

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Since most of the interesting examples of shift operators in the literature so far are those on function spaces, it is reasonable to study disjointness preserving shifts. Recall that T is *disjointness preserving* or *separating* if $Tf \cdot Tg = 0$ whenever $f \cdot g = 0$. Disjointness preserving shifts on Banach lattices are studied in [6], where the authors apply results in [1] and others to obtain the non-existence of such operators on Dedekind complete Banach lattices with at most finitely many atoms. However, they did not discuss disjointness preserving shifts on general $C_0(X)$; except for the special case when X is an extremely disconnected compact Hausdorff space. The authors of [6, 4] do not seem to be aware of the recent development of the theory of disjointness preserving linear operators. In particular, similar tools as those provided in [8], a major reference of [6, 4], have been established, especially the one that such operators are exactly weighted composition operators (see e.g., [10, 5, 11]).

In Sections 2, we shall discuss n-shift (resp. quasi-n-shift) operators. In [12], it was proved that every disjointness preserving Fredholm linear operator from $C_0(X)$ into $C_0(Y)$ with closed range is automatically continuous and can be written as a very special weighted composition operator. Since disjointness preserving quasi-n-shifts on $C_0(X)$ are Fredholm, they are automatically continuous. Moreover, tools developed in [12] is used to give a full description of them (Theorem 2.3). In fact, every disjointness preserving quasi-n-shift on $C_0(X)$ is implemented by a shift on a countable set with a tree-like structure, called φ-tree, with exactly n joints in the one-point compactification X_∞ of X (Theorem 2.5). The φ-tree arising from an n-shift is proved to be dense in X. In particular, X is separable whenever any disjointness preserving n-shift on $C_0(X)$ exists (Theorems 2.11).

In Section 3, we shall verify that all disjointness preserving (quasi-)n-shifts on $c_0 \cong C_0(\mathbb{N})$ can be written as a product of n (quasi-)shifts (Theorem 3.4). It is, however, not the case in general. We shall provide a counter example in Section 4 that some disjointness preserving isometric n-shifts cannot be written as products of n disjointness preserving shifts (Example 4.4). There is also a compact connected Hausdorff space X such that $C(X)$ admits a quasi-n-shift but not any quasi-k-shift for $n \geq 2$ and $k = 1, 2, \ldots, n-1$ (Example 4.5). Nevertheless, we show that a disjointness preserving quasi-n-shift can be dilated to a product of n quasi-shifts and corank one injections, provided for example that X is compact (Theorem 4.13).

We shall apply results in this paper and [12] to the study of isometric (quasi-)n-shifts on $C_0(X)$ in [13], which extends [6, 4, 2].

2. Disjointness preserving n-shifts and the related φ-tree structure

Let X be a locally compact Hausdorff space. Let X_∞ be the one-point compactification of X; namely $X_\infty = X \cup \{\infty\}$. The point ∞ is an isolated point in X_∞ if and only if X is compact. In this case, we write $C(X)$ for $C_0(X) = \{f \in C(X_\infty) : f(\infty) = 0\}$ as usual.

Definition 2.1. Let X be a locally compact Hausdorff space and let φ be a continuous map from X_∞ onto X_∞ with $\varphi(\infty) = \infty$. Define an equivalence relation $\sim$ in X_∞ by

$$x \sim x' \quad \text{if and only if} \quad \varphi(x) = \varphi(x').$$

(1) We call a point x in X_∞ a *φ-vanishing point* if $\varphi(x) = \infty$.
(2) We call x a *φ-merging point* and $\varphi(x)$ a *φ-merged point* if the equivalence class $[\varphi(x)] = \varphi^{-1}\{\varphi(x)\}$ contains at least two points. Denote by M_φ the set of all φ-merging points, and thus by $\varphi(M_\varphi)$ the set of all φ-merged points in X_∞.
(3) A *φ-branch* originated at a point x in X_∞ is defined to be the set

$$B_x = \bigcup \Big\{ \varphi^{-n}(x) \colon n = 0, 1, 2, \dots \Big\},$$

where $\varphi^0(x) = \{x\}$ and $\varphi^{-n}(x) = \{y \in X \colon \varphi^n(y) = x\}$ for $n = 1, 2, \dots$.
(4) The *φ-tree* is a directed graph in X_∞, whose vertex set is the union $\bigcup\{B_c : c \in \varphi(M_\varphi)\}$ of all φ-branches originated at φ-merged points, and there is a directed edge from a to b if and only if $\varphi(a) = b$.
(5) The *crown* of the φ-tree is the union $\bigcup\{B_a : a \in M_\varphi\}$ of all φ-branches originated at φ-merging points.
(6) The number $\#(M_\varphi) - \#(\varphi(M_\varphi))$ is called the *number of joints* of the φ-tree.
(7) A φ-tree is said to be *rooted at* ∞ if the φ-tree coincides with the φ-branch B_∞ originated at ∞.

We are interested in the φ-tree associated to a disjointness preserving quasi-n-shift T on $C_0(X)$. In fact, every such T gives rise to a unique map φ such that the action of T can be visualized as a shift on the φ-tree in X_∞, which has exactly n joints. Let us consider an example first.

Example 2.2. Let T be the disjointness preserving isometric 3-shift on c_0 ($\cong C_0(\mathbb{N})$) defined by

$$T\big((x_1, x_2, x_3, x_4, x_5, x_6, x_7, \dots)\big) = (0, x_1, x_1, x_2, x_2, x_3, x_4, x_5, x_6, x_7, \dots).$$

Every null sequence (x_n) in c_0 can be considered as a continuous function f on $\mathbb{N}_\infty$ such that $f(\infty) = \infty$ and $f(n) = x_n$ for all n in $\mathbb{N}$. Write $Tf = f \circ \varphi$ where the action of $\varphi \colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ can be visualized in the following φ-tree in which a directed edge $b \leftarrow a$

indicating $\varphi(a) = b$.

$$
\begin{tikzcd}
 & & & 4 \arrow[dl] & 7 \arrow[l] & 10 \arrow[l] & \cdots . \arrow[l] \\
 & & 2 \arrow[dl] & & & & \\
\infty \arrow[loop above] & 1 \arrow[l] & & 5 \arrow[ul] & 8 \arrow[l] & 11 \arrow[l] & \cdots \arrow[l] \\
 & & 3 \arrow[ul] & 6 \arrow[l] & 9 \arrow[l] & 12 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}
$$

Note that the set of all φ-merging points is $M_\varphi = \{\infty, 1, 2, 3, 4, 5\}$, and the set of all φ-merged points is $\varphi(M_\varphi) = \{\infty, 1, 2\}$. There are exactly $\#(M_\varphi) - \#(\varphi(M_\varphi)) = 3$ joints in the φ-tree at ∞, 1 and 2, respectively. Moreover, the φ-tree coincides with its crown and is rooted at ∞. In this case,

$$\varphi\colon \big(\mathbb{N}_\infty, \{\infty, 1, 2, 3, 4, 5\}\big) \to \big(\mathbb{N}_\infty, \{\infty, 1, 2\}\big)$$

is a relative homeomorphism, and the induced map

$$\widetilde{\varphi}\colon {}^{\mathbb{N}_\infty}\!\diagup_{\sim} \to \mathbb{N}_\infty$$

is a homeomorphism. In fact, ${}^{\mathbb{N}_\infty}\!\diagup_{\sim} = \{[\infty], [2], [4], [6], [7], [8], [9], \dots\}$ and $\widetilde{\varphi}([\infty]) = \infty$, $\widetilde{\varphi}([2]) = 1$, $\widetilde{\varphi}([4]) = 2$, $\widetilde{\varphi}([6]) = 3$, $\widetilde{\varphi}([7]) = 4$, $\widetilde{\varphi}([8]) = 5$, $\widetilde{\varphi}([9]) = 6, \dots$. $\square$

Denote by δ_x the evaluation at a point x in X. The following theorem is a special case of the results in [12].

Theorem 2.3. *Every disjointness preserving quasi-n-shift T on $C_0(X)$ is continuous. Let*

$$X_0 = \Big\{x \in X \colon \delta_x \circ T = 0\Big\} \quad \textit{and} \quad X_c = X \setminus X_0.$$

(1) There exist a continuous map φ from X_∞ onto X_∞ and a continuous bounded and away from zero scalar function h on X_c such that $\varphi(X_0 \cup \{\infty\}) = \{\infty\}$, $\varphi(X_c) = X$, and

$$\begin{aligned} Tf_{|X_c} &= h \cdot f \circ \varphi, \\ Tf_{|X_0} &\equiv 0. \end{aligned}$$

(2) The set M_φ of all φ-merging points in X_∞ is finite. In fact,

$$\#(M_\varphi) - \#(\varphi(M_\varphi)) = n.$$

(3) The map

$$\varphi\colon (X_\infty, M_\varphi) \to \big(X_\infty, \varphi(M_\varphi)\big)$$

is a relative homeomorphism, and the induced map

$$\widetilde{\varphi}\colon {}^{X_\infty}\!\diagup_{\sim} \to X_\infty$$

is a homeomorphism. Consequently, the finite set $X_0 = \varphi^{-1}(\infty) \cap X$ *consists of isolated points in* X *when* X *is compact, and* X_0 *is empty when* X *is compact and connected.*

The following example borrowed from [12] says that the last assertion in Theorem 2.3(3) can be false when X is not compact.

