Weighted Composition Operators of $C_0(X)$'s *†

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Abstract

In this paper, we prove that into isometries and disjointness preserving linear maps from $C_0(X)$ into $C_0(Y)$ are essentially weighted composition operators $Tf = h \cdot f \circ \varphi$ for some continuous map φ and some continuous scalar-valued function h.

1 Introduction.

Let X and Y be locally compact Hausdorff spaces. Let $C_0(X)$ (resp. $C_0(Y)$) be the Banach space of continuous scalar-valued (i.e. real- or complex-valued) functions defined on X (resp. Y) vanishing at infinity and equipped with the supremum norm. The classical Banach-Stone theorem gives a description of surjective isometries from $C_0(X)$ onto $C_0(Y)$. They are all *weighted composition operators* $Tf = h \cdot f \circ \varphi$ (i.e. $Tf(y) = h(y)f(\varphi(y))$, $\forall y \in Y$) for some homeomorphism φ from Y onto X and some continuous scalar-valued function h on Y with $|h(y)| \equiv 1$, $\forall y \in Y$. Different generalizations (see e.g. [1], [2], [4], [5], [7]) of the Banach-Stone Theorem have been studied in many years. Some of them discuss the structure of *into* isometries and disjointness preserving linear maps (see e.g. [3], [6]). A linear map from $C_0(X)$ into $C_0(Y)$ is said to be *disjointness preserving* if $f \cdot g = 0$ in $C_0(X)$ implies $Tf \cdot Tg = 0$ in $C_0(Y)$. In this paper, we shall discuss the structure of weighted composition operators from $C_0(X)$ into $C_0(Y)$.

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We prove that every into isometry and every disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$ is essentially a weighted composition operator.

Theorem 1. *Let X and Y be locally compact Hausdorff spaces and T a linear isometry from $C_0(X)$ into $C_0(Y)$. Then there exist a locally compact subset Y_1 (i.e. Y_1 is locally compact in the subspace topology) and a weighted composition operator T_1 from $C_0(X)$ into $C_0(Y_1)$ such that for all f in $C_0(X)$,*

$$Tf_{|Y_1} = T_1 f = h \cdot f \circ \varphi,$$

for some quotient map φ from Y_1 onto X and some continuous scalar-valued function h defined on Y_1 with $|h(y)| \equiv 1$, $\forall y \in Y_1$.

Theorem 2. *Let X and Y be locally compact Hausdorff spaces and T a bounded disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$. Then there exist an open subset Y_1 of Y and a weighted composition operator T_1 from $C_0(X)$ into $C_0(Y_1)$ such that for all f in $C_0(X)$, Tf vanishes outside Y_1 and*

$$Tf_{|Y_1} = T_1 f = h \cdot f \circ \varphi,$$

for some continuous map φ from Y_1 into X and some continuous scalar-valued function h defined on Y_1 with $h(y) \neq 0$, $\forall y \in Y_1$.

Since weighted composition operators from $C_0(X)$ into $C_0(Y)$ are disjointness preserving, Theorem 2 gives a complete description of all such maps. When X and Y are both compact, Theorems 1 and 2 reduce to the results of W. Holsztynski [3] and K. Jarosz [6], respectively. It is plausible to think that Theorems 1 and 2 could be easily obtained from their compact space versions by simply extending an into isometry (or a bounded disjointness preserving linear map) $T : C_0(X) \longrightarrow C_0(Y)$ to a bounded linear map $T_\infty : C(X_\infty) \longrightarrow C(Y_\infty)$ of the same type, where $X_\infty = X \bigcup \{\infty\}$ and $Y_\infty = Y \bigcup \{\infty\}$ are the one-point compactifications of the locally compact Hausdorff spaces X and Y, respectively. However, the example given in Section 4 will show that this idea is sometimes fruitless because T can have *no* such extensions at all. We thus have to modify, and in some cases give new arguments to, the proofs of W. Holsztynski [3] and K. Jarosz [6] to fit into our more general settings in this paper.

Recall that for f in $C_0(X)$, the *cozero* of f is $\mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}$ and the *support* $\mathrm{supp}(f)$ of f is the closure of $\mathrm{coz}(f)$ in X_∞. A linear map $T : C_0(X) \longrightarrow C_0(Y)$ is disjointness preserving if T maps functions with disjoint cozeroes to functions with disjoint cozeroes. For x in X, δ_x denotes the point evaluation at x, that is, δ_x is the linear functional on $C_0(X)$ defined by $\delta_x(f) = f(x)$. For y in Y, let $\mathrm{supp}(\delta_y \circ T)$ be the set of all x in X_∞ such that for any open neighborhood U of x in X_∞ there is an f in $C_0(X)$ with $Tf(y) \neq 0$ and $\mathrm{coz}(f) \subset U$. The kernel of a function f is denoted by $\ker f$.

