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On the degree theory for general mappings of monotone type ☆

Bui Trong Kien [a,*], Mu-Ming Wong [b], Ngai-Ching Wong [a]

[a] Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung, Taiwan 804
[b] Department of Information Technology, Meiho Institute of Technology, Pintaung, Taiwan

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Abstract

Degree theory has been developed as a tool for checking the solution existence of nonlinear equations. In his classic paper published in 1983, Browder developed a degree theory for mappings of monotone type $f + T$, where f is a mapping of class $(S)_+$ from a bounded open set Ω in a reflexive Banach space X into its dual X^*, and T is a maximal monotone mapping from X into X^*. This breakthrough paved the way for many applications of degree theoretic techniques to several large classes of nonlinear partial differential equations. In this paper we continue to develop the results of Browder on the degree theory for mappings of monotone type $f + T$. By enlarging the class of maximal monotone mappings and pseudo-monotone homotopies we obtain some new results of the degree theory for such mappings.

Keywords: Degree theory; Demicontinuity; Maximal monotonicity; Variational inequality

1. Introduction and an outline of Browder's degree theory

Many problems in analysis and in its various applications can be reduced to solving an equation $f(x) = y$ in an appropriate space. Degree theory has developed as a means of examining the solution existence and estimating the number of the solutions in a particular feasible region.

Suppose that X and Y are topological spaces. Let $\mathcal{O}$ be a class of open subsets Ω in X. For each Ω in $\mathcal{O}$, we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and boundary of Ω, respectively. Let $\mathcal{F}$ be a family of maps $f : \overline{\Omega} \to Y$ over which the degree theory is to be defined. For each Ω in $\mathcal{O}$, we consider a family of homotopies $\{f_t\colon 0 \leqslant t \leqslant 1\}$ of maps in $\mathcal{F}$, all having the common domain $\overline{\Omega}$. The collection of all such homotopies for the various Ω in $\mathcal{O}$ will be denoted by $\mathcal{H}$, which is called a class of permissible homotopies for the degree theory.

Definition 1.1. By a *degree theory* over the class $\mathcal{F}$, which is invariant with respect to the homotopies in $\mathcal{H}$ and which is normalized by a given map $f_0 : X \to Y$, we mean: For each y_0 in Y and for each $f : \overline{\Omega} \to Y$ in $\mathcal{F}$, with $y_0 \notin f(\partial\Omega)$, an integer $d(f, \Omega, y_0)$ is defined and satisfies the following three conditions:

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* Corresponding author. On leave from: National University of Civil Engineering, 55 Giai Phong, Hanoi, Vietnam.
E-mail addresses: btkien@math.nsysu.edu.tw (B.T. Kien), x2119@meiho.edu.tw (M.-M. Wong), wong@math.nsysu.edu.tw (N.-C. Wong).

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(a) (Normalization) If $d(f, \Omega, y_0) \neq 0$ then there exists x in Ω such that $f(x) = y_0$. For each Ω in $\mathcal{O}$, we have $f_0|_\Omega \in \mathcal{F}$, and if $y_0 \in f_0(\Omega)$ then $d(f_0, \Omega, y_0) = +1$.
(b) (Additivity) If $f \in \mathcal{F}$, $f : \overline{\Omega} \to Y$ and that Ω_1, Ω_2 are a pair of disjoint sets in $\mathcal{O}$ contained in Ω such that $y_0 \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$ then

$$d(f, \Omega, y_0) = d(f, \Omega_1, y_0) + d(f, \Omega_2, y_0).$$

(c) (Homotopy invariance) If $\{f_t\colon 0 \leqslant t \leqslant 1\}$ is a homotopy in $\mathcal{H}$ with domain $\overline{\Omega}$ and if $\{y_t\colon 0 \leqslant t \leqslant 1\}$ is a continuous path in Y with $y_t \notin f_t(\partial\Omega)$ for all t in [0, 1] then $d(f_t, \Omega, y_t)$ is independent of t.

The existence of a degree theory for a given class of mappings is not a trivial assertion. The classical example of an extension of the degree theory for the infinite dimensional case is the Leray–Schauder degree theory which is defined for the case in which $X = Y$ is an arbitrary Banach space, $\mathcal{O}$ is the class of bounded open subsets of X, $\mathcal{F}$ is the class of continuous maps $f : \overline{\Omega} \to X$ with $(I - f)(\overline{\Omega})$ relatively compact in X. Here $\mathcal{H}$ is the class of homotopies $\{f_t\colon 0 \leqslant t \leqslant 1\}$ which is restricted by the assumption that there exists a fixed compact set K in X such that $(I - f_t)(\overline{\Omega}) \subset K$ for all t in [0, 1].

Ever since the introduction of the Leray–Schauder degree theory in 1934, there have been various extensions and generalization in different directions (see, for instance, [2,3,5–11] and references given therein). One of the most important generalizations is due to Browder [2], where a degree theory was introduced for maps of class $(S)_+$ from a bounded open subset of a reflexive Banach space into its dual. In particular, Browder built the degree theory for more general mappings of monotone type $f + T$, where f is of class $(S)_+$ and T is a maximal monotone operator. This generalization paved the way for applications of the degree theory to a large number of nonlinear partial differential equations (see, e.g., [2,7]). For this important event, we will outline below some results in [2].

Throughout this paper we always assume that X is a reflexive Banach space. By the result which is due to Lindenstrauss, Asplund and Trojanski, it is true that X can be renormed so that X and X^* are both locally uniformly convex (see [4, Theorem 2.11]). In this case the duality mapping $J : X \to X^*$ defined by

$$\langle J(x), x\rangle = \|x\|^2 = \|J(x)\|^2,$$

is of class $(S)_+$, strictly monotone and a homeomorphism (see [2, Proposition 8] and [12, Corollary 32.24]).

Definition 1.2. Let G be a subset of a Banach space X and $f : G \to X^*$ be a mapping. Then

(a) f is said to be of *class* $(S)_+$ if for any sequence $\{x_n\}$ in G which converges weakly to x (written $x_n \rightharpoonup x$ briefly) and $\limsup_{n\to\infty}\langle f(x_n), x_n - x\rangle \leqslant 0$ we have $x_n \to x$ (in norm).
(b) f is said to be *demicontinuous* if for any sequence $\{x_n\}$ in G which converges to x we have $f(x_n)$ converges weakly to $f(x)$.

The following result on the existence and uniqueness of the degree theory for mappings of class $(S)_+$, was obtained in [2].

Theorem 1.3. *(See [2, Theorem 4].) Let X be a reflexive Banach space, $\mathcal{O}$ be a class of bounded open subsets Ω of X, $\mathcal{F}$ be a class of maps $f : \overline{\Omega} \to X^*$ such that f is of class $(S)_+$ and demicontinuous. Let $\mathcal{H}$ be the class of affine homotopies in $\mathcal{F}$ and J be the duality mapping form X to X^* corresponding to an equivalent norm on X in which both X and X^* are locally uniformly convex.*

Then there exists exactly one degree function d in the sense of Definition 1.1*, which defines on $\mathcal{F}$, to be invariant under $\mathcal{H}$ and normalized by J.*

Let $T : X \to 2^{X^*}$ be a multifunction. We will call the sets $\Gamma_T = \{(x, y)\colon y \in T(x)\}$ and $D(T) = \{x \in X\colon T(x) \neq \emptyset\}$, the *graph* and the *effective domain* of T, respectively. In the rest of the paper we always assume that $D(T)$ is nonempty.

Recall that the multifunction $T : X \to 2^{X^*}$ is said to be *monotone* if for any $(x_1, y_1), (x_2, y_2)$ in Γ_T one has $\langle y_2 - y_1, x_2 - x_1\rangle \geqslant 0$. T is said to be *maximal monotone* if T is monotone and it follows from

$$\langle v - y, u - x\rangle \geqslant 0, \quad \forall (x, y) \in \Gamma_T,$$

that $(u, v) \in \Gamma_T$.