Example 2.4. Let X be the disjoint union in $\mathbb{R}^2$ of $I_n^+ = \{(n,t)\colon 0 < t \leq 1\}$ and $I_n^- = \{(n,t) : -1 < t < 0\}$ for $n = 1, 2, \ldots$. Let p be the point $(1,1)$ and let $X_1 = X \setminus \{p\}$. Let φ be the homeomorphism from X_1 onto X by sending the intervals $I_1^+ \setminus \{p\}$ onto I_1^-, I_{n+1}^+ onto I_n^+, and I_n^- onto I_{n+1}^- in a canonical way for $n = 1, 2, \ldots$. Then the disjointness preserving isometric quasi-shift $Tf = f \circ \varphi$ on $C_0(X)$ has exactly one vanishing point, i.e. p, which is not an isolated point in X. In a similar manner, one can also construct an example in which X is connected (by adjoining each $I_n^\pm$ a common base point, for example). $\square$

From Theorem 2.3(2), we know that all equivalence classes in X_∞ induced by the relative homeomorphism φ are finite and at most finitely many of them consist of more than one points. Let all the possibly exceptional classes be

$$\begin{aligned} [\infty] &= \{p_1, p_2, \ldots, p_k, \infty\}, \\ \left[a_{l_1}^{(1)}\right] &= \left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{l_1}^{(1)}\right\}, \\ &\vdots \\ \left[a_{l_j}^{(j)}\right] &= \left\{a_1^{(j)}, a_2^{(j)}, \ldots, a_{l_j}^{(j)}\right\}. \end{aligned}$$

In other words, we have

$$\begin{gathered} \varphi(p_1) = \varphi(p_2) = \cdots = \varphi(p_k) = \varphi(\infty) = \infty, \\ \varphi\left(a_1^{(1)}\right) = \varphi\left(a_2^{(1)}\right) = \cdots = \varphi\left(a_{l_1-1}^{(1)}\right) = \varphi\left(a_{l_1}^{(1)}\right) = c_1, \\ \vdots \\ \varphi\left(a_1^{(j)}\right) = \varphi\left(a_2^{(j)}\right) = \cdots = \varphi\left(a_{l_j-1}^{(j)}\right) = \varphi\left(a_{l_j}^{(j)}\right) = c_j, \end{gathered}$$

for some distinct $c_1, c_2, \ldots, c_j$ in X. Then

$$M_\varphi = \left\{\infty, p_1, p_2, \ldots, p_k, a_1^{(1)}, a_2^{(1)}, \ldots, a_{l_1}^{(1)}, \ldots, a_1^{(j)}, a_2^{(j)}, \ldots, a_{l_j}^{(j)}\right\}.$$

In case $[\infty] = \varphi^{-1}(\infty) = \{\infty\}$, we have

$$M_\varphi = \left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{l_1}^{(1)}, \ldots, a_1^{(j)}, a_2^{(j)}, \ldots, a_{l_j}^{(j)}\right\}$$

instead.

The following theorem is again a consequence of the results in [12].

Theorem 2.5. *$C_0(X)$ admits a disjointness preserving quasi-n-shift if and only if X_∞ admits a φ-tree with exactly n joints. In this case, let*

$$X_0 = \{x \in X \colon \varphi(x) = \infty\} \quad \textit{and} \quad X_c = X \setminus X_0.$$

For any bounded and away from zero scalar function h on X_c, the disjointness preserving operator T defined by $Tf_{|X_c} = h \cdot f \circ \varphi$ and $Tf_{|X_0} = 0$ is a quasi-n-shift on $C_0(X)$. In above notations, we have

$$\operatorname{ran}(T) = \Bigg\{ g \in C_0(X) \colon g(p_1) = \cdots = g(p_k) = 0 \textit{ and } \frac{g(a_1^{(i)})}{h(a_1^{(i)})} = \frac{g(a_2^{(i)})}{h(a_2^{(i)})} = \cdots = \frac{g(a_{l_i}^{(i)})}{h(a_{l_i}^{(i)})}, \quad i = 1, 2, \ldots, j \Bigg\}.$$

In the following example, there are a quasi-shift (not shift) and a shift on $c_0 \cong C_0(\mathbb{N})$ such that they give rise to the same φ-tree. A necessary and sufficient condition on the weight function h to ensure $Tf = h \cdot f \circ \varphi$ defining an n-shift is given to this particular φ-tree.

Example 2.6. Let $T \colon c_0 \to c_0$ be a disjointness preserving linear operator defined by

$$T(x_1, x_2, \ldots) = (2x_1, x_1, x_2, \ldots).$$

Then $h(1) = 2$, $h(n) = 1$, $\varphi(1) = 1$, and $\varphi(n) = n - 1$ for $n \geq 2$. The φ-tree is

$$\begin{tikzcd} 1 \arrow[loop left] & 2 \arrow[l] & 3 \arrow[l] & \cdots \arrow[l] \end{tikzcd}.$$

It is clear that the φ-tree is the whole space $\mathbb{N}$, coincides with its crown and has one joint at 1. However, T is just a quasi-shift but not a shift, since $(1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots) \in \bigcap_{i=1}^\infty \operatorname{ran}(T^i)$. On the other hand, the operator sending $(x_1, x_2, \ldots)$ to $(x_1, x_1, x_2, \ldots)$ is a shift on c_0 giving rise to the same φ-tree.

In general, let h in $C(\mathbb{N})$ be bounded and away from zero. Then the weighted composition operator S on c_0 defined by $Sf = h \cdot f \circ \varphi$ is a quasi-shift. We shall show that S is a shift on c_0 if and only if

$$\limsup_{i \to \infty} \left| \frac{h(i+1) \cdots h(2)}{h(1)^i} \right| > 0.$$

Note that

$$S^i f(1) = h(1) \cdot S^{i-1} f(1) = h(1)^2 \cdot S^{i-2} f(1) = \cdots = h(1)^i \cdot f(1)$$

and

$$\begin{aligned} S^i f(i+1) &= h(i+1) \cdot S^{i-1} f(i) = h(i+1) \cdot h(i) \cdot S^{i-2} f(i-1) \\ &= \cdots = h(i+1) \cdots h(2) \cdot f(1). \end{aligned}$$

Hence $\frac{S^i f(i+1)}{h(i+1)\cdots h(2)} = f(1) = \frac{S^i f(1)}{h(1)^i}$ for all i in $\mathbb{N}$. Note also that if $g \in \operatorname{ran}(S^i)$ then $g \in \operatorname{ran}(S^j)$ for all $1 \leq j \leq i$. It follows that

$$\operatorname{ran}(S^i) = \Big\{ g \in C_0(\mathbb{N})\colon g(2) = \frac{h(2)}{h(1)} g(1), \\ g(3) = \frac{h(3)h(2)}{h(1)^2} g(1), \\ \vdots \\ g(i+1) = \frac{h(i+1)\cdots h(2)}{h(1)^i} g(1) \Big\}.$$

Therefore, $g \in \bigcap_{i=1}^{\infty} \operatorname{ran}(S^i)$ if and only if

$$g = \left(g(1), \frac{h(2)}{h(1)} g(1), \frac{h(3)h(2)}{h(1)^2} g(1), \dots, \frac{h(i+1)\cdots h(2)}{h(1)^i} g(1), \dots \right).$$

Consequently,

$$\bigcap_{i=1}^{\infty} \operatorname{ran}(S^i) = \{0\} \quad \text{if and only if} \quad \limsup_{i\to\infty} \left| \frac{h(i+1)\cdots h(2)}{h(1)^i} \right| > 0.$$

□

We are interested in the question of which φ-trees do provide us with a disjointness preserving n-shift regardless of the choice of the weight functions h. As a supplement to [6, Theorem 2.4], the following result states that every dense φ-tree rooted at ∞ does.

Theorem 2.7. *Suppose that a φ-tree is rooted at ∞, dense in X_∞ and has exactly n joints. Then for any bounded and away from zero continuous scalar function h on $X_c = X \setminus \{p_1, p_2, \dots, p_k\}$, where $p_1, \dots, p_k$ are all φ-vanishing points, the operator T, defined by $Tf_{|X_c} = h \cdot f \cdot \varphi$ and $Tf(p_1) = Tf(p_2) = \cdots = Tf(p_k) = 0$, is a disjointness preserving n-shift on $C_0(X)$.*

Proof. By Theorem 2.5, T is a quasi-n-shift. We only need to verify $\bigcap_{m=1}^{\infty} \operatorname{ran}(T^m) = \{0\}$. Suppose $g = T^m f$ for some f in $C_0(X)$ and $m \geq 1$. Then g vanishes at $\varphi^{-r}(p_i)$ for $r = 0, 1, 2, \dots, m-1$ and $i = 1, 2, \dots, k$. Consequently, every continuous function in $\bigcap_{m=1}^{\infty} \operatorname{ran}(T^m)$ vanishes on the whole φ-tree which is dense in X_∞. Hence, $\bigcap_{m=1}^{\infty} \operatorname{ran}(T^m) = \{0\}$ as asserted. Therefore, T is an n-shift. □

In the following example, we see that there are some φ-trees which provide us with no n-shift at all.