2 Isometries from $C_0(X)$ into $C_0(Y)$.

Definition. Let X and Y be locally compact Hausdorff spaces. A map φ from Y into X is said to be *proper* if preimages of compact subsets of X under φ are compact in Y.

It is obvious that φ is proper if and only if $\lim_{y\to\infty} \varphi(y) = \infty$. As a consequence, a proper continuous map φ from a locally compact Hausdorff space Y onto a locally compact Hausdorff space X is a quotient map, i.e. $\varphi^{-1}(O)$ is open in X if and only if O is open in Y. A quotient map from a locally compact space onto another is, however, not necessarily proper. For example, the quotient map φ from $(-\infty, +\infty)$ onto $[0, +\infty)$ defined by

$$\varphi(y) = \begin{cases} y, & y > 0, \\ 0, & y \leq 0 \end{cases}$$

is not proper.

Lemma 3. *Let X and Y be locally compact Hausdorff spaces, φ a map from Y into X, and h a continuous scalar-valued function defined on Y with bounds $M, m > 0$ such that $m \leq |h(y)| \leq M$, $\forall y \in Y$. Then the weighted composition $Tf = h \cdot f \circ \varphi$ defines a (necessarily bounded) linear map from $C_0(X)$ into $C_0(Y)$ if and only if φ is continuous and proper.*

Proof. For the sufficiency, we need to verify that $h \cdot f \circ \varphi$ vanishes at ∞ for all f in $C_0(X)$. For any $\epsilon > 0$, $|f(x)| < \epsilon/M$ outside some compact subset K of X. Since φ is proper, $\varphi^{-1}(K)$ is compact in Y. Now the fact that $|h(y) \cdot f(\varphi(y))| \leq M|f(\varphi(y))| < \epsilon$ outside $\varphi^{-1}(K)$ indicates that $h \cdot f \circ \varphi \in C_0(Y)$. The boundedness of T is trivial in this case.

For the necessity, we first check the continuity of φ. Suppose $y_\lambda \longrightarrow y$ in Y. We want to show that $x_\lambda = \varphi(y_\lambda) \longrightarrow \varphi(y)$ in X. Suppose not, by passing to a subnet if necessary, we can assume that x_λ either converges to some $x \neq \varphi(y)$ in

X or ∞. If $x_\lambda \longrightarrow x$ in X then for all f in $C_0(X)$,

$$\begin{aligned} h(y)f(x) &= \lim h(y_\lambda)f(x_\lambda) = \lim h(y_\lambda)f(\varphi(y_\lambda)) \\ &= \lim Tf(y_\lambda) = Tf(y) = h(y)f(\varphi(y)). \end{aligned}$$

As $h(y) \neq 0$, $f(x) = f(\varphi(y))$, $\forall f \in C_0(X)$. Consequently, we obtain a contradiction $x = \varphi(y)$! If $x_\lambda \longrightarrow \infty$ then a similar argument gives $f(\varphi(y)) = 0$ for all f in $C_0(X)$. Hence $\varphi(y) = \infty$, a contradiction again! Therefore, φ is continuous from Y into X. Finally, let K be a compact subset of X and we are going to see that $\varphi^{-1}(K)$ is compact in Y, or equivalently, closed in $Y_\infty = Y \cup \{\infty\}$, the one-point compactification of Y. To see this, suppose $y_\lambda \longrightarrow y$ in Y_∞ and $x_\lambda = \varphi(y_\lambda) \in K$. We want $y \in \varphi^{-1}(K)$, i.e. $y \neq \infty$ and $\varphi(y) \in K$. Without loss of generality, we can assume that $x_\lambda \longrightarrow x$ for some x in K. Now,

$$\lim |Tf(y_\lambda)| = \lim |h(y_\lambda)f(\varphi(y_\lambda))| \geq m \lim |f(x_\lambda)| = m|f(x)|$$

for all f in $C_0(X)$. This implies that $y \neq \infty$ and then a similar argument gives $\varphi(y) = x \in K$. $\square$

The assumption on the bounds of f in Lemma 3 is significant. For example, let $X = Y = \mathbb{R} = (-\infty, +\infty)$ and define

$$h(y) = \begin{cases} e^y, & y < 0, \\ 1, & y \geq 0, \end{cases} \qquad \text{and} \qquad \varphi(y) = \begin{cases} \sin y, & y < 0, \\ y, & y \geq 0. \end{cases}$$

Then the weighted composition operator $Tf = h \cdot f \circ \varphi$ from $C_0(\mathbb{R})$ into $C_0(\mathbb{R})$ is well-defined. It is not difficult to see that $\varphi^{-1}([-\frac{1}{2}, \frac{1}{2}])$ is not compact in $\mathbb{R}$. On the other hand, if we redefine $h(y) = e^y$ and $\varphi(y) = y$ for all y in $\mathbb{R}$ then the weighted composition operator T is not well-defined from $C_0(\mathbb{R})$ into $C_0(\mathbb{R})$, even though φ is proper and continuous in this case.