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For each $\epsilon > 0$, we consider the generalized *Yosida transformation* $T_{,\epsilon}$ corresponding to T, consider in [1] and defined by the formula

$$T_{,\epsilon} = \left(T^{-1} + \epsilon J^{-1}\right)^{-1},$$

which is a single-valued function (see, e.g., [4, Proposition 3.10]).

The following theorem is the basis for building the degree theory for mappings of type $f + T$.

Theorem 1.4. *(See [2, Theorem 8].) Assume $\Omega \in \mathcal{O}$ and conditions*:

(i) $f : \overline{\Omega} \to X^*$ *is of class* $(S)_+$, *demicontinuous and bounded.*
(ii) $T : X \to 2^{X^*}$ *is maximal monotone.*
(iii) $y_0 \notin (f + T)(\partial\Omega)$.

Suppose further that

(iv) $0 \in T(0)$.

Then there exists $\bar{\epsilon} > 0$ such that the following assertions are valid:

(a) $y_0 \notin (f + T_{,\epsilon})(\partial\Omega)$ *for all ϵ in* $(0, \bar{\epsilon}]$;
(b) $d(f + T_{,\epsilon}, \Omega, y_0) = d(f + T_{,\epsilon'}, \Omega, y_0)$ *for all ϵ, ϵ' in* $(0, \bar{\epsilon}]$.

Definition 1.5. Let $\Omega \in \mathcal{O}$, $f : \overline{\Omega} \to X^*$ be of class $(S)_+$, demicontinuous and bounded. Let $T : X \to 2^X$ be a maximal monotone operator and y_0 be a point in X^* such that $y_0 \notin (f + T)(\partial\Omega)$. Suppose, in addition, $0 \in T(0)$. The degree function $d_1(f + T, \Omega, y_0)$ is the common value of $d(f + T_{,\epsilon}, \Omega, y_0)$ for ϵ sufficiently small.

Definition 1.6. Let $\Omega \in \mathcal{O}$ and $\{f_t, 0 \leqslant t \leqslant 1\}$ be a family of maps from $\overline{\Omega}$ into X^*. Then $\{f_t\}$ is said to be a *homotopy of class* $(S)_+$ if for any sequence $\{x_n\}$ in $\overline{\Omega}$ converging weakly to x and for any sequence $\{t_n\}$ in $[0, 1]$ converging to t for which

$$\limsup_{n\to\infty} \left\langle f_{t_n}(x_n), x_n - x\right\rangle \leqslant 0,$$

then $x_n \to x$ and $f_{t_n}(x_n) \rightharpoonup f_t(x)$.

Definition 1.7. Let $\{T_t, 0 \leqslant t \leqslant 1\}$ be a family of maximal monotone maps from X into 2^{X^*}. Suppose, in addition, that $0 \in T_t(0)$ for all t in $[0, 1]$. Then $\{T_t\}$ is called a *pseudo-monotone homotopy* if it satisfies the following mutually equivalent properties [2, Proposition 15]:

(a) Suppose that $t_n \to t$ in $[0, 1]$ and $(x_n, y_n) \in \Gamma_{T_{t_n}}$ such that $x_n \rightharpoonup x$ in X, $y_n \rightharpoonup y$ in X^* and $\limsup \langle y_n, x_n\rangle \leqslant \langle y, x\rangle$. Then $(x, y) \in \Gamma_{T_t}$ and $\lim \langle y_n, x_n\rangle = \langle y, x\rangle$.
(b) The mapping φ of $X^* \times [0, 1]$ into X defined by

$$\varphi(w, t) = (T_t + J)^{-1}(w),$$

is continuous, with X and X^* given the norm topologies.
(c) For each w in X^*, the mapping φ_w of $[0, 1]$ into X defined by

$$\varphi_w(t) = (T_t + J)^{-1}(w),$$

is continuous from $[0, 1]$ into X in the norm topology.
(d) For any (x, y) in Γ_{T_t} and a sequence $t_n \to t$ in $[0, 1]$, there exists a sequence (x_n, y_n) in $\Gamma_{T_{t_n}}$ such that $x_n \to x$ and $y_n \to y$.

The following results establish the homotopy invariance of the degree theory for maps of type $f + T$.

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Theorem 1.8. *(See [2, Theorem 9].) Assume $\Omega \in \mathcal{O}$ and conditions:*

(i) $\{f_t\colon 0 \leqslant t \leqslant 1\}$ *is a homotopy of class* $(S)_+$ *of maps of* $\overline{\Omega}$ *into a bounded subset of* X^*.
(ii) $\{T_t\colon 0 \leqslant t \leqslant 1\}$ *is a pseudo-monotone homotopy from* X *into* 2^{X^*}.
(iii) $\{y_t\colon 0 \leqslant t \leqslant 1\}$ *is a continuous path in* X^* *such that* $y_t \notin (f_t + T_t)(\partial\Omega)$ *for all* t *in* $[0, 1]$.

Suppose, in addition,

(iv) $0 \in T_t(0)$ *for all* t *in* $[0, 1]$.

Then there exists $\bar{\epsilon} > 0$ *such that the following assertions are valid:*

(a) $y_t \notin (f_t + T_{t,\epsilon})(\partial\Omega)$ *for all* $\epsilon \in (0, \bar{\epsilon}]$ *and* t *in* $[0, 1]$.
(b) $d(f_t + T_{t,\epsilon}, \Omega, y_t)$ *is independent of* t *and* ϵ.

Theorem 1.9. *(See [2, Theorem 12].) Let X be a reflexive Banach space. Then there exists exactly one degree function in the sense of Definition* 1.1 *on the class $\mathcal{F}$ of maps $f + T$, where f is of class $(S)_+$, demicontinuous and bounded, and T is maximal monotone* satisfying $0 \in T(0)$, *which is normalized by J and invariant under class $\mathcal{H}$ of all affine homotopies of the form $(1-t)(T+f)+tg$, where f, g are of class $(S)_+$, demicontinuous and T is maximal monotone.*

Theorems 1.4, 1.8, and 1.9 are nice results on the degree theory for mappings of monotone type $f + T$. After more than twenty years, it is now well known and has been cited by many papers. However, in our opinion, the class of mappings of monotone type $f + T$ given by Browder is rather narrow. Namely, in the above theorems, the conditions

$$0 \in T(0) \quad \text{and} \quad 0 \in T_t(0) \quad \text{were required.}$$

Browder used these conditions in an essential way to prove his theorems. Meanwhile, several applications, where the graphs of operators T_t are supposed not to contain $(0, 0)$. It is noted that, if we take any (x_0, x_0^*) in Γ_T and put $T'(x) = T(x + x_0) - x_0^*$ and $f'(x) = f(x + x_0)$ then T' is maximal monotone and satisfies $0 \in T'(0)$. It is plausible to define the degree $d(f + T, \Omega, y_0)$ via $d(f' + T', \Omega - x_0, y_0 - x_0^*)$. However, in this case, the degree depends on (x_0, x_0^*) and we will encounter easily a trouble with the homotopy invariance since it might happen that $x_0^* \notin \bigcap_t T_t(x_0)$. To illustrate this we give the following example.

Example 1.10. Let K be a nonempty closed convex set in X such that $0 \notin K$ and $2K \subset K$. Assume that $N_K(x)$ is the normal cone of K at x defined by

$$N_K(x) = \begin{cases} \{x^* \in X^*\colon \langle x^*, y - x\rangle \leqslant 0 \ \forall y \in K\} & \text{if } x \in K, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Put $T_t(x) = (1 - t)(-J(x) + N_K(x))$. Since $0 \notin K$, $0 \notin T_t(0)$. Indeed, we have $\bigcap_t T_t(x) = \emptyset$ for all x in K. Hence $\{T_t,\ 0 \leqslant t \leqslant 1\}$ is not a pseudo-monotone homotopy in the sense of Definition 1.7.