Example 2.8. Let

$$X = \Big\{ (-n, 0), (n, 1), (n, -1)\colon n = 1, 2, 3, \dots \Big\} \cup \Big\{ (0, 0) \Big\}$$

in $\mathbb{R}^2$. Let $\varphi\colon (X,\{(1,1),(1,-1)\}) \to (X,\{(0,0)\})$ be the relative homeomorphism defined by $\varphi(1,\pm 1)=(0,0)$, $\varphi(n+1,\pm 1)=(n,\pm 1)$ and $\varphi(-n+1,0)=(-n,0)$ for $n=1,2,\ldots$. The φ-tree has one joint at $(0,0)$, and is given below:

$$
\begin{array}{ccl}
 & & (1,1) \longleftarrow (2,1) \longleftarrow (3,1) \longleftarrow \cdots \quad . \\
 & \swarrow & \\
(0,0) & & \\
 & \nwarrow & \\
 & & (1,-1) \leftarrow (2,-1) \leftarrow (3,-1) \leftarrow \cdots
\end{array}
$$

We shall show that there is not any disjointness preserving shift T on $C_0(X)$ associated with this φ-tree; no matter how we define $Tf = h\cdot f\circ\varphi$ for any bounded and away from zero continuous scalar function h on X. To this end, we first note that

$$\operatorname{ran}(T) = \left\{ g\in C_0(X)\colon \frac{g(1,1)}{h(1,1)} = \frac{g(1,-1)}{h(1,-1)} \right\}$$

by Theorem 2.5. Similarly,

$$\operatorname{ran}(T^2) = \left\{ g\in C_0(X)\colon \frac{g(1,1)}{h(1,1)} = \frac{g(1,-1)}{h(1,-1)} \text{ and } \frac{g(2,1)}{h(2,1)h(1,1)} = \frac{g(2,-1)}{h(2,-1)h(1,-1)} \right\}.$$

In the same manner, we have for each positive integer m that

$$
\begin{aligned}
\operatorname{ran}(T^m) = \Big\{ g\in C_0(X)\colon & \frac{g(1,1)}{h(1,1)} = \frac{g(1,-1)}{h(1,-1)}, \\
& \frac{g(2,1)}{h(2,1)h(1,1)} = \frac{g(2,-1)}{h(2,-1)h(1,-1)}, \\
& \vdots \\
& \frac{g(m,1)}{h(m,1)\cdots h(1,1)} = \frac{g(m,-1)}{h(m,-1)\cdots h(1,-1)} \Big\}.
\end{aligned}
$$

It is then easy to see that the nonzero continuous function g_0 in $C_0(X)$, defined by $g_0(0,0)=1$ and $g_0=0$ elsewhere, does belong to all $\operatorname{ran}(T^m)$ for $m=1,2,\ldots$. In fact, $\bigcap_{m=1}^{\infty}\operatorname{ran}(T^m)$ has infinite codimension in $C_0(X)$. □

In dealing with the range space $\operatorname{ran}(T^m)$ of a power of a disjointness preserving quasi-n-shift T on $C_0(X)$, it is useful to consider the notion of "h-equipotential functions". Suppose $Tf = h\cdot f\circ\varphi$ on X_c. Denote

$$h\circ\varphi^{k!}(x) = h(x)h\big(\varphi(x)\big)\cdots h\left(\varphi^{k-1}(x)\right), \quad \forall x\in X_\infty, \forall k=1,2,\ldots.$$

We set $h_{|X_0\cup\{\infty\}}=1$ for convenience.

Definition 2.9. A function g in $C_0(X)$ is said to be *h-equipotential* on the φ-tree at level k, for $k=1,2,\ldots$, if $\frac{g(a)}{h\circ\varphi^{k!}(a)} = \frac{g(b)}{h\circ\varphi^{k!}(b)}$ whenever a, b are vertices in the φ-tree such that $\varphi^k(a)=\varphi^k(b)$.

Examples 2.6 and 2.8 are two demonstrations of the following lemma which is a consequence of Theorem 2.5.

Lemma 2.10. *Let T be a disjointness preserving quasi-n-shift on $C_0(X)$ and m be a positive integer. Then*

$$\mathrm{ran}(T^m) = \big\{g \in C_0(X)\colon g \textit{ is } h\textit{-equipotential on the } \varphi\textit{-tree arising from } T \textit{ up to level } m\big\}.$$

The following theorem says that $C_0(X)$ admits no disjointness preserving n-shift if X is inseparable for any $n = 1, 2, \dots$.

Theorem 2.11. *Let T be a disjointness preserving n-shift on $C_0(X)$. Then the crown of the φ-tree arising from T is dense in X. In particular, X is separable.*

Proof. Recall that the crown of the φ-tree is the union of all φ-branches originated at φ-merging points. Suppose g in $C_0(X)$ vanishes on the crown of the φ-tree. Then g is in $\mathrm{ran}(T^m)$ for $m = 1, 2, \dots$, by Lemma 2.10. As a result, $\bigcap_{m=1}^{\infty} \mathrm{ran}(T^m)$ contains the subspace $\big\{g \in C_0(X)\colon g$ vanishes on the crown of the φ-tree$\big\}$. If T is an n-shift, then $\bigcap_{m=1}^{\infty} \mathrm{ran}(T^m) = \{0\}$, and thus, the crown of the φ-tree is dense in X. In this case, X is separable since the crown of the φ-tree is a countable set. □

3. Writing (quasi-)n-shifts on c_0 as products of n (quasi)-shifts

This section is devoted to a comprehensive study of disjointness preserving n-shifts on c_0 ($\cong C_0(\mathbb{N})$). In the following two lemmas, $\varphi\colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ will be a continuous surjective map such that $\varphi(\infty) = \infty$.

Lemma 3.1. *Suppose a φ-tree in $\mathbb{N}_\infty$ has exactly n joints and its crown contains $\mathbb{N}$. Then*

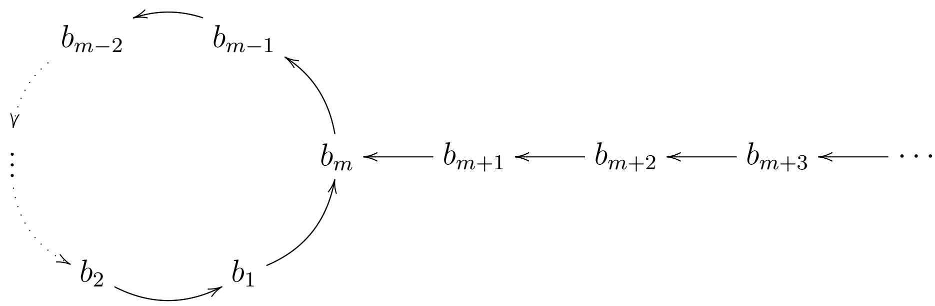
(1) the set $\{\varphi^l(x)\colon l \in \mathbb{N}\}$ is finite for all x in $\mathbb{N}$;
(2) $\mathbb{N}$ is the disjoint union $\bigcup B_{a_i}$ of the branches $B_{a_i} = \{\varphi^{-n}(a_i)\colon n \in \mathbb{N}\}$ of the φ-tree originated at some merging points a_i for which either $\varphi(a_i) = \infty$ or $\varphi^m(a_i) = a_i$ for some m in $\mathbb{N}$.

Proof. Suppose the set $\{\varphi^l(x)\colon l \in \mathbb{N}\}$ is infinite for some x in $\mathbb{N}$. Since the number of merging points is finite, there exists N in $\mathbb{N}$ such that $\{\varphi^l(x)\colon l \geq N\} \cap M_\varphi = \emptyset$. As the crown of the φ-tree contains $\mathbb{N}$, we have $\varphi^N(x) \in B_a = \bigcup\{\varphi^{-l}(a)\colon l \in \mathbb{N}\}$, the branch of the φ-tree originated at some merging point a in M_φ. Then $\varphi^N(x) \in \varphi^{-m}(a)$ for some m in $\mathbb{N}$. Hence $\varphi^{N+m}(x) = a \in M_\varphi$, a contradiction. Thus $\{\varphi^l(x)\colon l \in \mathbb{N}\}$ is finite for all x in $\mathbb{N}$. This gives (1), and in particular, $\{\varphi^l(a)\colon l \in \mathbb{N}\}$ is finite for all merging points a. It is easy to see if there is no other merging point in $\{\varphi^l(a)\colon l \in \mathbb{N}\}$ and $\varphi(a) \neq \infty$, then there exists a positive integer m such that $\varphi^m(a) = a$. We take $a_1, a_2, \dots, a_k$ to be such merging points together with those $a \neq \infty$ such that $\varphi(a) = \infty$. Finally, we note that if a_i and a_j are two distinct merging points satisfying $a_i \notin B_{a_j}$ and $a_j \notin B_{a_i}$ then $B_{a_i} \cap B_{a_j} = \emptyset$. This gives (2). $\square$

Lemma 3.2. *Suppose a φ-tree in $\mathbb{N}_\infty$ has n joints. Then the crown of the φ-tree contains $\mathbb{N}$ if and only if any (and thus every) weighted composition operator $Tf = h \cdot f \circ \varphi$ is an n-shift on c_0, where h is a unimodular function on $\mathbb{N}$, i.e. $|h(x)| \equiv 1$ for all x in $\mathbb{N}$.*

Proof. By Theorem 2.11, we need to verify the necessity only. Suppose that the crown of the φ-tree contains $\mathbb{N}$. By Theorem 2.5, it is enough to show that $\bigcap_{l=1}^\infty \operatorname{ran}(T^l) = \{0\}$. By Lemma 3.1, $\mathbb{N}$ is a disjoint union of the φ-branches originated at some φ-merging points $a_1, \ldots, a_j$, where either $\varphi(a_i) = \infty$ or $\varphi^m(a_i) = a_i$ for some $m > 0$.