Recall that a bounded linear map T from a Banach space E into a Banach space F is called an *injection* if there is an $m > 0$ such that $\|Tx\| \geq m\|x\|$, $\forall x \in E$. It follows from the open mapping theorem that T is an injection if and only if T is one-to-one and has closed range.

Proposition 4. *Let X and Y be locally compact Hausdorff spaces, φ a map from Y into X and h a continuous scalar-valued function defined on Y. The weighted composition operator $Tf = h \cdot f \circ \varphi$ from $C_0(X)$ into $C_0(Y)$ is an injection if and only if φ is continuous, proper and onto and h has bounds $M, m > 0$ such* *that* $m \leq |h(y)| \leq M$, $\forall y \in Y$. *In this case,* φ *is a quotient map and thus* X *is a quotient space of* Y.

Proof. The sufficiency follows easily from Lemma 3 and the observation that $\|Tf\| = \|h \cdot f \circ \varphi\| \geq m\|f\|$, $\forall f \in C_0(X)$. For the necessity, we first note that there are constants $M, m > 0$ such that $m\|f\| \leq \|Tf\| \leq M\|f\|$ for all f in $C_0(X)$. It is then obvious that $m \leq |h(y)| \leq M$, $\forall y \in Y$. By Lemma 3, φ is continuous and proper. Finally, we check that φ is onto. It is not difficult to see that φ has dense range. In fact, if $\varphi(Y)$ were not dense in X, then there were an x in X and a neighborhood U of x in X such that $U \cap \varphi(Y) = \emptyset$. Choose an f in $C_0(X)$ such that $f(x) = 1$ and f vanishes outside U. Then $Tf(y) = h(y)f(\varphi(y)) = 0$ for all y in Y, i.e. $Tf = 0$. Since T is an injection, we get a contradiction that $f = 0$! We now show that $\varphi(Y) = X$. Let $x \in X$ and K a compact neighborhood of x in X. By the density of $\varphi(Y)$ in X, there is a net $\{y_\lambda\}$ in Y such that $x_\lambda = \varphi(y_\lambda) \longrightarrow x$ in X. Without loss of generality, we can assume that x_λ belongs to K for all λ. Since $\varphi^{-1}(K)$ is compact in Y, $\varphi(\varphi^{-1}(K))$ is a compact subset of X containing the net $\{x_\lambda\}$. Consequently, $x = \lim x_\lambda$ belongs to $\varphi(\varphi^{-1}(K)) \subset \varphi(Y)$. □

Proof of Theorem 1. We adopt some notations from W. Holsztynski [3] and K. Jarosz [6]. Let $X_\infty = X \cup \{\infty\}$ and $Y_\infty = Y \cup \{\infty\}$ be the one-point compactifications of X and Y, respectively. For each x in X and y in Y, put

$$\begin{aligned} S_x &= \{f \in C_0(X) : |f(x)| = \|f\| = 1\}, \\ R_y &= \{g \in C_0(Y) : |g(y)| = \|g\| = 1\}, \text{ and} \\ Q_x &= \{y \in Y : T(S_x) \subset R_y\}. \end{aligned}$$

We first claim that $\{Q_x\}_{x \in X}$ is a disjoint family of non-empty subsets of Y. In fact, for f_1, $f_2, \cdots$, f_n in S_x, let $h = \sum_{i=1}^n \overline{f_i(x)} f_i$. Then $\|h\| = n$ and thus $\|Th\| = n$. Hence there is a y in Y such that $|\sum_{i=1}^n \overline{f_i(x)} Tf_i(y)| = |Th(y)| = n$. This implies $|Tf_i(y)| = 1$ for all $i = 1, 2, \cdots, n$. In other words, $y \in \bigcap_{i=1}^n (Tf_i)^{-1}(\Gamma)$, where $\Gamma = \{z : |z| = 1\}$. We have just proved that the family $\{(Tf)^{-1}(\Gamma) : f \in S_x\}$ of closed subsets of the compact space Y_∞ has finite intersection property. It is plain that $\infty \notin (Tf)^{-1}(\Gamma)$ for all f in S_x. Hence $Q_x = \bigcap_{f \in S_x} (Tf)^{-1}(\Gamma)$ is non-empty for all x in X. Moreover, $Q_{x_1} \cap Q_{x_2} = \emptyset$ if $x_1 \neq x_2$ in X. In fact, f_1 in S_{x_1} and f_2 in S_{x_2} exist such that $\operatorname{coz}(f_1) \cap \operatorname{coz}(f_2) = \emptyset$. If there is a y in $Q_{x_1} \cap Q_{x_2}$ then it follows from $Tf_1 \in R_y$ and $Tf_2 \in R_y$ that $1 = \|f_1 + f_2\| = \|T(f_1 + f_2)\| = |T(f_1 + f_2)(y)| = 2$, a contradiction!