The aim of the paper is to relax the assumptions that $0 \in T(0)$ and $0 \in T_t(0)$, and enlarges the class of pseudo-monotone homotopies $\{T_t\}$ which is given by Browder. To this end, we have to develop a new scheme in proving our theorems. By using this scheme we will be able to obtain enhanced versions of Theorems 1.4, 1.8, and 1.9 under relaxed conditions for a broader class of $\{T_t\}$. We think this is significant, since our results can be applied to a degree theory of variational inequalities in reflexive Banach spaces and solving nonlinear PDEs, for some of which the original Browder's degree theory does not seem to be very handy.

2. Main results

In this section we will keep all the notations of the preceding section. Below extends Theorem 1.4.

Theorem 2.1. *Assume X is a reflexive Banach space and the following conditions:*

(i) *$f:\overline{\Omega}\to X^*$ is of class $(S)_+$, demicontinuous and bounded.*
(ii) *$T:X\to 2^{X^*}$ is maximal monotone.*
(iii) *$y_0\notin(f+T)(\partial\Omega)$.*

Then there exists $\bar\epsilon>0$ such that the following assertions are valid:

(a) *$y_0\notin(f+T_{,\epsilon})(\partial\Omega)$ for all ϵ in $(0,\bar\epsilon]$*;
(b) *$d(f+T_{,\epsilon},\Omega,y_0)=d(f+T_{,\epsilon'},\Omega,y_0)$ for all ϵ,ϵ' in $(0,\bar\epsilon]$.*

Definition 2.2. Let $\Omega\in\mathcal{O}$ and $f:\overline{\Omega}\to X^*$ be of class $(S)_+$, demicontinuous and bounded. Let $T:X\to 2^{X^*}$ be a maximal monotone operator and y_0 be a point such that $y_0\notin(f+T)(\partial\Omega)$. The degree $d_1(f+T,\Omega,y_0)$ is assigned to be the common value of $d(f+T_{,\epsilon},\Omega,y_0)$ for ϵ sufficiently small.

The next theorems will be proved for the class of maximal monotone operators and pseudo-monotone homotopies which enlarges the class given by Browder. For such classes, we introduce the following definition.

Definition 2.3. Let $\{T_t\colon 0\leqslant t\leqslant 1\}$ be a family of maximal monotone maps from X into 2^{X^*} such that their effective domains are nonempty. Then $\{T_t\}$ is called a *pseudo-monotone homotopy* if for any (x,y) in Γ_{T_t} and a sequence $t_n\to t$ in $[0,1]$, there exists a sequence (x_n,y_n) in $\Gamma_{T_{t_n}}$ such that $x_n\to x$ and $y_n\to y$.

It is obvious that if $\{T_t,\ 0\leqslant t\leqslant 1\}$ is a pseudo-monotone homotopy in the sense of Definition 1.7 then it is also a pseudo-monotone homotopy in the sense of Definition 2.3. Example 1.10 indicates that there exists a pseudo-monotone homotopy in the sense of Definition 2.3 but it is not a pseudo-monotone homotopy in the sense of Definition 1.7.

Based on the class of pseudo-monotone homotopies in the sense of Definition 2.3 we obtain the following results extending Theorems 1.8 and 1.9.

Theorem 2.4. *Assume X is a reflexive Banach space and conditions*:

(i) *$\{f_t\colon 0\leqslant t\leqslant 1\}$ is a homotopy of class $(S)_+$ of maps of $\overline{\Omega}$ into a bounded subset of X^*.*
(ii) *$\{T_t\colon 0\leqslant t\leqslant 1\}$ is a pseudo-monotone homotopy from X into 2^{X^*}.*
(iii) *$\{y_t\colon 0\leqslant t\leqslant 1\}$ is a continuous path in X^* such that $y_t\notin(f_t+T_t)(\partial\Omega)$ for all t in $[0,1]$.*

Then there exists $\bar\epsilon>0$ such that the following assertions are valid:

(a) *$y_t\notin(f_t+T_{t,\epsilon})(\partial\Omega)$ for all ϵ in $(0,\bar\epsilon]$ and t in $[0,1]$.*
(b) *$d(f_t+T_{t,\epsilon},\Omega,y_t)$ is independent of t and ϵ.*

Theorem 2.5. *Let X a reflexive Banach space, $\Omega\in\mathcal{O}$ and $f:\Omega\to X^*$ be of class $(S)_+$, demicontinuous and bounded. Let $T:X\to 2^{X^*}$ be a maximal monotone operator and y_0 be a point of X^* such that $y_0\notin(f+T)(\partial\Omega)$. Then the following assertions hold*:

(a) (Existence) *If $d_1(f+T,\Omega,y_0)\neq 0$ then there exists x in Ω such that $y_0\in f(x)+T(x)$.*
(b) (Additivity) *If Ω_1, Ω_2 are a pair disjoint subsets of $\mathcal{O}$ contained in Ω such that $y_0\notin(f+T)(\overline{\Omega}\setminus(\Omega_1\cup\Omega_2))$ then*

$$d_1(f+T,\Omega,y_0)=d_1(f+T,\Omega_1,y_0)+d_1(f+T,\Omega_2,y_0).$$

(c) (Homotopy invariance) *If $\{f_t\colon 0\leqslant t\leqslant 1\}$ is a homotopy of class $(S)_+$ of maps of $\overline{\Omega}$ into a bounded set of X^*, $\{T_t\colon 0\leqslant t\leqslant 1\}$ is a pseudo-monotone homotopy from X into 2^{X^*} and $\{y_t\colon 0\leqslant t\leqslant 1\}$ is a continuous path in X^* such that $y_t\notin(f_t+T_t)(\partial\Omega)$ for all t in $[0,1]$, then $d_1(f_t+T_t,\Omega,y_t)$ is independent of t.*

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Theorem 2.6. *Let X be a reflexive Banach space. Then there exists exactly one degree function in the sense of Definition* 1.1 *on the class $\mathcal{F}$ of maps $f+T$, where f is of class $(S)_+$, demicontinuous and bounded, and T is maximal monotone, which is normalized by J and invariant under class $\mathcal{H}$ of all affine homotopies of the form $(1-t)(f+T)+tg$, where f and g are of class $(S)_+$, demicontinuous and bounded, T is maximal monotone with $\overline{D(T)}=X$.*

3. Proofs

We notice that Theorem 2.4 includes Theorem 2.1 as a special case when $f_t=f$, $y_t=y_0$ and $T_t=T$ for all t in $[0,1]$. Hence it suffices to prove Theorem 2.4. We will complete the proof of Theorem 2.4 by proving some lemmas.

Lemma 3.1. *Let f be of class $(S)_+$ and demicontinuous. Assume that g is demicontinuous and monotone. Then $f+g$ is of class $(S)_+$ and demicontinuous.*

Proof. It is easily seen that $f+g$ is demicontinuous. Let $\{x_n\}$ be a sequence such that $x_n \rightharpoonup x_0$ and

$$\limsup_{n\to\infty}\langle f(x_n)+g(x_n), x_n-x_0\rangle \leqslant 0.$$

By the monotonicity of g we get

$$\langle f(x_n)+g(x_n), x_n-x_0\rangle \geqslant \langle f(x_n), x_n-x_0\rangle + \langle g(x_0), x_n-x_0\rangle.$$

This implies that $\limsup_{n\to\infty}\langle f(x_n), x_n-x_0\rangle \leqslant 0$. Since f belongs to class $(S)_+$, we have $x_n \to x_0$. Hence $f+g$ is of class $(S)_+$. □

Lemma 3.2. *Suppose that f satisfies conditions in Theorem* 2.4*, $x_n\in\Omega$ such that $x_n \rightharpoonup x_0$ and $t_n\in[0,1]$. Then* $\liminf_{n\to\infty}\langle f_{t_n}(x_n), x_n-x_0\rangle \geqslant 0$.