Let $g \in \bigcap_{i=1}^\infty \operatorname{ran}(T^i)$. If $\varphi(a_1) = \infty$ then g vanishes on the branch of the φ-tree originated at a_1 by Lemma 2.10. Suppose otherwise $\varphi^m(a_1) = a_1$ for some positive integer m. Choose distinct $b_1 = a_1$ and $b_i \in \varphi^{-i+1}(a_1)$ with $\varphi(b_{i+1}) = b_i$. A part of the branch of the φ-tree originated at $b_1 = a_1$ looks like:



Then $\varphi^i(b_i) = \varphi^i(b_{m+i}) = b_m$ for all i in $\mathbb{N}$. By Lemma 2.10 again, $|g(b_i)| = |g(b_{m+i})|$ for all i in $\mathbb{N}$. Note that $\{b_{nm+i}\}_{n=1}^\infty$ converges to ∞ for $i = 1, \ldots, m$. This implies $g(b_i) = 0$ for $i = 1, 2, 3, \ldots$. We thus conclude again in this case that g vanishes on the branch of the φ-tree originated at a_1. In a similar manner, we assert that g vanishes on the branches of the φ-tree originated at all other merging points $a_2, \ldots, a_j$, and thus on the crown of the φ-tree which contains $\mathbb{N}$. This gives $g = 0$. Consequently, T is an n-shift. $\square$

Corollary 3.3. *Isometric disjointness preserving shifts T on c_0 are exactly those in one of the following forms:*

$$T((x_1, x_2, \ldots, x_m, \ldots)) = (0, \lambda_2 x_1, \lambda_3 x_2, \ldots, \lambda_{m+1} x_m, \ldots),$$

or

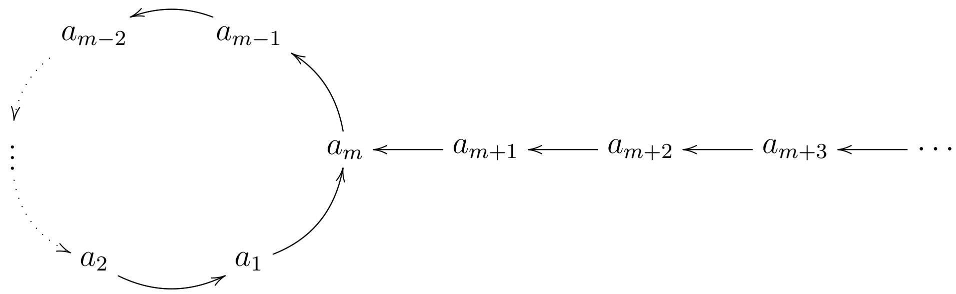
$$T((x_1, x_2, \ldots, x_m, \ldots)) = (\lambda_1 x_m, \lambda_2 x_1, \lambda_3 x_2, \ldots, \lambda_{m+1} x_m, \ldots), \quad m = 1, 2, 3, \ldots,$$

after reordering the standard basis of c_0, if necessarily, where $|\lambda_k| = 1$ for $k = 1, 2, 3, \ldots$.

Proof. It is indeed a direct consequence of Lemmas 3.1 and 3.2. More precisely, if $Tf = h \cdot f \circ \varphi$ then the φ-tree will be either rooted at ∞ or the one with a loop of m elements. In other words, the φ-tree of T is in either one of the following two forms.

$$\begin{tikzcd} \infty \arrow[loop left] & a_1 \arrow[l] & a_2 \arrow[l] & a_3 \arrow[l] & \cdots, \arrow[l] \end{tikzcd}$$

or



After reordering the standard basis of c_0, if necessarily, and then setting $\lambda_k = h(k)$ for $k = 1, 2, 3, \ldots$, we will arrive at the desired conclusion. $\square$

Theorem 3.4. *Let T be an isometric disjointness preserving n-shift on c_0. Then T can be written as a product of n isometric disjointness preserving shifts on c_0.*

Proof. Let $\varphi\colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ be a continuous surjective map with $\varphi(\infty) = \infty$ such that $Tf_{|X_c} = h \cdot f \circ \varphi$, where $X_c = \{p \in \mathbb{N}\colon \varphi(p) \neq \infty\}$ and h is continuous on X_c with $|h(x)| \equiv 1$. Since $\mathbb{N}$ is discrete, we may extend h continuously to $\mathbb{N}$ by setting $h_{|\mathbb{N}\setminus X_c} \equiv 1$. By Theorem 2.5 and Lemma 3.2, the φ-tree has exactly n joints and the crown of the φ-tree contains $\mathbb{N}$.

We claim that there exist n continuous maps $\varphi_1, \ldots, \varphi_n$ from $\mathbb{N}_\infty$ onto $\mathbb{N}_\infty$ sending ∞ to ∞ such that every φ_i-tree has exactly 1 joint, the crown of the φ_i-tree contains $\mathbb{N}$ and $\varphi = \varphi_n \circ \cdots \circ \varphi_1$.

By Lemma 3.1, there exist n disjoint sequences $\left\{a_m^{(i)}\right\}_{m=1}^{\infty}$, $i = 1, \ldots, n$, such that $a_m^{(i)} = \varphi\left(a_{m+1}^{(i)}\right)$ and $\mathbb{N} = \bigcup_{i=1}^{n}\left\{a_m^{(i)}\colon m \in \mathbb{N}\right\}$. Note that the φ-tree has n joints and we may need to make a cut at each subsequent merged point in those initial φ-branches given in Lemma 3.1. Let φ^* be the continuous map from $\mathbb{N}_\infty \setminus \{a_1^{(1)}\}$ onto $\mathbb{N}_\infty$ defined by

$$\begin{aligned} \varphi^*(a_j^{(i)}) &= a_j^{(i-1)}, \quad i = 2, \ldots, n \text{ and } j \in \mathbb{N}, \\ \varphi^*(a_{j+1}^{(1)}) &= a_j^{(n)}, \quad j \in \mathbb{N}, \end{aligned}$$

and

$$\varphi^*(\infty) = \infty.$$

It is easy to see that φ^* is bijective,

$$(\varphi^*)^n(a) = \varphi(a) \quad \text{for all } a \neq a_1^{(i)}, \tag{1}$$

and

$$(\varphi^*)^{(i-1)}(a_1^{(i)}) = a_1^{(1)} \quad \text{for } i = 2, \ldots, n. \tag{2}$$

Now define $\varphi_i \colon \mathbb{N}_\infty \to \mathbb{N}_\infty$, $i = 1, \ldots, n$, by

$$\varphi_{i_{|\mathbb{N}_\infty \setminus \{a_1^{(1)}\}}} = \varphi^*$$

and

$$\varphi_i(a_1^{(1)}) = \begin{cases} \infty, & \text{if } \varphi(a_1^{(i)}) = \infty, \\ (\varphi^*)^{-(n-i)}(b_i), & \text{if } \varphi(a_1^{(i)}) = b_i \neq \infty. \end{cases}$$

By (1), to see $\varphi = \varphi_n \circ \cdots \circ \varphi_2 \circ \varphi_1$ we only need to check $\varphi_n \circ \cdots \circ \varphi_1(a_1^{(i)}) = \varphi(a_1^{(i)})$ for all $i = 1, \ldots, n$. In fact, if $\varphi(a_1^{(i)}) = \infty$ then by (2) we have

$$\varphi_n \circ \cdots \circ \varphi_1(a_1^{(i)}) = \varphi_n \circ \cdots \circ \varphi_i(a_1^{(1)}) = \varphi_n \circ \cdots \circ \varphi_{i+1}(\infty) = \infty.$$

If $\varphi(a_1^{(i)}) = b_i \neq \infty$, then

$$\begin{aligned} \varphi_n \circ \cdots \circ \varphi_1(a_1^{(i)}) &= \varphi_n \circ \cdots \circ \varphi_i(a_1^{(1)}) = \varphi_n \circ \cdots \circ \varphi_{i+1}((\varphi^*)^{-(n-i)}(b_i)) \\ &= (\varphi^*)^{n-i}((\varphi^*)^{-(n-i)}(b_i)) = b_i. \end{aligned}$$

Hence, $\varphi = \varphi_n \circ \cdots \circ \varphi_2 \circ \varphi_1$.

It is clear that φ_i is continuous from $\mathbb{N}_\infty$ onto $\mathbb{N}_\infty$ satisfying that $\varphi_i(\infty) = \infty$, the φ_i-tree has exactly 1 joint at $\varphi_i(a_1^{(1)})$, and the crown of the φ_i-tree contains $\mathbb{N}$. Now define $T_i \colon c_0 \to c_0$, $i = 1, \ldots, n$, by

$$T_1 f(x) = \begin{cases} h(x) \cdot f \circ \varphi_1(x), & \text{if } \varphi_1(x) \neq \infty, \\ 0, & \text{if } \varphi_1(x) = \infty, \end{cases}$$

and

$$T_i f(x) = \begin{cases} f \circ \varphi_i(x), & \text{if } \varphi_i(x) \neq \infty, \\ 0, & \text{if } \varphi_i(x) = \infty, \end{cases} \quad i = 2, 3, \ldots, n.$$

By Lemma 3.2, $T_1, \ldots, T_n$ are isometric disjointness preserving shifts. It is plain that $T = T_1 \circ \cdots \circ T_n$. □

The following example demonstrates the idea we employed in the proof above.

Example 3.5. Let T be a 5-shift on c_0 defined by

$$\begin{aligned} &T(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, \ldots) = \\ &\quad (0, x_{13}, x_1, x_2, x_2, x_3, x_3, x_4, x_5, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, \ldots). \end{aligned}$$

Then the φ-tree is

$$\begin{tikzcd}[column sep=small, row sep=small]
\infty \arrow[loop left] & 1 \arrow[l] & 3 \arrow[l] & 6 \arrow[l] & 11 \arrow[l] & 16 \arrow[l] & 21 \arrow[l] & \cdots \arrow[l] . \\
 & & & 7 \arrow[ul] & 12 \arrow[l] & 17 \arrow[l] & 22 \arrow[l] & \cdots \arrow[l] \\
 & 2 \arrow[rrr, bend left] & 4 \arrow[l] & 8 \arrow[l] & 13 \arrow[l] & 18 \arrow[l] & 23 \arrow[l] & \cdots \arrow[l] \\
 & & 5 \arrow[ul] & 9 \arrow[l] & 14 \arrow[l] & 19 \arrow[l] & 24 \arrow[l] & \cdots \arrow[l] \\
 & & & 10 \arrow[ul] & 15 \arrow[l] & 20 \arrow[l] & 25 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

First, we relabel the φ-tree as in the following. Note that the vertices a_1, a_2, a_3, a_4 and a_5 are the pivots in our machinery.