Let $Y_1 = \bigcup_{x\in X} Q_x$. It is not difficult to see that $\operatorname{supp}(\delta_y \circ T) = \{x\}$ whenever $y \in Q_x$. So we can define a surjective map $\varphi : Y_1 \to X$ by

$$\{\varphi(y)\} = \operatorname{supp}(\delta_y \circ T).$$

Note that for all f in $C_0(X)$ and for all y in Y_1,

$$\varphi(y) \notin \operatorname{supp}(f) \Longrightarrow T(f)(y) = 0. \tag{1}$$

In fact, if $Tf(y) \neq 0$, without loss of generality, we can assume $Tf(y) = r > 0$ and $\|f\| = 1$. Since $\varphi(y) \notin \operatorname{supp}(f)$, there is a g in $C_0(X)$ such that $\operatorname{coz}(f) \bigcap \operatorname{coz}(g) = \emptyset$ and $Tg(y) = \|g\| = 1$. Hence $1+r = T(f+g)(y) > \|f+g\| = 1$, a contradiction!

Now, we want to show that φ is continuous. Suppose φ were not continuous at some y in Y_1, without loss of generality, let $\{y_\lambda\}$ be a net converging to y in Y_1 such that $\varphi(y_\lambda) \to x \neq \varphi(y)$ in X_∞. Then there exist disjoint neighborhoods U_1 and U_2 of x and $\varphi(y)$ in X_∞, respectively, and a λ_0 such that $\varphi(y_\lambda) \in U_1$, $\forall \lambda \geq \lambda_0$. Let $f \in C_0(X)$ such that $\operatorname{coz}(f) \subseteq U_2$ and $T(f)(y) = \|f\| = 1$. As $\operatorname{supp}(f) \bigcap U_1 = \emptyset$, we have $\varphi(y_\lambda) \notin \operatorname{supp}(f)$, $\forall \lambda \geq \lambda_0$. By (1), $T(f)(y_\lambda) = 0$, $\forall \lambda \geq \lambda_0$. This implies $T(f)$ is not continuous at y, a contradiction!

For each y in Y_1, put

$$\begin{aligned} J_y &= \{f \in C_0(X) : \varphi(y) \notin \operatorname{supp}(f)\}, \text{ and} \\ K_y &= \{f \in C_0(X) : f(\varphi(y)) = 0\}. \end{aligned}$$

For f in K_y and $\varepsilon > 0$, let $X_1 = \{x \in X : |f(x)| \geq \varepsilon\}$ and $X_2 = \{x \in X : |f(x)| \leq \varepsilon/2\}$. Let g be a continuous function defined on X such that $0 \leq g(x) \leq 1, \forall x \in X$, $g(x) = 1, \forall x \in X_1$, and $g(x) = 0, \forall x \in X_2$. Let $f_\varepsilon = g \cdot f$. Then $f_\varepsilon \in J_y$ and $\|f_\varepsilon - f\| \leq 2\varepsilon$. One thus can show that J_y is a dense subset of K_y. By (1), $J_y \subset \ker(\delta_y \circ T)$, and hence $\ker(\delta_{\varphi(y)}) = K_y \subset \ker(\delta_y \circ T)$. Consequently, there exists a scalar $h(y)$ such that $\delta_y \circ T = h(y) \cdot \delta_{\varphi(y)}$, i.e.

$$T(f)(y) = h(y) \cdot f(\varphi(y)), \qquad \forall f \in C_0(X).$$

It follows from the definition of Y_1 that h is continuous on Y_1 and $|h(y)| = 1$, $\forall y \in Y_1$.

It is the time to see that Y_1 is locally compact. For each y_1 in Y_1 and a neighborhood U_1 of y_1 in Y_1, we want to find a compact neighborhood K_1 of y_1 in Y_1 such that $y_1 \in K_1 \subset U_1$. Let $x_1 = \varphi(y_1)$ in X. Then

$$|Tf(y_1)| = |f(x_1)|, \quad \forall f \in C_0(X).$$

Fix f_1 in S_{x_1}. Then $V_1 = \varphi^{-1}(\{x \in X : |f_1(x)| > \frac{1}{2}\}) \cap U_1$ is an open neighborhood of y_1 in Y_1 and contained in U_1. Since $V_1 = W \bigcap Y_1$ for some neighborhood W of y_1 in Y, there exists a compact neighborhood K of y_1 in Y such that $y_1 \in K \subset W$. We are going to verify that $K_1 = K \bigcap Y_1$ is a compact neighborhood of y_1 in Y_1. Let $\{y_\lambda\}$ be a net in $K_1 \subset V_1$. By passing to a subnet, we can assume that y_λ converges to y in K and we want to show $y \in Y_1$. Let $x_\lambda = \varphi(y_\lambda)$ in X. Since X_∞ is compact, by passing to a subnet again, we can assume that x_λ converges to x in X or $x_\lambda \to \infty$. If $x_\lambda \to x$ in X, $|Tf(y)| = \lim |Tf(y_\lambda)| = \lim |h(y_\lambda) f(\varphi(y_\lambda))| = \lim |f(x_\lambda)| = |f(x)|$, for all f in $C_0(X)$. Hence $y \in Q_x$, and thus $y \in Y_1$. If $x_\lambda \to \infty$, $|Tf_1(y)| = \lim |Tf_1(y_\lambda)| = \lim |h(y_\lambda) f_1(\varphi(y_\lambda))| = \lim |f_1(x_\lambda)| = 0$. However, the fact that $y_\lambda \in V_1$ ensures $|Tf_1(y_\lambda)| = |f_1(x_\lambda)| > \frac{1}{2}$ for all λ, a contradiction! Hence Y_1 is locally compact.