Proof. Suppose on the contrary that

$$\alpha := \liminf_{n\to\infty}\langle f_{t_n}(x_n), x_n-x_0\rangle < 0. \tag{1}$$

Since $\langle f_{t_n}(x_n), x_n-x_0\rangle \geqslant -\|f(x_n)\|\|x_n-x_0\|$, by the boundedness of sequences $\{f_{t_n}(x_n)\}$ and $\{x_n-x_0\}$, we have $\alpha > -\infty$. It follows that there exist subsequence $\{x_{n_k}\}$ and $\{t_{n_k}\}$ such that $\lim_{k\to\infty}\langle f_{t_{n_k}}(x_{n_k}), x_{n_k}-x_0\rangle = \alpha < 0$. Hence there exists a number $k_0>0$ such that $\langle f_{t_{n_k}}(x_{n_k}), x_{n_k}-x_0\rangle < 0$ for all $k \geqslant k_0$. This implies that $\limsup_{k\to\infty}\langle f_{t_{n_k}}(x_{n_k}), x_{n_k}-x_0\rangle \leqslant 0$. Since f is of class $(S)_+$, we get $x_{n_k}\to x_0$. By using the inequality

$$\langle f_{t_{n_k}}(x_{n_k}), x_{n_k}-x_0\rangle \geqslant -\|f_{t_{n_k}}(x_{n_k})\|\|x_{n_k}-x_0\|$$

we obtain $\alpha = \lim_{k\to\infty}\langle f_{t_{n_k}}(x_{n_k}), x_{n_k}-x_0\rangle \geqslant 0$. This contradicts (1). The proof is complete. □

Proof of Theorem 2.4. (a) Suppose that the assertion is false. Then we could find a sequence $\epsilon_n \to 0^+$, a sequence x_n in $\partial\Omega$ and a sequence t_n in $[0,1]$ such that

$$y_{t_n} = f_{t_n}(x_n) + z_n, \tag{2}$$

where $z_n = T_{t_n,\epsilon_n}(x_n)$. Since $\{f_{t_n}(x_n)\}$ and $\{y_{t_n}\}$ are bounded, the sequence $\{z_n\}$ is also bounded. Hence we can assume that $x_n \rightharpoonup x_0$, $t_n \to t$ in $[0,1]$, $z_n \rightharpoonup z_0$ and $y_{t_n} \to y_t$. Since

$$\langle z_n, x_n-x_0\rangle = \langle y_{t_n} - f_{t_n}(x_n), x_n-x_0\rangle,$$

we have

$$\langle z_n, x_n\rangle = \langle z_n, x_0\rangle + \langle y_{t_n}, x_n-x_0\rangle - \langle f_{t_n}(x_n), x_n-x_0\rangle. \tag{3}$$

This implies that

$$\limsup_{n\to\infty}\langle z_n, x_n\rangle \leqslant \limsup_{n\to\infty}\langle z_n, x_0\rangle + \limsup_{n\to\infty}\langle y_{t_n}, x_n-x_0\rangle - \liminf_{n\to\infty}\langle f_{t_n}(x_n), x_n-x_0\rangle. \tag{4}$$

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By the Lemma 3.2, we have $\liminf_{n\to\infty}\langle f_{t_n}(x_n), x_n - x_0\rangle \geqslant 0$. Hence it follows from (4) that

$$\limsup_{n\to\infty}\langle z_n, x_n\rangle \leqslant \langle z_0, x_0\rangle. \tag{5}$$

Note that the equality $z_n = T_{t_n,\epsilon_n}(x_n)$ means that $z_n \in T_{t_n}(x_n - \epsilon_n J^{-1}(z_n))$. We now take any (u, v) in Γ_{T_t}. By the definition of $\{T_t\}$, there exists sequences $u_n \to u$ and $v_n \to v$ such that $(u_n, v_n) \in \Gamma_{T_{t_n}}$. Using the monotonicity of T_{t_n} we get

$$\bigl\langle z_n - v_n, x_n - \epsilon_n J^{-1}(z_n) - u_n\bigr\rangle \geqslant 0.$$

Hence

$$\langle z_n, x_n\rangle \geqslant \langle z_n, u_n\rangle + \langle v_n, x_n - u_n\rangle + \epsilon_n\|z_n\|^2 - \epsilon_n\|v_n\|\|z_n\|. \tag{6}$$

From (5) and (6) we have

$$\begin{aligned}\langle z_0, x_0\rangle &\geqslant \limsup_{n\to\infty}\langle z_n, x_n\rangle\\ &\geqslant \liminf_{n\to\infty}\langle z_n, x_n\rangle\\ &\geqslant \liminf_{n\to\infty}\langle z_n, u_n\rangle + \liminf_{n\to\infty}\langle v_n, x_n - u_n\rangle + \liminf_{n\to\infty}\bigl(\epsilon_n\|z_n\|^2 - \epsilon_n\|v_n\|\|z_n\|\bigr)\\ &\geqslant \langle z_0, u\rangle + \langle v, x_0 - u\rangle.\end{aligned} \tag{7}$$

This implies that $\langle z_0 - v, x_0 - u\rangle \geqslant 0$ for all (u, v) in Γ_{T_t}. By the maximal monotonicity of T_t we obtain $z_0 \in T_t(x_0)$. Substituting $(u, v) = (x_0, z_0)$ into (7) we get $\lim_{n\to\infty}\langle z_n, x_n\rangle = \langle z_0, x_0\rangle$. Hence it follows from (3) that $\lim_{n\to\infty}\langle f_{t_n}(x_n), x_n - x_0\rangle = 0$. Since $\{f_s, 0 \leqslant s \leqslant 1\}$ is homotopy of class $(S)_+$ and $\partial\Omega$ is closed, $x_n \to x_0 \in \partial\Omega$ and $f_{t_n}(x_n) \rightharpoonup f_t(x_0)$. By letting $n \to \infty$, from (2) we obtain $y_t = f_t(x_0) + z_0$, where $z_0 \in T_t(x_0)$ and $x_0 \in \partial\Omega$. This means that $y_t \in (f_t + T_t)(\partial\Omega)$ for some t in [0, 1]. We obtain a contradiction.

(b) We first show that for each ϵ in $(0, \bar\epsilon]$ and t in [0, 1], the degree $d_1(f_t + T_{t,\epsilon}, \Omega, y_t)$ is well defined. Recall that

$$T_{t,\epsilon} = \bigl(T_t^{-1} + \epsilon J^{-1}\bigr)^{-1}.$$

Since T_t is maximal monotone, T_t^{-1} is also maximal monotone (see [12, Proposition 32.5]). By [4, Proposition 5.5] (see also [12, Corollary 32.24]), J^{-1} has the same properties of J. According to Proposition 3.10 in [4], $(T_t^{-1} + \epsilon J^{-1})^{-1}$ is a single-valued and demicontinuous maximal monotone operator. Hence we obtain from Lemma 3.1 that $f_t + T_{t,\epsilon}$ is of class $(S)_+$ and demicontinuous. Consequently, the degree $d(f_t + T_{t,\epsilon}, \Omega, y_t)$ is well defined.