$$\begin{tikzcd}[column sep=small, row sep=small]
\infty \arrow[loop left] & a_1 \arrow[l] & a_6 \arrow[l] & a_{11} \arrow[l] & a_{16} \arrow[l] & a_{21} \arrow[l] & a_{26} \arrow[l] & \cdots \arrow[l] . \\
 & & & a_2 \arrow[ul] & a_7 \arrow[l] & a_{12} \arrow[l] & a_{17} \arrow[l] & \cdots \arrow[l] \\
 & a_3 \arrow[rrr, bend left] & a_8 \arrow[l] & a_{13} \arrow[l] & a_{18} \arrow[l] & a_{23} \arrow[l] & a_{28} \arrow[l] & \cdots \arrow[l] \\
 & & a_4 \arrow[ul] & a_9 \arrow[l] & a_{14} \arrow[l] & a_{19} \arrow[l] & a_{24} \arrow[l] & \cdots \arrow[l] \\
 & & & a_5 \arrow[ul] & a_{10} \arrow[l] & a_{15} \arrow[l] & a_{20} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Let φ^* be the continuous map from $\mathbb{N}_\infty \setminus \{a_1\}$ onto $\mathbb{N}_\infty$ defined by

$$\varphi^*(\infty) = \infty \quad \text{and} \quad \varphi^*(a_{n+1}) = a_n, \quad \forall n \in \mathbb{N}.$$

Observe that $\varphi(a_1) = \infty$, $\varphi(a_2) = a_6$, $\varphi(a_3) = a_{18}$, $\varphi(a_4) = a_3$ and $\varphi(a_5) = a_4$. Following the proof of Theorem 3.4, we let $\varphi_1(a_1) = \infty$, $\varphi_2(a_1) = (\varphi^*)^{-3}(a_6) = a_9$, $\varphi_3(a_1) = (\varphi^*)^{-2}(a_{18}) = a_{20}$, $\varphi_4(a_1) = (\varphi^*)^{-1}(a_3) = a_4$ and $\varphi_5(a_1) = a_4$. Moreover, we set $\varphi_i = \varphi^*$ elsewhere for $i = 1, 2, 3, 4, 5$. The φ_i-trees are given below.

$$\begin{tikzcd}[row sep=tiny]
\varphi_1 : & & \infty \arrow[loop left] & a_1 \arrow[l] & a_2 \arrow[l] & a_3 \arrow[l] & a_4 \arrow[l] & \cdots, \arrow[l] \\
\varphi_2 : & a_1 \arrow[rrrrrr, bend left] & a_2 \arrow[l] & a_3 \arrow[l] & \cdots \arrow[l] & a_8 \arrow[l] & a_9 \arrow[l] & a_{10} \arrow[l] & \cdots, \arrow[l] \\
\varphi_3 : & a_1 \arrow[rrrrrr, bend left] & a_2 \arrow[l] & a_3 \arrow[l] & \cdots \arrow[l] & a_{19} \arrow[l] & a_{20} \arrow[l] & a_{21} \arrow[l] & \cdots, \arrow[l] \\
\varphi_4 : & a_1 \arrow[rrr, bend left] & a_2 \arrow[l] & a_3 \arrow[l] & a_4 \arrow[l] & a_5 \arrow[l] & \cdots, \arrow[l] \\
\varphi_5 : & a_1 \arrow[rrr, bend left] & a_2 \arrow[l] & a_3 \arrow[l] & a_4 \arrow[l] & a_5 \arrow[l] & \cdots. \arrow[l]
\end{tikzcd}$$

It is easy to see that $\varphi = \varphi_5 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$. In its original notations, we have

$$\begin{tikzcd}[row sep=tiny]
\varphi_1 : & & \infty \arrow[loop left] & 1 \arrow[l] & 7 \arrow[l] & 2 \arrow[l] & 5 \arrow[l] & \cdots, \arrow[l] \\
\varphi_2 : & 1 \arrow[rrrrrr, bend left] & 7 \arrow[l] & 2 \arrow[l] & \cdots \arrow[l] & 4 \arrow[l] & 9 \arrow[l] & 15 \arrow[l] & \cdots, \arrow[l] \\
\varphi_3 : & 1 \arrow[rrrrrr, bend left] & 7 \arrow[l] & 2 \arrow[l] & \cdots \arrow[l] & 19 \arrow[l] & 25 \arrow[l] & 16 \arrow[l] & \cdots, \arrow[l] \\
\varphi_4 : & 1 \arrow[rrr, bend left] & 7 \arrow[l] & 2 \arrow[l] & 5 \arrow[l] & 10 \arrow[l] & \cdots, \arrow[l] \\
\varphi_5 : & 1 \arrow[rrr, bend left] & 7 \arrow[l] & 2 \arrow[l] & 5 \arrow[l] & 10 \arrow[l] & \cdots. \arrow[l]
\end{tikzcd}$$

Let $T_i f = f \circ \varphi_i$ for $i = 1, 2, \ldots, 5$. Then we have 5 isometric disjointness preserving shifts on c_0 such that $T = T_1 \circ T_2 \circ T_3 \circ T_4 \circ T_5$. □

Corollary 3.6. *Let $Tf = h \cdot f \circ \varphi$ be a disjointness preserving n-shift on c_0, or a quasi-n-shift on c_0 such that the crown of the φ-tree contains $\mathbb{N}$. Then there exist n isometric disjointness preserving shifts $S_1, \ldots, S_n$ on c_0 such that*

$$T = h \cdot S_1 \circ \cdots \circ S_n.$$

Proof. Apply Lemma 3.2 and Theorem 3.4 to the isometry $f \mapsto f \circ \varphi$. □

In Example 2.6, there is a disjointness preserving quasi-shift on c_0, which is not a shift but the crown of its φ-tree contains $\mathbb{N}$. On the other hand, in Example 4.2 below, we shall have an isometric disjointness preserving quasi-2-shift on c_0, which can be written as a product of two isometric disjointness preserving shifts on c_0 but its φ-tree does not contain the whole of $\mathbb{N}$. In particular, the converse of Corollary 3.6 does not hold.

4. Simplifying (quasi-)n-shifts on $C_0(X)$

By definition, we have

Proposition 4.1. *The product of a disjointness preserving quasi-m-shift and a disjointness preserving quasi-n-shift on $C_0(X)$ is a disjointness preserving quasi-$(m+n)$-shift on $C_0(X)$.*

In contrast to Proposition 4.1, the following example tells us that the product of any n 1-shifts may not be an n-shift.

Example 4.2. Let S_1 and S_2 be the isometric disjointness preserving shifts on c_0 defined by

$$S_1(x_1, x_2, x_3, x_4, \cdots) = (x_2, x_1, x_2, x_3, x_4, \cdots),$$
$$S_2(x_1, x_2, x_3, x_4, \cdots) = (x_2, x_1, x_1, x_3, x_4, \cdots).$$

Then

$$(S_1 \circ S_2)(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (x_1, x_2, x_1, x_1, x_3, x_4, x_5, x_6, \ldots)$$

is a quasi-2-shift, but not a 2-shift, since $(0, 1, 0, 0, 0, \ldots) \in \bigcap_{m=1}^{\infty} \operatorname{ran}(T^m)$. □

Modifying the proof given for the case $n = 1$ in [3], we have the following result which says every n-shift on a Banach space E is similar to a 'classical' n-shift on a sequence space E_S.

Proposition 4.3. *Suppose T is an n-shift on the Banach space E. Then there exists a Banach space E_S of scalar sequences, isomorphic and isometric to E, such that on E_S the n-shift T corresponds to the operator T_S defined by*

$$T_S(a_1, a_2, \dots) = (\underbrace{0, \cdots, 0}_{n}, a_1, a_2, \dots).$$

Proof. Since T has closed range and corank n, there exist n elements $x_1, x_2, \dots, x_n$ in E linear independent modulo TE such that E is the Banach space direct sum

$$E = \operatorname{span}\{x_1, x_2, \dots, x_n\} \oplus TE.$$

Let $y \in E$. Then there exist n unique scalars $a_1(y), a_2(y), \dots, a_n(y)$ and an element y_1 in E such that

$$y = a_1(y)x_1 + a_2(y)x_2 + \cdots + a_n(y)x_n + Ty_1.$$

Since T is injective, the choice of y_1 is unique. Similarly, there exist another n unique scalars $a_{n+1}(y), a_{n+2}(y), \dots, a_{2n}(y)$ and a unique element y_2 in E such that

$$y_1 = a_{n+1}(y)x_1 + a_{n+2}(y)x_2 + \cdots + a_{2n}(y)x_n + Ty_2.$$

Thus,

$$\begin{aligned} y =& a_1(y)x_1 + a_2(y)x_2 + \cdots + a_n(y)x_n \\ &+ a_{n+1}(y)Tx_1 + a_{n+2}(y)Tx_2 + \cdots + a_{2n}(y)Tx_n + T^2y_2. \end{aligned}$$

By induction, there exist a unique sequence of scalars $\{a_m(y)\}_{m=1}^{\infty}$ and a unique sequence of vectors $\{y_m\}_{m=1}^{\infty}$ in E such that for $m = 1, 2, \dots,$ we have

$$y = \sum_{k=0}^{m} \left(a_{kn+1}(y)T^k x_1 + a_{kn+2}(y)T^k x_2 + \cdots + a_{kn+n}(y)T^k x_n\right) + T^{m+1}y_{m+1}. \tag{3}$$

Let E_S denote the vector space of sequences $\{a_m(y)\}_{m=1}^{\infty}$. The mapping $y \mapsto \{a_m(y)\}_{m=1}^{\infty}$ is linear and maps E onto E_S. Since $\bigcap_{k=1}^{\infty} \operatorname{ran}(T^k) = \{0\}$, no non-zero vector is maped to the zero sequence. Thus the correspondence is a linear isomorphism.