Let $T_1 : C_0(X) \to C_0(Y_1)$ defined by $T_1 f = h \cdot f \circ \varphi$. It is clear that T_1 is a linear isometry and $Tf_{|Y_1} = T_1 f$. By Proposition 4, the surjective continuous map φ is proper and thus a quotient map. The proof is complete. □

In Theorem 1, Y_1 can be neither open nor closed in Y and φ may not be an open map. See the following examples.

Example 5. Let $X = [0, +\infty)$ and $Y = [-\infty, +\infty]$. Let T be a linear isometry from $C_0(X)$ into $C_0(Y)$ defined for all f in $C_0(X)$ by

$$Tf(y) = \begin{cases} f(y), & 0 \le y < +\infty, \\ \frac{e^y}{2}(f(-y) + f(0)), & -\infty < y \le 0, \\ 0, & y = \pm\infty. \end{cases}$$

Then in notations of Theorem 1, $Y_1 = [0, +\infty)$ is neither closed nor open in Y. In this case, $\varphi(y) = y$ for all y in $[0, +\infty)$, and X and Y_1 are homeomorphic. □

Example 6. Let $X = \mathbb{R}$ and $Y = \{(x, y) \in \mathbb{R}^2 : y = 0\} \bigcup \{(x, y) \in \mathbb{R}^2 : 0 \le x, 0 \le y \le 1\}$. Let $\varphi : Y \to X$ defined by $\varphi(u_1, u_2) = u_1$. Then φ is continuous, onto and proper, and thus a quotient map. Moreover, $T : C_0(X) \to C_0(Y)$ defined by $Tf = f \circ \varphi$ is a linear isometry. Note that $O = \{(x, y) \in \mathbb{R}^2 : 0 \le x < 1, 0 < y \le 1\}$ is open in Y, but $\varphi(O) = [0, 1)$ is not open in X. Hence φ is not an open map. □

3 Disjointness preserving linear maps from $C_0(X)$ to $C_0(Y)$.

It is clear that Theorem 2 follows from the following more general result in which discontinuity of the linear disjointness preserving map T is allowed. The payoff of the discontinuity is a finite subset F of X at which the behaviour of T is not under control.

Theorem 7. *Let X and Y be locally compact Hausdorff spaces and T a disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$. Then Y can be written as a disjoint union $Y = Y_1 \cup Y_2 \cup Y_3$, in which Y_2 is open and Y_3 is closed. A continuous map φ from $Y_1 \cup Y_2$ into X_∞ exists such that for every f in $C_0(X)$,*

$$\varphi(y) \notin \operatorname{supp}(f) \Longrightarrow T(f)(y) = 0. \tag{2}$$

Moreover, a continuous bounded non-vanishing scalar-valued function h on Y_1 exists such that

$$\begin{aligned} Tf_{|Y_1} &= h \cdot f \circ \varphi, \text{ and} \\ Tf_{|Y_3} &= 0. \end{aligned}$$

Furthermore, $F = \varphi(Y_2)$ is a finite set and the functionals $\delta_y \circ T$ are discontinuous on $C_0(X)$ for all y in Y_2.

Proof. We shall follow the plan of K. Jarosz in his compact space version [6]. Set

$$\begin{aligned} Y_3 &= \{y \in Y | \delta_y \circ T \equiv 0\}, \\ Y_2 &= \{y \in Y | \delta_y \circ T \text{ is discontinuous}\}, \text{ and} \\ Y_1 &= Y \setminus (Y_2 \bigcup Y_3). \end{aligned}$$

First, we claim that $\operatorname{supp}(\delta_y \circ T)$ contains exactly one point for every y in $Y_1 \bigcup Y_2$. Suppose on the contrary that $\operatorname{supp}(\delta_y \circ T)$ contains two distinct points x_1 and x_2 in X_∞. Let U_1 and U_2 be neighborhoods of x_1 and x_2 in X_∞, respectively, such that $U_1 \cap U_2 = \emptyset$. Let f_1 and f_2 in $C_0(X)$ with $\operatorname{coz}(f_1) \subset U_1$ and $\operatorname{coz}(f_2) \subset U_2$ be such that $Tf_1(y) \neq 0$ and $Tf_2(y) \neq 0$. However, $f_1 f_2 = 0$ implies $Tf_1 Tf_2 = 0$, a contradiction! Suppose $\operatorname{supp}(\delta_y \circ T)$ is empty. Then we can write the compact