We now show that there exists $\bar\epsilon > 0$ such that $d(f_t + T_{t,\epsilon}, \Omega, y_t)$ is independent of ϵ in $(0, \bar\epsilon]$ and t in [0, 1]. Suppose on the contrary that the assertion is false. Then there exist $\epsilon_n \to 0^+$, $\epsilon'_n \to 0^+$ and t_n in [0, 1] such that

$$d(f_{t_n} + T_{t_n,\epsilon_n}, \Omega, y_{t_n}) \neq d(f_{t_n} + T_{t_n,\epsilon'_n}, \Omega, y_{t_n}). \tag{8}$$

For each fixed n, we consider the homotopy H_n defined by

$$H_n(s, x) = sT_{t_n,\epsilon_n}(x) + (1 - s)T_{t_n,\epsilon'_n}(x) + f_{t_n}(x). \tag{9}$$

If $y_{t_n} \notin H_n(s, \partial\Omega)$ for all s in [0, 1] then

$$d\bigl(H_n(1, .), \Omega, y_{t_n}\bigr) = d\bigl(H_n(0, .), \Omega, y_{t_n}\bigr).$$

Hence

$$d(f_{t_n} + T_{t_n,\epsilon_n}, \Omega, y_{t_n}) = d(f_{t_n} + T_{t_n,\epsilon'_n}, \Omega, y_{t_n}),$$

which contradicts (8). Thus there exists x_n in $\partial\Omega$ and s_n in [0, 1] such that

$$y_{t_n} = s_nT_{t_n,\epsilon_n}(x_n) + (1 - s_n)T_{t_n,\epsilon'_n}(x_n) + f_{t_n}(x_n). \tag{10}$$

Put $z_n = s_nT_{t_n,\epsilon_n}(x_n) + (1 - s_n)T_{t_n,\epsilon'_n}(x_n)$. Then $\{z_n\}$ is a bounded sequence because the sequences $\{f_{t_n}(x_n)\}$ and $\{y_{t_n}\}$ are bounded. Without loss of generality we can assume that $t_n \to t$, $s_n \to s$, $x_n \rightharpoonup x_0$, $z_n \rightharpoonup z_0$ and $y_{t_n} \to y_t$. Putting $a_n = T_{t_n,\epsilon_n}(x_n)$, $b_n = T_{t_n,\epsilon'_n}(x_n)$, we have

$$a_n \in T_{t_n}\big(x_n - \epsilon_n J^{-1}(a_n)\big), \qquad b_n \in T_{t_n}\big(x_n - \epsilon'_n J^{-1}(b_n)\big). \tag{11}$$

As $y_{t_n} = f_{t_n}(x_n) + z_n$, we have

$$\langle z_n, x_n\rangle = \langle z_n, x_0\rangle + \langle y_{t_n}, x_n - x_0\rangle - \big\langle f_{t_n}(x_n), x_n - x_0\big\rangle. \tag{12}$$

By using the similar arguments in the proof of part (a) we obtain

$$\limsup_{n\to\infty}\langle z_n, x_n\rangle \leqslant \langle z_0, x_0\rangle. \tag{13}$$

We now consider the following cases:

Case 1. The sequence $\{s_n a_n\}$ is bounded.

Since the sequence $\{s_n a_n\}$ is bounded, $\{(1 - s_n)b_n\}$ is also bounded. Take any (u, v) in Γ_{T_t}, by the definition of the family $\{T_s, 0 \leqslant s \leqslant 1\}$ we can find sequences $u_n \to u$ and $v_n \to v$ such that $(u_n, v_n) \in \Gamma_{T_{t_n}}$. Using the monotonicity of T_{t_n} we get

$$\big\langle a_n - v_n, x_n - \epsilon_n J^{-1}(a_n) - u_n\big\rangle \geqslant 0.$$

Hence

$$\langle a_n, x_n\rangle \geqslant \langle a_n, u_n\rangle + \langle v_n, x_n - u_n\rangle + \epsilon_n\|a_n\|^2 - \epsilon_n\|v_n\|\|a_n\|. \tag{14}$$

Similarly, we also have

$$\langle b_n, x_n\rangle \geqslant \langle b_n, u_n\rangle + \langle v_n, x_n - u_n\rangle + \epsilon'_n\|b_n\|^2 - \epsilon'_n\|v_n\|\|b_n\|. \tag{15}$$

Multiplying (14) with s_n, (15) with $1 - s_n$ and summing up, we get

$$\langle z_n, x_n\rangle \geqslant \langle z_n, u_n\rangle + \langle v_n, x_n - u_n\rangle + s_n\epsilon_n\big(\|a_n\|^2 - \|v_n\|\|a_n\|\big) + (1 - s_n)\epsilon'_n\big(\|b_n\|^2 - \|v_n\|\|b_n\|\big). \tag{16}$$

It follows that

$$\langle z_n, x_n\rangle \geqslant \langle z_n, u_n\rangle + \langle v_n, x_n - u_n\rangle - \epsilon_n s_n\|v_n\|\|a_n\| - \epsilon'_n(1 - s_n)\|v_n\|\|b_n\|. \tag{17}$$

From (13) and (17) we have

$$\begin{aligned}
\langle z_0, x_0\rangle &\geqslant \limsup_{n\to\infty}\langle z_n, x_n\rangle \\
&\geqslant \liminf_{n\to\infty}\langle z_n, x_n\rangle \\
&\geqslant \liminf_{n\to\infty}\langle z_n, u_n\rangle + \liminf_{n\to\infty}\langle v_n, x_n - u_n\rangle + \liminf_{n\to\infty}\big(-\epsilon_n s_n\|v_n\|\|a_n\| - (1 - s_n)\epsilon'_n\|v_n\|\|b_n\|\big) && (18)\\
&\geqslant \langle z_0, u\rangle + \langle v, x_0 - u\rangle. && (19)
\end{aligned}$$

This implies that $\langle z_0 - v, x_0 - u\rangle \geqslant 0$ for all (u, v) in Γ_{T_t}. Hence $z_0 \in T_t(x_0)$ by the maximal monotonicity. Substituting $(u, v) = (x_0, z_0)$ into (19), we get $\lim_{n\to\infty}\langle z_n, x_n\rangle = \langle z_0, x_0\rangle$. Hence it follows from (12) that $\lim_{n\to\infty}\langle f_{t_n}(x_n), x_n - x_0\rangle = 0$. Since $\{f_s, 0 \leqslant s \leqslant 1\}$ is homotopy of class $(S)_+$, we get $x_n \to x_0 \in \partial\Omega$ and $f_{t_n}(x_n) \rightharpoonup f_t(x)$. Hence it yields $y_t = f_t(x_0) + z_0$. This means that $y_t \in (f_t + T_t)(\partial\Omega)$ for some t in [0, 1]. We obtain a contradiction.

Case 2. The sequence $\{s_n a_n\}$ is unbounded.

Since $\{s_n a_n\}$ is unbounded, $\{(1 - s_n)b_n\}$ is also unbounded and we have

$$\lim_{n\to\infty} s_n\|a_n\| = \lim_{n\to\infty}(1 - s_n)\|b_n\| = +\infty.$$

The next lemma will finish the proof of the theorem.