Let $\|\{a_m(y)\}_{m=0}^{\infty}\|$ be defined as $\|y\|$. Then the two spaces are isometric, and E_S is a Banach space. Equation (3) implies that

$$Ty = \sum_{k=0}^{m} \left(a_{kn+1}(y)T^{k+1}x_1 + a_{kn+2}(y)T^{k+1}x_2 + \cdots + a_{kn+n}(y)T^{k+1}x_n\right) + T^{m+2}y_{m+1}.$$

Therefore, the corresponding sequence for Ty is $\{\underbrace{0, 0, \dots, 0}_{n}, a_1(y), a_2(y), \dots\}$. Thus T is similar to the 'classical' n-shift T_S on E_S. □

It is plausible that $T = T_1^n$ where T_1 is induced by the unilateral shift sending $(x_1, x_2, \dots)$ to $(0, x_1, x_2, \dots)$. However, it is not necessarily true that $(0, x_1, x_2, \dots)$ belongs to E_S when $(x_1, x_2, \dots)$ does. Even if it is the case, the shift operator T_1 need not be disjointness preserving on E when T is. Thus, this idea may not be implementable in some cases. We shall see in the following two examples that such a hope is indeed fruitless.

Example 4.4. This example tells us that there exists a compact Hausdorff space X such that $C(X)$ admits an isometric disjointness preserving 2-shift which cannot be written as a product of two disjointness preserving shifts.

Let $X = \big\{(\frac{1}{n}, i)\colon n \in \mathbb{N} \text{ and } i = 0, 1, 2\big\} \cup \big\{(0,0), (0,1), (0,2)\big\}$. Then X is a compact Hausdorff space contained in $\mathbb{R}^2$. Note that ∞ is an isolated point in $X_\infty = X \cup \{\infty\}$. Define $\varphi\colon X_\infty \to X_\infty$ by

$$\varphi(1,0) = \varphi(1,2) = \varphi(\infty) = \infty,$$

$$\varphi(\frac{1}{n+1}, 0) = (\frac{1}{n}, 0),\ \ \varphi(\frac{1}{n}, 1) = (\frac{1}{n}, 2) \text{ and } \varphi(\frac{1}{n+1}, 2) = (\frac{1}{n}, 1), \quad \forall n \in \mathbb{N},$$

and

$$\varphi(0,0) = (0,0), \quad \varphi(0,1) = (0,2), \quad \varphi(0,2) = (0,1). \tag{4}$$

Define $T\colon C(X) \to C(X)$ by

$$\begin{aligned} Tf(x) &= f(\varphi(x)), \quad \forall x \neq (1,0), (1,2), \\ Tf(1,0) &= 0 \quad \text{and} \quad Tf(1,2) = 0. \end{aligned}$$

By Theorem 2.7, T is a disjointness preserving 2-shift. We shall show that T cannot be written as a product of two disjointness preserving shifts.

We first make some general observations. Let $\psi\colon (X_\infty, M_\psi) \to (X_\infty, \psi(M_\psi))$ be a relative homeomorphism induced by a shift on $C(X)$, where $M_\psi = \{a, b\}$ is the set of all two ψ-merging points with $b \neq \infty$. Since ψ maps cluster points to cluster points, we have

$$\big\{\psi(0,0), \psi(0,1), \psi(0,2)\big\} \subseteq \big\{(0,0), (0,1), (0,2)\big\}. \tag{5}$$

We are going to show that ψ maps $\big\{(0,0), (0,1), (0,2)\big\}$ onto itself without fixing any point.

We claim

$$M_\psi = \{a, b\} \not\subset \{(0,0), (0,1), (0,2)\}. \tag{6}$$

If it is not the case then $X_\infty \setminus \{a, b\}$ has only one cluster point while $X_\infty \setminus \{\psi(b)\}$ has two cluster points. It is impossible since ψ is a homeomorphism from $X_\infty \setminus \{a, b\}$ onto $X_\infty \setminus \{\psi(b)\}$. As a consequence, the equality in (5) holds.

Since the ψ-tree contains $X \setminus \{(0,0), (0,1), (0,2)\}$ (Theorem 2.11), a similar argument as in the proof of Lemma 3.1 will give us that $\{\psi^n(x)\colon n \in \mathbb{N}\}$ is a finite set for every x

in X. Hence we can assume $\psi^m(a) = a$ for some positive integer m. Note that a can be the isolated point ∞. The ψ-tree is exactly the branch originated at a, i.e.,

$$\begin{tikzcd}
\vdots \arrow[rr, bend left] & & a \arrow[dl] & & & \\
& \psi(a) \arrow[ul, bend left] & b \arrow[l] & \psi^{-1}(b) \arrow[l] & \psi^{-2}(b) \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{7}$$
.

In this case, b must be an isolated point in X. In fact, if $b \in \{(0,0),(0,1),(0,2)\}$ then $\psi^i(b) \in \{(0,0),(0,1),(0,2)\}$ for all i in $\mathbb{N}$ by (5). As $a = \psi^m(a) = \psi^m(b)$, we have $\{a,b\} \subset \{(0,0),(0,1),(0,2)\}$, a contradiction to (6). Since $\psi\colon \big(X_\infty,\{a,b\}\big) \to \big(X_\infty,\{\psi(b)\}\big)$ is a relative homeomorphism and both b and ∞ are isolated, $\psi\colon X\setminus\{b\} \to X$ is a homeomorphism. Let Ψ be the inverse of $\psi_{|X\setminus\{b\}}$. Then Ψ is a homeomorphism from X onto $X \setminus \{b\}$.

Now, we claim

$$\psi(x) \neq x \quad \text{ if } x = (0,0),(0,1) \text{ or } (0,2).$$

Suppose not and assume, for example, that $\psi(0,0) = (0,0)$ and thus $\Psi(0,0) = (0,0)$. Let

$$A_0 = \{(\frac{1}{n},0)\colon n \in \mathbb{N}\} \cup \{(0,0)\}.$$

By the continuity of Ψ, there are at most finitely many points $z_1,\dots,z_k$ in the *open* set A_0 such that

$$\Psi(x) \in A_0, \qquad \text{for } x \in A_0 \setminus \{z_1,\dots,z_k\}. \tag{8}$$

Recall that the ψ-tree $\big\{a,\psi(a),\dots,\psi^{m-1}(a)\big\} \cup \big\{b,\Psi(b),\Psi^2(b),\dots\big\}$ displayed in (7) contains $X \setminus \{(0,0),\ (0,1),\ (0,2)\}$. Then there exist positive integers N_0 and N_1 with $N_1 > N_0$ such that

$$z_1,\dots,z_k \in \big\{a,\psi(a),\dots,\psi^{m-1}(a)\big\} \cup \big\{b,\Psi(b),\dots,\Psi^{N_0}(b)\big\} \tag{9}$$

and

$$\Psi^{N_1}(b) \in A_0.$$

It follows from (8) and (9) that

$$\Psi^{N_1+k}(b) \in A_0, \quad \forall k = 1,2,\dots .$$

This implies the whole ψ-tree is contained in A_0 eventually. Thus it is not dense in X, a contradiction.

At this moment, we arrive at the conclusion that for every relative homeomorphism ψ arising from a disjointness preserving shift on $C(X)$ either one of the following two alternatives holds; namely,

$$\psi(0,0) = (0,1), \quad \psi(0,1) = (0,2), \quad \psi(0,2) = (0,0) \tag{10}$$

or

$$\psi(0,0) = (0,2), \quad \psi(0,1) = (0,0), \quad \psi(0,2) = (0,1). \tag{11}$$

We are now ready to verify that T cannot be written as a product of two disjointness preserving shifts on $C(X)$. Suppose, on the contrary, there were two disjointness preserving shifts S_1 and S_2 on $C(X)$ such that $T = S_1 \circ S_2$. Let $\psi_i \colon X_\infty \to X_\infty$ be the relative homeomorphism induced by S_i for $i = 1, 2$. This gives $\varphi(x) = \psi_2(\psi_1(x))$. However, this cannot be true by (4), (10) and (11). Hence T cannot be written as a product of two disjointness preserving shifts. $\square$

In Example 4.4, although the 2-shift T cannot be written as a product of two shifts, there are anyway two quasi-shifts (not shifts) T_1 and T_2 on $C(X)$ such that $T = T_1 \circ T_2$. In fact, let $\varphi_1 \colon X \setminus \{(1,0)\} \to X$ and $\varphi_2 \colon X \setminus \{(1,2)\} \to X$ be homeomorphisms defined by

$$\varphi_{1|A_0} = \varphi_{|A_0} \quad \text{and} \quad \varphi_1(x) = x, \ \forall x \in X \setminus A_0,$$

and

$$\varphi_{2|X\setminus A_0} = \varphi_{|X\setminus A_0} \quad \text{and} \quad \varphi_2(x) = x, \ \forall x \in A_0.$$

Then the weighted composition operators $T_i f = f \circ \varphi_i$, $i = 1, 2$, are quasi-shifts on $C(X)$. It is easy to see that $T = T_1 \circ T_2$. Nevertheless, we can have a situation even worse than this.

Example 4.5. This example tells us that there is a compact connected Hausdorff space X such that $C(X)$ admits an isometric disjointness preserving quasi-2-shift but no disjointness preserving quasi-shift at all. As a result, a disjointness preserving quasi-2-shift need not be a product of two disjointness preserving quasi-shifts.