Hausdorff space X_∞ as a finite union of open sets $X_\infty = \bigcup_{i=1}^n U_i$ such that $Tf(y) = 0$ whenever $\mathrm{coz}(f) \subset U_i$ for some $i = 1, 2, \ldots, n$. Let $\mathbf{1} = \sum_{i=1}^n f_i$ be a continuous decomposition of the identity coordinate to $\{U_i\}_{i=1}^n$. Then for all f in $C_0(X)$, $Tf(y) = \sum_{i=1}^n T(ff_i)(y) = 0$. This says $\delta_y \circ T \equiv 0$ and thus $y \in Y_3$.

Next we define a map φ from $Y_1 \bigcup Y_2$ into X_∞ by

$$\{\varphi(y)\} = \mathrm{supp}(\delta_y \circ T).$$

We now prove (2). Assume $\varphi(y) \notin \mathrm{supp}(f)$. Then there is an open neighborhood U of $\varphi(y)$ disjoint from $\mathrm{coz}(f)$. Let $g \in C_0(X)$ such that $\mathrm{coz}(g) \subset U$ and $Tg(y) \neq 0$. Since $fg = 0$ and T is disjointness preserving, $Tf(y) = 0$ as asserted.

It then follows from (2) the continuity of φ as one can easily modify an argument of the proof of Theorem 1 for this goal. Similarly, it also follows from (2) the desired representation

$$Tf(y) = h(y)f(\varphi(y)), \qquad \forall f \in C_0(X), \forall y \in Y_1, \tag{3}$$

where h is a continuous non-vanishing scalar-valued function defined on Y_1.

Claim. *Let $\{y_n\}_{n=1}^\infty$ be a sequence in $Y_1 \bigcup Y_2$ such that $x_n = \varphi(y_n)$'s are distinct points of X. Then*

$$\limsup \|\delta_{y_n} \circ T\| < \infty.$$

In particular, only finitely many $\delta_y \circ T$ can have infinite norms.

Assume the contrary and, by passing to a subsequence if necessary, we have

$$\|\delta_{y_n} \circ T\| > n^4, \quad n = 1, 2, \cdots.$$

Let $f_n \in C_0(X)$ with $\|f_n\| \leq 1$ such that

$$|Tf_n(y_n)| \geq n^3, \ n = 1, 2, \cdots.$$

Let V_n, W_n and U_n be open subsets of X such that $x_n \in V_n \subseteq \overline{V_n} \subseteq W_n \subseteq \overline{W_n} \subseteq U_n$ and $U_n \bigcap U_m = \emptyset$ if $n \neq m$, $n, m = 1, 2, \cdots$, and let $g_n \in C(X_\infty)$ such that $0 \leq g_n \leq 1$, $g_{n|_{V_n}} \equiv 1$ and $g_{n|_{X_\infty \setminus W_n}} \equiv 0$, $n = 1, 2, \cdots$. Then (2) implies

$$\begin{aligned} Tf_n(y_n) &= T(f_n g_n)(y_n) + T(f_n(1 - g_n))(y_n) \\ &= T(f_n g_n)(y_n), \quad n = 1, 2, \cdots. \end{aligned}$$

Therefore, we can assume $\text{supp} f_n \subset U_n$. Let $f = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n$ in $C_0(X)$. By (2) again, $|Tf(y_n)| = |\frac{1}{n^2} Tf_n(y_n)| \geq n$ for $n = 1, 2, \cdots$. This conflicts with the boundedness of Tf in $C_0(Y)$, and the claim is thus verified.

The assertion $F = \varphi(Y_2)$ is a finite subset of X is clearly a consequence of the claim while the boundedness of h follows from the claim and (3). It is also plain that $Y_3 = \bigcap\{\ker Tf : f \in C_0(X)\}$ is closed in Y. Finally, to see that Y_2 is open, we consider for every f in $C_0(X)$,

$$\begin{aligned}
\sup\{|Tf(y)| : y \in \overline{Y_1 \cup Y_3}\} &= \sup\{|Tf(y)| : y \in Y_1 \cup Y_3\} \\
&= \sup\{|Tf(y)| : y \in Y_1\} \\
&= \sup\{|h(y)f(\varphi(y))| : y \in Y_1\} \leq M\|f\|,
\end{aligned}$$

where $M > 0$ is a bounded of h on Y_1. It follows that the linear functional $\delta_y \circ T$ is bounded for all y in $\overline{Y_1 \cup Y_3}$, and thus $Y_2 \cap \overline{Y_1 \cup Y_3} = \emptyset$. Hence, $Y_1 \cup Y_3 = \overline{Y_1 \cup Y_3}$ is closed. In other words, Y_2 is open. □