Lemma 3.3.

$$\lim_{n\to\infty}\epsilon_n\|a_n\| = \lim_{n\to\infty}\epsilon'_n\|b_n\| = 0.$$

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Proof. Fix (u_0, v_0) in Γ_{T_t}. Then there exist a sequence $u_n \to u_0$ and a sequence $v_n \to v_0$ such that $v_n \in T_{t_n}(u_n)$. Using the monotonicity of T_{t_n} we get

$$\langle a_n - v_n, x_n - \epsilon_n J^{-1}(a_n) - u_n \rangle \geqslant 0.$$

Hence

$$\langle a_n, x_n - u_n\rangle \geqslant \langle v_n, x_n - u_n\rangle + \epsilon_n\|a_n\|^2 - \epsilon_n\|v_n\|\|a_n\|.$$

This implies that

$$\|a_n\|\|x_n - u_n\| \geqslant \langle v_n, x_n - u_n\rangle + \epsilon_n\|a_n\|^2 - \epsilon_n\|v_n\|\|a_n\|.$$

It is equivalent to

$$\epsilon_n\|a_n\|^2 - \|a_n\|\big(\|x_n - u_n\| + \epsilon_n\|v_n\|\big) + \langle v_n, x_n - u_n\rangle \leqslant 0. \tag{20}$$

Put

$$\Delta_n = \big(\epsilon_n\|v_n\| + \|x_n - u_n\|\big)^2 - 4\epsilon_n\langle v_n, x_n - u_n\rangle.$$

We get

$$\Delta_n \geqslant \big(\|x_n - u_n\| - \epsilon_n\|v_n\|\big)^2.$$

Hence it follows from (20) that

$$\|a_n\| \leqslant \frac{\epsilon_n\|v_n\| + \|x_n - u_n\| + \sqrt{\Delta_n}}{2\epsilon_n}.$$

Consequently,

$$\epsilon_n\|a_n\| \leqslant \frac{\epsilon_n\|v_n\| + \|x_n - u_n\| + \sqrt{\Delta_n}}{2}. \tag{21}$$

Similarly, we have

$$\epsilon'_n\|b_n\| \leqslant \frac{\epsilon'_n\|v_n\| + \|x_n - u_n\| + \sqrt{\Delta_n}}{2}. \tag{22}$$

Since $x_n \rightharpoonup x_0$, $u_n \to u_0$, $v_n \to v_0$, it follows from (21) and (22) that the sequences $\{\epsilon_n\|a_n\|\}$ and $\{\epsilon'_n\|b_n\|\}$ are bounded. We now can assume that

$$\lim_{n\to\infty} \epsilon_n\|a_n\| = \alpha, \qquad \lim_{n\to\infty} \epsilon'_n\|b_n\| = \beta.$$

It remains to show that $\alpha = \beta = 0$. Conversely, suppose that the assertion is false. Then we must have either $\alpha > 0$ or $\beta > 0$. Fixing (u, v) in Γ_{T_t}, we can find out a sequence $u_n \to u$ and a sequence $v_n \to v$ such that $v_n \in T_{t_n}(u_n)$. Using the monotonicity of T_{t_n} and the similar arguments we obtain (16) again. From (13) and (16) we get

$$\begin{aligned}
\langle z_0, x_0\rangle &\geqslant \limsup_{n\to\infty}\langle z_n, x_n\rangle \\
&\geqslant \liminf_{n\to\infty}\langle z_n, x_n\rangle \\
&\geqslant \liminf_{n\to\infty}\langle z_n, u_n\rangle + \liminf_{n\to\infty}\langle v_n, x_n - u_n\rangle + \liminf_{n\to\infty}\big(-s_n\epsilon_n\|a_n\|\|v_n\| - (1-s_n)\epsilon'_n\|b_n\|\|v_n\|\big) \\
&\quad + \liminf_{n\to\infty}\big(\epsilon_n\|a_n\|s_n\|a_n\| + \epsilon'_n\|b_n\|(1-s_n)\|b_n\|\big).
\end{aligned} \tag{23}$$

Note that

$$\begin{aligned}
&\liminf_{n\to\infty}\langle z_n, u_n\rangle = \langle z_0, u\rangle, \\
&\liminf_{n\to\infty}\langle v_n, x_n - u_n\rangle = \langle v, x_0 - u\rangle, \\
&\liminf_{n\to\infty}\big(-s_n\epsilon_n\|a_n\|\|v_n\| - (1-s_n)\epsilon'_n\|b_n\|\|v_n\|\big) = -s\alpha\|v\| - (1-s)\beta\|v\|,
\end{aligned}$$

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and

$$\liminf_{n\to\infty}\big(\epsilon_n\|a_n\|s_n\|a_n\|+\epsilon'_n\|b_n\|(1-s_n)\|b_n\|\big)=+\infty.$$

Hence from (23) we get $\langle z_0, x_0\rangle \geqslant +\infty$ which is an absurd. The proof of the lemma is complete. □

We now take any (u, v) in Γ_{T_t}. Let $u_n \to u$, $v_n \to v$ and $v_n \in T_{t_n}(u_n)$. Using (23) again we get

$$\begin{aligned}\langle z_0, x_0\rangle &\geqslant \limsup_{n\to\infty}\langle z_n, x_n\rangle\\ &\geqslant \liminf_{n\to\infty}\langle z_n, x_n\rangle\\ &\geqslant \liminf_{n\to\infty}\langle z_n, u_n\rangle+\liminf_{n\to\infty}\langle v_n, x_n-u_n\rangle+\liminf_{n\to\infty}\big(-s_n\epsilon_n\|a_n\|\|v_n\|-(1-s_n)\epsilon'_n\|b_n\|\|v_n\|\big)\\ &\geqslant \langle z_0, x_0\rangle+\langle v, x_0-u\rangle. \qquad (24)\end{aligned}$$

This implies that $\langle z_0 - v, x_0 - u\rangle \geqslant 0$. Since (u, v) is arbitrary and T_t is maximal monotone, we obtain $z_0 \in T_t(x_0)$. Substituting $(u, v) = (x_0, z_0)$ into (24) we obtain again $\lim_{n\to\infty}\langle z_n, x_n\rangle = \langle z_0, x_0\rangle$. In the same manner of the proof at the end of Case 1 we show that $y_t \in (f_t + T_t)(\partial\Omega)$ for some t in [0, 1]. This contradicts our assumptions. The proof of the theorem is now complete. □

Remark 3.4. The scheme of our proof is different from the proofs of Theorems 8 and 9 in [2]. In his approach, Browder had used heavily the condition $0 \in T(0)$ and the condition $0 \in T_t(0)$ for all t in [0, 1], to show that there exists a constant $M > 0$ such that $s_n\epsilon_n\|a_n\|^2 \leqslant M$ and $(1 - s_n)\epsilon'_n\|b_n\|^2 \leqslant M$. Then he concluded that $s_n\epsilon_n\|a_n\| \to 0$ and $(1 - s_n)\epsilon'_n\|b_n\| \to 0$. However, this scheme will collapse if those conditions are absent. In our proof above, we do not assume these conditions and give instead a new scheme of reasoning which based on a intrinsic characterization of maximal monotone mappings as well as the structure of the generalized Yosida transformation $T_{,\epsilon}$.

Proof of Theorem 2.5. (a) Choose a sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0^+$. By the definition, there exists $n_0 > 0$ such that

$$d_1(f + T, \Omega, y_0) = d(f + T_{\epsilon_n}, \Omega, y_0)$$

for all $n \geqslant n_0$. Since $d(f + T_{\epsilon_n}, \Omega, y_0) \neq 0$, there exists an x_n in Ω such that $f(x_n) + T_{\epsilon_n}(x_n) = y_0$. Since $\{f(x_n)\}$ is bounded, $\{T_{\epsilon_n}(x_n)\}$ is also bounded. Putting $z_n = T_{\epsilon_n}(x_n)$, we get $z_n \in T(x_n - \epsilon_n J^{-1}(z_n))$ and $f(x_n) + z_n = y_0$. Without loss of generality, we can assume that $x_n \rightharpoonup x_0$, $z_n \rightharpoonup z_0$. By using the similar arguments in the proof of Theorem 2.4 we can show that

$$\limsup_{n\to\infty}\langle z_n, x_n\rangle \leqslant \langle z_0, x_0\rangle. \qquad (25)$$