For x, y in $\mathbb{R}^2$, let

$$l(x,y) = \{tx + (1-t)y \colon 0 \le t \le 1\}$$

be the line segment joining x and y in $\mathbb{R}^2$. Denote by $re^{i\theta}$ the point $(r\cos\theta, r\sin\theta)$ in $\mathbb{R}^2$ and by

$$\mathrm{arc}(re^{i\theta_1}, re^{i\theta_2}) = \left\{ re^{i(t\theta_1 + (1-t)\theta_2)} \colon 0 \le t \le 1 \right\}$$

the circular arc joining $re^{i\theta_1}$ to $re^{i\theta_2}$ in $\mathbb{R}^2$. Let O denote the origin $(0,0)$ in $\mathbb{R}^2$. We are going to construct a compact connected space X contained in the closed united disk in $\mathbb{R}^2$. Let

$$\begin{aligned} A_1 &= l(O, e^{i\frac{3\pi}{4}}) \cup l(\tfrac{1}{2}e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{8}}) \cup l(\tfrac{1}{2}e^{i\frac{3\pi}{4}}, e^{i\frac{7\pi}{8}}), \\ A_2 &= l(O, \tfrac{1}{2}e^{i\frac{3\pi}{8}}) \cup l(\tfrac{1}{4}e^{i\frac{3\pi}{8}}, \tfrac{1}{2}e^{i\frac{5\pi}{16}}) \cup l(\tfrac{1}{4}e^{i\frac{3\pi}{8}}, \tfrac{1}{2}e^{i\frac{7\pi}{16}}), \end{aligned}$$

and, in general,

$$A_n = l(O, \tfrac{1}{2^{n-1}}e^{i\frac{3\pi}{2^{n+1}}}) \cup l(\tfrac{1}{2^n}e^{i\frac{3\pi}{2^{n+1}}}, \tfrac{1}{2^{n-1}}e^{i\frac{5\pi}{2^{n+2}}}) \cup l(\tfrac{1}{2^n}e^{i\frac{3\pi}{2^{n+1}}}, \tfrac{1}{2^{n-1}}e^{i\frac{7\pi}{2^{n+2}}}),$$

for $n = 1, 2, \ldots$. Similarly, we let

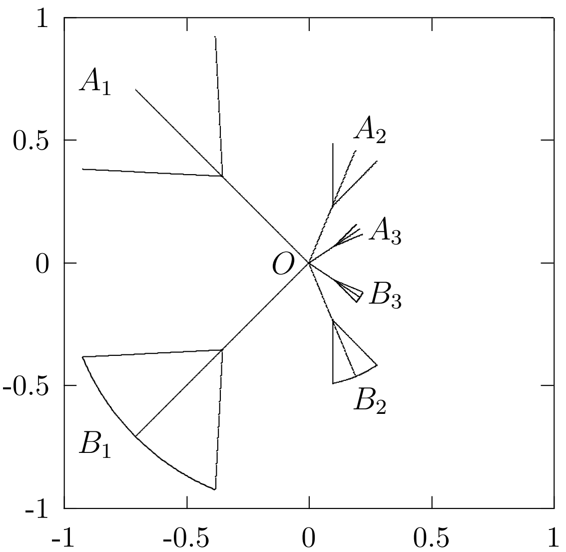
$$B_1 = l(O, e^{-i\frac{3\pi}{4}}) \cup l(\tfrac{1}{2}e^{-i\frac{3\pi}{4}}, e^{-i\frac{5\pi}{8}}) \cup l(\tfrac{1}{2}e^{-i\frac{3\pi}{4}}, e^{-i\frac{7\pi}{8}}) \cup \operatorname{arc}(e^{-i\frac{5\pi}{8}}, e^{-i\frac{7\pi}{8}}),$$

$$B_2 = l(O, \tfrac{1}{2}e^{-i\frac{3\pi}{8}}) \cup l(\tfrac{1}{4}e^{-i\frac{3\pi}{8}}, \tfrac{1}{2}e^{-i\frac{5\pi}{16}}) \cup l(\tfrac{1}{4}e^{-i\frac{3\pi}{8}}, \tfrac{1}{2}e^{-i\frac{7\pi}{16}}) \cup \operatorname{arc}(\tfrac{1}{2}e^{-i\frac{5\pi}{16}}, \tfrac{1}{2}e^{-i\frac{7\pi}{16}}),$$

and, in general,

$$B_n = l(O, \tfrac{1}{2^{n-1}}e^{-i\frac{3\pi}{2^{n+1}}}) \cup l(\tfrac{1}{2^n}e^{-i\frac{3\pi}{2^{n+1}}}, \tfrac{1}{2^{n-1}}e^{-i\frac{5\pi}{2^{n+2}}}) \cup l(\tfrac{1}{2^n}e^{-i\frac{3\pi}{2^{n+1}}}, \tfrac{1}{2^{n-1}}e^{-i\frac{7\pi}{2^{n+2}}})$$
$$\cup \operatorname{arc}(\tfrac{1}{2^{n-1}}e^{-i\frac{5\pi}{2^{n+2}}}, \tfrac{1}{2^{n-1}}e^{-i\frac{7\pi}{2^{n+2}}}),$$

for $n = 1, 2, \ldots$. The following is the picture of A_1, A_2, A_3, B_1, B_2 and B_3.



Set

$$X = \bigcup_{n=1}^{\infty} A_n \cup B_n.$$

It is clear that each pair of $A_1, A_2, \ldots, B_1, B_2, \ldots$ intersects exactly at the origin O. Let

$$\varphi\colon \big(X, \{e^{i\frac{5\pi}{8}}, e^{i\frac{3\pi}{4}}, e^{i\frac{7\pi}{8}}\}\big) \to \big(X, \{e^{-i\frac{3\pi}{4}}\}\big)$$

be a relative homeomorphism such that φ is onto X, and one-to-one from X except for

$$\varphi(e^{i\frac{5\pi}{8}}) = \varphi(e^{i\frac{3\pi}{4}}) = \varphi(e^{i\frac{7\pi}{8}}) = e^{-i\frac{3\pi}{4}}.$$

Moreover, we assume that $\varphi(A_1) = B_1$, $\varphi(A_{n+1}) = A_n$, and $\varphi(B_n) = B_{n+1}$ for $n = 1, 2, \ldots$. Then the φ-tree has exactly two joints (both at $e^{-i\frac{3\pi}{4}}$) and the composition operator $Tf = f \circ \varphi$ is an isometric disjointness preserving quasi-2-shift on $C(X)$.

On the other hand, there is no disjointness preserving quasi-shift on $C(X)$ at all. In fact, suppose there were one. By Theorem 2.3(3), there would be two points a and b in

X such that the quotient space ${}^{X}\!/_{\sim_{a,b}}$ is homeomorphic to X, where the equivalence relation $\sim_{a,b}$ in X is defined by identifying a and b. But this is impossible.

With a trivial modification, one can also obtain examples of compact connected Hausdorff spaces X_n such that $C(X_n)$ admits isometric disjointness preserving quasi-n-shifts but not any disjointness preserving quasi-k-shift for $k = 1, 2, \ldots, n-1$ and $n = 2, 3, \ldots$. □

Remark 4.6. When X does not contain isolated points, it is shown in [13] that every isometric quasi-n-shift on $C_0(X)$ is disjointness preserving. Therefore, Example 4.5 gives also an example of an isometric quasi-n-shift which cannot be written as a product of n isometric quasi-shifts.

Question 4.7. How can we study (quasi-)n-shifts in term of (quasi-)shifts?

For a partial answer to Question 4.7, we show below that every "simple" disjointness preserving quasi-n-shifts on $C_0(X)$ can be dilated to a product of n quasi-shifts.

Definition 4.8. A φ-tree is said to be *simple* if all φ-vanishing points in X are isolated points. A disjointness preserving quasi-n-shift T is said to be *simple* if its associated φ-tree is simple.

We note that all disjointness preserving quasi-n-shifts on a *compact* Hausdorff space are simple by Theorem 2.3(3).

Lemma 4.9. *Let T be a simple disjointness preserving quasi-n-shift on $C_0(X)$ with exactly n vanishing points. Let*

$$\widetilde{X} = X \cup \mathbb{N} \quad (\textit{disjoint union}),$$

and thus $C_0(\widetilde{X}) = C_0(X) \oplus c_0$. Then the simple quasi-$n$-shift $\widetilde{T} = T \oplus I$ can be written as a product of n simple quasi-shifts on $C_0(\widetilde{X})$. In case T is an isometry, we can assume that these quasi-shifts are also isometries.

Proof. Let $X_0 = \{p \in X : \delta_p \circ T = 0\} = \{p_1, \ldots, p_n\}$ and $X_c = X \setminus X_0$. Write $\widetilde{f}$ in $C_0(\widetilde{X}) = C_0(X) \oplus c_0$ as $f \oplus (f_k)$; namely,

$$\widetilde{f}_{|X} = f \quad \text{and} \quad \widetilde{f}(k) = f_k \text{ for } k \text{ in } \mathbb{N}.$$

Let s be the unilateral shift on c_0, i.e.,

$$s((x_1, x_2, \ldots)) = (0, x_1, x_2, \ldots).$$

Define $S_1 : C_0(\widetilde{X}) \to C_0(\widetilde{X})$ by $S_1 = I \oplus s$, i.e.,

$$S_1(\widetilde{f}) = f \oplus (f_{k-1}),$$

where we set $f_0 = 0$ for convenience. Define $S_2\colon C_0(\widetilde{X}) \to C_0(\widetilde{X})$ by

$$\begin{aligned}
&(S_2\widetilde{f})_{|X_c} = (Tf)_{|X_c},\\
&(S_2\widetilde{f})(p_1) = 0,\\
&(S_2\widetilde{f})(p_{k+1}) = f_k \quad \text{for } k = 1, 2, \dots, n-1,
\end{aligned}$$

and

$$S_2\widetilde{f}(k) = f_{n+k-1} \quad \text{for } k = 1, 2, \dots .$$

Clearly, both S_1 and S_2 are simple disjointness preserving quasi-shifts on $C_0(\widetilde{X})$. S_1 is always an isometry, and so is S_2 whenever T is. Observe that $S_1^{n-1} = I \oplus s^{n-1}$. It then follows

$$\begin{aligned}
&(S_2S_1^{n-1}\widetilde{f})_{|X_c} = (T(S_1^{n-1}\widetilde{f})_{|X})_{|X_c} = (Tf)_{|X_c},\\
&(S_2S_1^{n-1}\widetilde{f})(p_1) = 0,\\
&(S_2S_1^{n-1}\widetilde{f})(p_{k+1}) = S_1^{n-1}\widetilde{f}(k) = 0 \quad \text{for } k = 1, 2, \dots, n-1,
\end{aligned}$$

and

$$(S_2S_1^{n-1}\widetilde{f})(k) = (S_1^{n-1}\widetilde{f})(n+k-1) = f_k = \widetilde{f}(k) \quad \text{for } k = 1, 2, \dots .$$