Theorem 8. *Let X and Y be locally compact Hausdorff spaces and T a bijective disjointness preserving linear map from $C_0(X)$ onto $C_0(Y)$. Then T is a bounded weighted composition operator, and X and Y are homeomorphic.*

Proof. We adopt the notations used in Theorem 7. Since T is surjective, $Y_3 = \emptyset$. We are going to verify that $Y_2 = \emptyset$, too. First, we note that the finite set $F \setminus \{\infty\}$ consists of non-isolated points in X. In fact, if $y \in Y_2$ such that $x = \varphi(y)$ is an isolated point in X then it follows from (2) that for every f in $C_0(X)$, $f(x) = 0$ implies $\varphi(y) = x \notin \text{supp} f$ and thus $Tf(y) = 0$. Hence, $\delta_y \circ T = \lambda\delta_x$ for some scalar λ. Therefore, $\delta_y \circ T$ is continuous, a contradiction to the assumption that $y \in Y_2$. We then claim that $\varphi(Y) = \varphi(Y_1 \bigcup Y_2)$ is dense in X. In fact, if a nonzero f in $C_0(X)$ exists such that $\text{supp} f \bigcap \varphi(Y) = \emptyset$ then $Tf = 0$ by (2), conflicting with the injectivity of T. Since

$$X = \overline{\varphi(Y)} = \overline{\varphi(Y_1) \bigcup \varphi(Y_2)} = \overline{\varphi(Y_1) \bigcup F} = \overline{\varphi(Y_1)} \text{ or } \overline{\varphi(Y_1)} \bigcup \{\infty\},$$

for every f in $C_0(X)$,

$$Tf_{|_{Y_1}} = 0 \Longrightarrow f_{|_{\varphi(Y_1)}} = 0 \Longrightarrow f = 0 \Longrightarrow Tf_{|_{Y_2}} = 0.$$

Therefore, the open set $Y_2 = \emptyset$ by the surjectivity of T. Theorem 7 then gives

$$Tf = h \cdot (f \circ \varphi), \quad \forall f \in C_0(X).$$

This representation implies that T^{-1} is also a bijective disjointness preserving linear map from $C_0(Y)$ onto $C_0(X)$. The above discussion provides that

$$T^{-1}g = h_1 \cdot g \circ \varphi_1, \quad \forall g \in C_0(Y),$$

for some continuous non-vanishing scalar-valued function h_1 on X and continuous function φ_1 from X into Y. It is plain that $\varphi_1 = \varphi^{-1}$ and thus X and Y are homeomorphic. □

4 A counter example.

The following example shows that not every into isometry or bounded disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$ can be extended to a bounded linear map from $C(X_\infty)$ into $C(Y_\infty)$ of the same type. Here X and Y are locally compact Hausdorff spaces with one-point compactifications X_∞ and Y_∞, respectively.

Example 9. Let $X = [0, +\infty)$, $Y = (-\infty, +\infty)$ and the underlying scalar field is the field $\mathbb{R}$ of real numbers. Let

$$h(y) = \begin{cases} 1, & y > 2, \\ y - 1, & 0 \leq y \leq 2, \\ -1, & y < 0, \end{cases}$$

and

$$\varphi(y) = \begin{cases} y, & y \geq 0, \\ -y, & y < 0. \end{cases}$$

Then the weighted composition operator $Tf = h \cdot f \circ \varphi$ is simultaneously an into isometry and a bounded disjointness preserving linear map from $C_0([0, +\infty))$ into $C_0((-\infty, +\infty))$. However, no bounded linear extension T_∞ from $C([0, \infty])$ into $C((-\infty, +\infty) \cup \{\infty\})$ of T can be an into isometry or a disjointness preserving linear map.

Suppose, on the contrary, T_∞ were an into isometry. Consider f_n in $C_0([0, +\infty))$ defined by

$$f_n(x) = \begin{cases} 1, & 0 \leq x \leq n, \\ \frac{2n-x}{n}, & n < x < 2n, \\ 0, & 2n \leq x \leq +\infty, \end{cases} \quad n = 1, 2, \ldots .$$

Note that $\delta_y \circ T_\infty$ can be considered as a bounded Borel measure m_y on $[0, +\infty]$ for all point evaluation δ_y at y in $(-\infty, +\infty) \cup \{\infty\}$ with total variation $\|m_y\| =$

$\|\delta_y \circ T_\infty\| \leq 1$. Let $\mathbf{1}$ be the constant function $\mathbf{1}(x) \equiv 1$ in $C([0, +\infty])$. For all y in $(-\infty, +\infty)$,

$$\begin{aligned} T_\infty \mathbf{1}(y) &= \delta_y \circ T_\infty(\mathbf{1}) = \int_{[0,+\infty]} \mathbf{1}\, dm_y \\ &= \lim_{n\to\infty} \int_{[0,+\infty]} f_n\, dm_y + m_y(\{\infty\}) = \lim_{n\to\infty} \delta_y \circ T_\infty(f_n) + m_y(\{\infty\}) \\ &= \lim_{n\to\infty} Tf_n(y) + m_y(\{\infty\}) = \lim_{n\to\infty} h(y) \cdot f_n(\varphi(y)) + m_y(\{\infty\}) \\ &= h(y) + m_y(\{\infty\}). \end{aligned}$$