We now utilize a variant of the proof of Theorem 2.4. Taking any (u, v) in Γ_T and using the monotonicity of T, we have

$$\big\langle z_n - v, x_n - \epsilon_n J^{-1}(z_n) - u\big\rangle \geqslant 0.$$

This implies that

$$\langle z_n, x_n\rangle \geqslant \langle z_n, u\rangle + \langle v, x_n - u\rangle + \epsilon\|z_n\|^2 - \epsilon_n\|z_n\|\|v\|.$$

By (25) we get

$$\begin{aligned}\langle z_0, x_0\rangle &\geqslant \limsup_{n\to\infty}\langle z_n, x_n\rangle\\ &\geqslant \liminf_{n\to\infty}\langle z_n, x_n\rangle\\ &\geqslant \liminf_{n\to\infty}\langle z_n, u\rangle+\liminf_{n\to\infty}\langle v, x_n-u\rangle+\liminf_{n\to\infty}\big(-\epsilon_n\|v\|\|z_n\|\big)\\ &\geqslant \langle z_0, u\rangle+\langle v, x_0-u\rangle. \qquad (26)\end{aligned}$$

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Consequently, $\langle z_0 - v, x_0 - u\rangle \geqslant 0$ for all (u, v) in Γ_T. By the maximal monotonicity of T, it yields $z_0 \in T(x_0)$. Substituting $(u, v) = (z_0, x_0)$ into (26), we get $\lim_{n\to\infty}\langle z_n, x_n\rangle = \langle z_0, x_0\rangle$. Since

$$\langle z_n, x_n\rangle = \langle z_n, x_0\rangle + \langle y_0, x_n - x_0\rangle - \big\langle f(x_n), x_n - x_0\big\rangle,$$

we get $\limsup_{n\to\infty}\langle f(x_n), x_n - x_0\rangle = 0$. Hence $x_n \to x_0$ because f is of class $(S)_+$. On the other hand f is demicontinuous, it follows $f(x_n) \rightharpoonup f(x_0)$. Using the equality $f(x_n) + z_n = y_0$ we obtain $y_0 = f(x_0) + z_0$, where $z_0 \in T(x_0)$ for some x_0 in Ω. Thus we have showed that $y_0 \in f(x_0) + T(x_0)$ for some x_0 in Ω.

(b) It easy to show that there exists $\bar\epsilon > 0$ such that

$$y_0 \notin (f + T_{,\epsilon})\big(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\big)$$

for all ϵ in $(0, \bar\epsilon]$. By Theorem 1.3, we have

$$d(f + T_{,\epsilon}, \Omega, y_0) = d(f + T_{,\epsilon}, \Omega_1, y_0) + d(f + T_{,\epsilon}, \Omega_2, y_0)$$

for all ϵ in $(0, \bar\epsilon]$. Hence

$$d_1(f + T, \Omega, y_0) = d_1(f + T, \Omega_1, y_0) + d_1(f + T, \Omega_2, y_0).$$

(c) The assertion follows directly from Theorem 2.4. The proof of the theorem is complete. □

Proof of Theorem 2.6. *Existence*. Let d_1 be the degree function in Definition 2.2. Put $T_t := (1 - t)T$ and $f_t := (1 - t)f + tg$. Then $\{T_t\}$ is a pseudo-monotone homotopy in the sense of Definition 2.3 (see Appendix A) and $\{f_t\}$ is a homotopy of class $(S)_+$ in the sense of Definition 1.6. By Theorem 2.5, d_1 meets all requirements of the theorem.

Uniqueness. Suppose that there exists two degree functions d_1 and d_2 satisfying the conclusion of our theorem. By Theorem 1.3 on the uniqueness of the degree function for mappings of class $(S)_+$, the restrictions of d_1 and d_2 on this class are equal. Let f be of class $(S)_+$, demicontinuous and bounded, T be maximal monotone and y_0 be a point of X^* with $y_0 \notin (f + T)(\partial\Omega)$. Assume that

$$d_1(f + T, \Omega, y_0) \neq d_2(f + T, \Omega, y_0). \tag{27}$$

By Theorem 2.1, there exists $\bar\epsilon > 0$ such that

$$y_0 \notin (f + T_{,\epsilon})(\partial\Omega), \quad \forall\epsilon \in (0, \bar\epsilon].$$

Since $f + T_{,\epsilon}$ is of class $(S)_+$ and demicontinuous, we have

$$d_1(f + T, \Omega, y_0) = d_1(f + T_{,\epsilon}, \Omega, y_0) = d_2(f + T_{,\epsilon}, \Omega, y_0).$$

By this and (27), it yields

$$d_2(f + T, \Omega, y_0) \neq d_2(f + T_{,\epsilon}, \Omega, y_0). \tag{28}$$

Consider the affine homotopy between $(f + T)$ and $(f + T_{,\epsilon})$ for $\epsilon \in (0, \bar\epsilon]$. In view of (28), we can find $\epsilon_n \to 0^+$, $u_n \in \partial\Omega$ and $t_n \in [0, 1]$ such that

$$y_0 \in (1 - t_n)T(u_n) + t_n T_{,\epsilon_n}(u_n) + f(u_n). \tag{29}$$

Hence there exists w_n in $T_{,\epsilon_n}(u_n)$ such that

$$y_0 = (1 - t_n)w_n + t_n T_{,\epsilon_n}(u_n) + f(u_n). \tag{30}$$

Put $z_n = (1 - t_n)w_n + t_n T_{,\epsilon_n}(u_n)$ and $v_n = T_{,\epsilon_n}(u_n)$. Then $\{z_n\}$ is a bounded sequence because $\{f(u_n)\}$ is a bounded sequence and

$$y_0 = z_n + f(u_n). \tag{31}$$

Without loss of generality we can assume that $u_n \rightharpoonup u_0$, $z_n \rightharpoonup z_0$ and $t_n \to t$.

In the sequel we will use a variant of arguments used in the proof of Theorem 2.4. In particular, we can show that

$$\langle z_0, u_0\rangle \geqslant \limsup\langle z_n, u_n\rangle. \tag{32}$$

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Fixing any (x, y) in Γ_T, we claim that

$$\langle z_0, u_0\rangle \geqslant \limsup\langle z_n, u_n\rangle \geqslant \liminf\langle z_n, u_n\rangle \geqslant \langle y, u_0 - x\rangle + \langle z_0, x\rangle. \tag{33}$$

Since $v_n = T_{,\epsilon_n}(u_n)$, $v_n \in T(u_n - \epsilon_n J^{-1}(v_n))$. By the monotonicity of T, we get

$$\left\langle v_n - y, u_n - \epsilon_n J^{-1}(v_n) - x\right\rangle \geqslant 0$$

and

$$\langle w_n - y, u_n - x\rangle \geqslant 0.$$

It follows that

$$\langle v_n, u_n\rangle \geqslant \langle v_n, x\rangle + \langle y, u_n - x\rangle + \epsilon_n\|v_n\|^2 - \epsilon_n\|y\|\|v_n\| \tag{34}$$

and

$$\langle w_n, u_n\rangle \geqslant \langle w_n, x\rangle + \langle y, u_n - x\rangle. \tag{35}$$

Multiplying (33) with t_n, (34) with $(1 - t_n)$ and summing up, we get

$$\begin{aligned}\langle z_n, u_n\rangle &\geqslant \langle z_n, x\rangle + \langle y, u_n - x\rangle + \epsilon_n t_n\|v_n\|^2 - \epsilon_n t_n\|y\|\|v_n\| \\ &\geqslant \langle z_n, x\rangle + \langle y, u_n - x\rangle - \epsilon_n t_n\|y\|\|v_n\|.\end{aligned} \tag{36}$$

Case 1. The sequence $\{t_n v_n\}$ is bounded.