Hence

$$S_2S_1^{n-1} = T \oplus I = \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}$$

in $C_0(\widetilde{X}) = C_0(X) \oplus c_0$. $\square$

Lemma 4.10. *Let T be a simple disjointness preserving quasi-n-shift on $C_0(X)$ with m vanishing points $p_1, \dots, p_m$. Let $l = n - m$. Let $\widetilde{X} = X \bigcup \mathbb{N}$ be a disjoint union. Then the simple quasi-n-shift $T \oplus I$ on $C_0(\widetilde{X}) = C_0(X) \oplus c_0$ can be written as*

$$T \oplus I = T_l S_1^m,$$

where S_1 is a simple isometric quasi-shift on $C_0(\widetilde{X})$ and T_l is a quasi-l-shift on $C_0(\widetilde{X})$ without vanishing points. In case T is an isometry, we can assume that T_l is an isometry as well.

Proof. Let $X_c = X \setminus \{p_1, \dots, p_m\}$. Define $T_l\colon C_0(\widetilde{X}) \to C_0(\widetilde{X})$ by

$$\begin{aligned}
&(T_l\widetilde{f})_{|X_c} = (Tf)_{|X_c},\\
&(T_l\widetilde{f})(p_k) = f_k \quad \text{for } k = 1, \dots, m,
\end{aligned}$$

and

$$(T_l\widetilde{f})(k) = f_{m+k} \quad \text{for } k = 1, 2, \dots .$$

Let $S_1 = I \oplus s$ as in Lemma 4.9. Then

$$(T_l S_1^m \widetilde{f})_{|X_c} = T(S_1^m \widetilde{f}_{|X})_{|X_c} = Tf_{|X_1},$$
$$(T_l S_1^m \widetilde{f})(p_k) = S_1^m \widetilde{f}(k) = 0 \quad \text{for } k = 1, 2, \ldots, m,$$

and

$$(T_l S_1^m \widetilde{f})(k) = S_1^m \widetilde{f}(m+k) = \widetilde{f}(k) \quad \text{for } k = 1, 2, \ldots.$$

Hence

$$T_l S_1^m = T \oplus I = \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}.$$

Finally, we note that T_l is a quasi-l-shift without vanishing point, and isometric whenever T is. □

Remark 4.11. If $l = 0$ in Lemma 4.10 then

$$T \oplus I = T_0 S_1^n;$$

that is, every simple disjointness preserving quasi-n-shift on $C_0(X)$ with exactly n vanishing points can be dilated to a product of an invertible (composition) operator T_0 and n copies of the isometric quasi-shift $S_1 = I \oplus s$. We note that $S_2 = T_0 \circ S_1$ is the one given in Lemma 4.9.

Recall that a bounded linear operator T between Banach spaces is an injection if and only if it is injective and has closed range.

Lemma 4.12. *Let T be a disjointness preserving injection (resp. isometry) from $C_0(X)$ into $C_0(Y)$ of corank n. Suppose there is no vanishing points of T. Then T can be written as a product of n disjointness preserving injections (resp. isometries) of corank* 1.

Proof. By a result in [12], we can suppose $Tf = h \cdot f \circ \varphi$ for some continuous map φ from Y onto X and continuous bounded and away from zero scalar function h on Y. Moreover, if $M_\varphi = \{y \in Y \colon \#\varphi^{-1}(\varphi(y)) \geq 2\}$ is the set of all merging points of T then $\#(M_\varphi) - \#\varphi(M_\varphi) = n$. Fix two distinct points a and b in M_φ with $\varphi(a) = \varphi(b)$. Let $Y/\sim_{a,b}$ be the quotient space of Y by identifying a and b. Define $\widetilde{\varphi}^{a,b} \colon Y/\sim_{a,b} \to X$ by $\widetilde{\varphi}^{a,b}([y]) = \varphi(y)$. Let $M_{\widetilde{\varphi}^{a,b}} = \{[y] \in Y/\sim_{a,b} \colon \#(\widetilde{\varphi}^{a,b})^{-1}(\widetilde{\varphi}^{a,b}([y])) \geq 2\}$. Then $M_{\widetilde{\varphi}^{a,b}} \subset [M_\varphi]$ and $\#(M_{\widetilde{\varphi}^{a,b}}) - \#\widetilde{\varphi}^{a,b}(M_{\widetilde{\varphi}^{a,b}}) = n - 1$. On the other hand, we define $\varphi_1 \colon Y \to Y/\sim_{a,b}$ by $\varphi_1(y) = [y]$. Note that $M_{\varphi_1} = \{a, b\}$ is the set of all φ_1-merging points in Y. Clearly,

$$\varphi = \widetilde{\varphi}^{a,b} \circ \varphi_1.$$

Let g be a continuous scalar function on Y satisfying either one of the following conditions:

$$g(a) = \left|\frac{h(b)}{h(a)}\right|, \ g(b) = 1, \ \text{and} \ \left|\frac{h(b)}{h(a)}\right| \leq g \leq 1 \text{ when } \left|\frac{h(b)}{h(a)}\right| \leq 1;$$
$$g(a) = 1, \ g(b) = \left|\frac{h(a)}{h(b)}\right|, \ \text{and } 1 \geq g \geq \left|\frac{h(a)}{h(b)}\right| \text{ when } \left|\frac{h(b)}{h(a)}\right| \geq 1.$$

Define $h_2(y) = |h(y)|g(y)$ and $h_1(y) = \frac{h(y)}{h_2(y)}$ for y in Y. Then

$$h_2(a) = h_2(b) \quad \text{and} \quad h_1(y) \cdot h_2(y) = h(y), \quad \forall y \in Y.$$

Define a scalar function $\widetilde{h_2}^{a,b}$ on $Y/\sim_{a,b}$ by $\widetilde{h_2}^{a,b}([y]) = h_2(y)$. Then h_1 and $\widetilde{h_2}^{a,b}$ are continuous, bounded and away from zero on Y and $Y/\sim_{a,b}$, respectively. Moreover,

$$h_1(y) \cdot (\widetilde{h_2}^{a,b} \circ \varphi_1)(y) = h_1(y)\widetilde{h_2}^{a,b}([y]) = h(y), \quad \forall y \in Y.$$

Define $\widetilde{T}^{a,b} \colon C_0(X) \to C_0\left(Y/\sim_{a,b}\right)$ by

$$\widetilde{T}^{a,b} f = \widetilde{h_2}^{a,b} \cdot f \circ \widetilde{\varphi}^{a,b},$$

and $\widetilde{Q}^{a,b} \colon C_0\left(Y/\sim_{a,b}\right) \to C_0(Y)$ by

$$\widetilde{Q}^{a,b} \widetilde{f}^{a,b} = h_1 \cdot \widetilde{f}^{a,b} \circ \varphi_1.$$

Then, $\widetilde{Q}^{a,b} \circ \widetilde{T}^{a,b} \colon C_0(X) \to C_0(Y)$ satisfies that

$$\begin{aligned}(\widetilde{Q}^{a,b} \circ \widetilde{T}^{a,b}) f &= h_1 \cdot (\widetilde{T}^{a,b} f) \circ \varphi_1 = h_1 \cdot (\widetilde{h_2}^{a,b} \cdot f \circ \widetilde{\varphi}^{a,b}) \circ \varphi_1 \\ &= h_1 \cdot (\widetilde{h_2}^{a,b} \circ \varphi_1) \cdot f \circ (\widetilde{\varphi}^{a,b} \circ \varphi_1) \\ &= h \cdot f \circ \varphi = Tf, \quad \forall f \in C_0(X).\end{aligned}$$

Hence $T = \widetilde{Q}^{a,b} \circ \widetilde{T}^{a,b}$.

Clearly $\widetilde{Q}^{a,b}$ is a disjointness preserving injection of corank one, and $\widetilde{T}^{a,b}$ is a disjointness preserving injection of corank $n-1$. Both $\widetilde{Q}^{a,b}$ and $\widetilde{T}^{a,b}$ will be isometries whenever T is. The above construction can be applied to further decompose $\widetilde{T}^{a,b}$ into $n-1$ disjointness preserving injections (resp. isometries) of corank one. □

Theorem 4.13. *Let T be a simple disjointness preserving quasi-n-shift on $C_0(X)$ with m vanishing points. Let $l = n - m$ and let $\widetilde{X} = X \cup \mathbb{N}$ (disjoint union). Then $T \oplus I$ on $C_0(\widetilde{X}) = C_0(X) \oplus c_0$ is a product of m copies of the isometric disjointness preserving quasi-shift $S_1 = I \oplus s$ and l corank one disjointness preserving injections $Q_1, Q_2, \ldots, Q_l$, i.e.*

$$\begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} = Q_1 Q_2 \cdots Q_l S_1^m.$$

Here, s is the unilateral shift on c_0. In case $m = n$, the right hand side becomes QS_1^n for some invertible composition operator Q on $C_0(\widetilde{X})$. Moreover, all $Q_1, Q_2, \ldots, Q_l$ can be chosen to be isometries whenever T is.

Proof. It follows from Lemmas 4.9, 4.10 and 4.12. $\square$

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