Let $g(y) = m_y(\{\infty\})$ for all y in $(-\infty, +\infty)$. Then $g(y) = T_\infty \mathbf{1}(y) - h(y)$ is continuous on $(-\infty, +\infty)$ and $|g(y)| = |m_y(\{\infty\})| \leq \|m_y\| \leq 1, \forall y \in (-\infty, +\infty)$. Note that $\|T_\infty \mathbf{1}\| = 1$. Therefore, $g(y) = T_\infty \mathbf{1}(y) - 1 \leq 0$ when $y > 2$, and $g(y) = T_\infty \mathbf{1}(y) + 1 \geq 0$ when $y < -2$. We claim that $g(y)g(-y) = 0$ whenever $|y| > 2$. In fact, if for example $g(y_0) < -\delta$ for some $y_0 > 2$ and some $\delta > 0$, then for each small $\epsilon > 0$, $0 \leq T_\infty \mathbf{1}(y) < 1 - \delta$ for all y in $(y_0 - \epsilon, y_0 + \epsilon)$. We can choose an f in $C_0([0, +\infty))$ satisfying that $f(y_0) = \|f\| = 1$ and f vanishes outside $(y_0 - \epsilon, y_0 + \epsilon) \subset (2, +\infty)$. Now,

$$\begin{aligned} T_\infty(\mathbf{1} + \delta f)(y) &= T_\infty(\mathbf{1})(y) + \delta T_\infty(f)(y) \\ &= T_\infty(\mathbf{1})(y) + \delta T(f)(y) \\ &= h(y) + g(y) + \delta h(y) f(\varphi(y)) \\ &= \begin{cases} 1 + g(y) + \delta f(y), & y > 2, \\ T_\infty \mathbf{1}(y), & -2 \leq y \leq 2, \\ -1 + g(y) - \delta f(-y), & y < -2. \end{cases} \end{aligned}$$

Since $\|T_\infty(\mathbf{1} + \delta f)\| = \|1 + \delta f\| = 1 + \delta$ and $|T_\infty(\mathbf{1} + \delta f)(y)| \leq 1$ unless $-y \in (y_0 - \epsilon, y_0 + \epsilon)$, there is a y_1 in $(y_0 - \epsilon, y_0 + \epsilon)$ such that $|-1 + g(-y_1) - \delta f(y_1)| = 1 + \delta$. It forces that $g(-y_1) = 0$. Since ϵ can be arbitrary small, we have $g(-y_0) = 0$ and our claim that $g(y)g(-y) = 0$ whenever $|y| > 2$ has thus been verified. As $T_\infty \mathbf{1}$ is continuous on $(-\infty, +\infty) \cup \{\infty\}$, we must have

$$\lim_{y\to+\infty} T_\infty \mathbf{1}(y) = \lim_{y\to-\infty} T_\infty \mathbf{1}(y),$$

that is,

$$\lim_{y\to+\infty} -1 + g(y) = \lim_{y\to-\infty} 1 + g(y).$$

Let L be their common (finite) limit. Then

$$\lim_{y\to+\infty} g(y) = L + 1, \qquad \lim_{y\to-\infty} g(y) = L - 1.$$

Consequently,

$$0 = \lim_{y\to+\infty} g(y)g(-y) = L^2 - 1.$$

It follows that $L = \pm 1$, and thus either $\lim_{y\to+\infty} g(y) = 2$ or $\lim_{y\to-\infty} g(y) = -2$. Both of them contradicts the fact that $|g(y)| \leq 1$, $\forall y \in (-\infty, +\infty)$.

On the other hand, suppose T_∞ were disjointness preserving. Since $f_n(\mathbf{1} - f_{2n}) = 0$, we have $T_\infty f_n \cdot T_\infty(\mathbf{1} - f_{2n}) = 0$. That is,

$$T_\infty f_n(y) \cdot T_\infty(\mathbf{1} - f_{2n})(y) = 0, \quad \forall y \in (-\infty, +\infty) \cup \{\infty\}.$$

When $|y| < n$ and $y \neq 1$, $T_\infty f_n(y) = Tf_n(y) = h(y) \neq 0$ and hence $T_\infty(\mathbf{1})(y) = T_\infty(f_{2n})(y) = T(f_{2n})(y) = h(y)$. Since $T_\infty \mathbf{1}$ is continuous on $(-\infty, +\infty) \cup \{\infty\}$, we must have

$$+1 = \lim_{y\to+\infty} h(y) = \lim_{y\to-\infty} h(y) = -1,$$

a contradiction again! □

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