From (32) and (36) we get

$$\begin{aligned}\langle z_0, u_0\rangle &\geqslant \limsup\langle z_n, u_n\rangle \\ &\geqslant \liminf\langle z_n, u_n\rangle \\ &\geqslant \liminf\langle z_n, x\rangle + \liminf\langle y, u_n - x\rangle + \liminf\big(-\epsilon_n t_n\|v_n\|\|y\|\big) \\ &\geqslant \langle y, u_0 - x\rangle + \langle z_0, x\rangle.\end{aligned}$$

This implies (33).

Case 2. The sequence $\{t_n v_n\}$ is unbounded.

From (34) we have

$$\|v_n\|\|u_n - x\| \geqslant \langle y, u_n - x\rangle + \epsilon_n\|v_n\|^2 - \epsilon_n\|y\|\|v_n\|.$$

It is equivalent to

$$\epsilon_n\|v_n\|^2 - \|v_n\|\big(\|u_n - x\| + \epsilon_n\|y\|\big) + \langle y, u_n - x\rangle \leqslant 0.$$

This implies that

$$\epsilon_n\|v_n\| \leqslant \frac{(\|u_n - x\| + \epsilon_n\|y\|) + \sqrt{\Delta_n}}{2}, \tag{37}$$

where

$$\Delta_n = \big(\|u_n - x\| + \epsilon_n\|y\|\big)^2 - 4\epsilon\langle y, u_n - x\rangle$$

which satisfies the inequality

$$\Delta_n \geqslant \big(\|u_n - x\| - \epsilon_n\|y\|\big)^2.$$

Since $u_n \rightharpoonup u_0$, (37) implies that the sequence $\{\epsilon_n\|v_n\|\}$ is bounded. We now can assume that $\lim \epsilon_n\|v_n\| = \alpha$. If $\alpha > 0$ then (36) implies

$$\begin{aligned}\langle z_0, u_0\rangle &\geqslant \limsup\langle z_n, u_n\rangle\\ &\geqslant \liminf\langle z_n, u_n\rangle\\ &\geqslant \liminf\langle z_n, x\rangle + \liminf\langle y, u_n - x\rangle + \liminf\bigl(-\epsilon_n t_n\|v_n\|\|y\|\bigr) + \liminf t_n\|v_n\|\epsilon_n\|v_n\|\\ &\geqslant \langle y, u_0 - x\rangle + \langle z_0, x\rangle - t\alpha\|y\| + \infty,\end{aligned}$$

which is impossible. Hence we must have $\alpha = 0$. Using (36) again, it yields

$$\begin{aligned}\langle z_0, u_0\rangle &\geqslant \limsup\langle z_n, u_n\rangle\\ &\geqslant \liminf\langle z_n, u_n\rangle\\ &\geqslant \liminf\langle z_n, x\rangle + \liminf\langle y, u_n - x\rangle + \liminf\bigl(-\epsilon_n t_n\|v_n\|\|y\|\bigr)\\ &\geqslant \langle y, u_0 - x\rangle + \langle z_0, x\rangle - t\|y\|\alpha.\end{aligned}$$

This implies (33). Thus we have proved that for any (x, y) in Γ_T, (33) holds. From (33) we get $\langle z_0 - y, u_0 - x\rangle \geqslant 0$. By the maximal monotonicity of T we have $z_0 \in T(u_0)$. Substituting $(x, y) = (u_0, z_0)$ into (33) we find that $\lim\langle z_n, u_n\rangle = \langle z_0, u_0\rangle$. Since

$$\langle z_n, u_n\rangle = \langle z_n, u_0\rangle + \langle y_0, u_n - u_0\rangle - \bigl\langle f(u_n), u_n - u_0\bigr\rangle,$$

we have $\lim\langle f(u_n), u_n - u_0\rangle = 0$. Since f is of class $(S)_+$ and $\partial\Omega$ is closed, $u_n \to u_0 \in \partial\Omega$. On the other hand f is demicontinuous, we have $f(u_n) \rightharpoonup f(u_0)$. From (31), it yields $y_0 = z_0 + f(u_0)$ for some z_0 in $T(u_0)$. Hence $y_0 \in (f + T)(\partial\Omega)$, which contradicts our assumption. Thus we must have $d_1(f + T, \Omega, y_0) = d_2(f + T, \Omega, y_0)$. The proof is complete. □

4. Concluding remark

In summary, we have proved the existence and uniqueness of the degree function in the sense of Definition 1.1 for mappings of type $f + T$. Our results improve and extend those in [2] to a larger class of maximal monotone mappings T and homotopy $\{T_t\}$, namely, without assuming the graph of homotopy $\{T_t\colon 0 \leqslant t \leqslant 1\}$ having a common intersection. By putting $T = N_K$, where N_K is the normal cone of a closed convex set K, which is a maximal monotone mapping, we will obtain results concerning a degree theory for variational inequalities of type $f + N_K$. There are many applications and examples of the above results in this direction, and we refer the readers to [7].

Acknowledgments

In the first online version of this paper, the condition $\overline{D(T)} = X$ has not appeared in Theorem 2.6. It was Professor A.G. Kartsatos who showed us a gap in our proofs and suggested this important additional assumption. The authors wish to express their sincere gratitude to Professor A.G. Kartsatos for many helpful suggestions and comments.

Appendix A

The following result is due to A.G. Kartsatos.

Let $T : X \to X^*$ be a maximal monotone operator and $\lambda > 0$. For each fixed $x \in X$ we consider the equation

$$0 \in \lambda T(y) + J(y - x).$$

This equation has unique solution x_λ (see proof of Theorem 3.13 in [4]. Denote by $J_\lambda x = x_\lambda$ and $\hat{T}_\lambda x = -\frac{1}{\lambda}J(x_\lambda - x)$, $x \in X$ and $\lambda > 0$. Then the maps $J_\lambda : X \to X$ and $\hat{T}_\lambda : X \to X^*$ are single-valued. Moreover, for every $x \in \operatorname{conv} D(T)$ we have $J_\lambda x \to x$ as $\lambda \to 0$.

Theorem. *Let $T : X \to 2^{X^*}$ be a maximal monotone operator with $\overline{D(T)} = X$. Then the family of mappings T_s with $T_s x = sTx$, $(s, x) \in [0, 1] \times D(T)$, is a pseudomonotone homotopy.*

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Proof. Let $s_n \to s_0 \in [0,1]$ and $(x, x^*) \in \Gamma_{T_{s_0}}$. We have to show that there exists a sequence $\{(x_n, x_n^*)\}$ such that $(x_n, x_n^*) \in \Gamma_{T_{s_n}}$ and $(x_n, x_n^*) \to (x, x^*)$ as $n \to \infty$.

Let $s_0 > 0$. Since $x^* \in s_0 Tx$, $x^* = s_0 y^*$ for some $y^* \in Tx$. Putting $x_n = x$ and $x_n^* = s_n y^*$, we have $(x_n, x_n^*) \in \Gamma_{T_{s_n}}$ and $(x_n, x_n^*) \to (x, x^*)$.

Now, we consider the crucial case $s_0 = 0$. Since $T_0 x = \{0\}$, $x^* = 0$. Put

$$x_n = J_{t_n} x, \quad x_n^* = s_n \hat{T}_{t_n} x, \quad \text{where } t_n = \sqrt{s_n}.$$

Since $\hat{T}_{t_n} x \in T x_n$, $x_n^* \in s_n T x_n = T_{s_n} x_n$. Besides, we have $x_n \to x$ because of $x \in X = \overline{D(T)}$. We now have

$$\|x_n^*\| = s_n \|\hat{T}_{t_n} x\| \leqslant s_n \frac{1}{t_n} \|x_n - x\| \to 0.$$

Hence $x_n^* \to x^*$. $\square$

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