WEAK AND STRONG CONVERGENCE THEOREMS FOR COMMUTATIVE FAMILIES OF POSITIVELY HOMOGENEOUS NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we first prove a weak convergence theorem by Mann's iteration for a commutative family of positively homogeneous nonexpansive mappings in a Banach space. Next, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota, we prove a strong convergence theorem for such a family of the mappings. These results are new even if the mappings are linear and contractive.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers. Let E be a real Banach space with norm $\|\cdot\|$ and let C be a closed and convex subset of E. Let T be a mapping of C into itself. We denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. Let C be a closed convex cone of E. A mapping $T : C \to C$ is called *positively homogeneous* if $T(\alpha x) = \alpha T(x)$ for all $x \in C$ and $\alpha \geq 0$. From Reich [27] we know a weak convergence theorem by Mann's iteration [20] for nonexpansive mappings in a Banach space: Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then, $\{x_n\}$ converges weakly to $z \in F(T)$.

In this theorem, the fixed point z is characteraized under any projections in a Banach space. Recently, Takahashi and Yao [45] proved a theorem for positively homogeneous nonexpansive mappings in a Banach space. In the theorem, the limit of weak convergence is characteraized by using a sunny generalized nonexpansive retraction in the sense of Ibaraki and Takahashi [9]. On the other hand, Nakajo and Takahashi [25] proved a strong convergence theorem for nonexpansive mappings in a Hilbert space by using the hybrid method in mathematical programming: Let C be a closed and convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ such

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that $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ and

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|u_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$. Then, $\{x_n\}$ converges strongly to $z \in F(T)$, where $z = P_{F(T)}x$ and $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Such a strong convergence theorem for nonexpansive mappings has not extended to Banach spaces. Takahashi and Yao [45] also proved such a theorem for positively homogeneous nonexpansive mappings.

Our purpose in this paper is first to prove a weak convergence theorem by Mann's iteration for a commutative family of positively homogeneous nonexpansive mappings in a Banach space. In the theorem, the limit of weak convergence is also characteraized by using a sunny generalized nonexpansive retraction. Furthermore, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota, we prove a strong convergence theorem for a commutative family of positively homogeneous nonexpansive mappings in a Banach space. These results are new even if the mappings are linear and contractive.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to C$ is quasi-nonexpansive if $F(T) \ne \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a closed convex subset of E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [11]. The following result was proved by Browder; see [34].

Lemma 2.1. *Let E be a uniformly convex Banach space and let C be a bounded closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$, then u is a fixed point of T.*

Let C be a nonempty closed convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C(x)$, we call P_C the *metric projection* of E onto C. The *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called *smooth*. We know that E is smooth if and only if J ia a single valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. It is known that if the norm of E is Fréchet differentiable, then J is norm to norm continuous. For more details, see [34]. We know the following result;

Lemma 2.2. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

The following result was proved by Xu [46].

Lemma 2.3 (Xu [46])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|ax + (1-a)y\|^2 \leq a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)g(\|x - y\|)$$

for all $x, y \in B_r$ and $a \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space. The function $\phi\colon E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [14]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.2}$$

for all $x, y, z \in E$. From $(\|x\|^2 - \|y\|^2) \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.3}$$

If C is a nonempty closed convex subset of a smooth, strictly and reflexive Banach space E, then for all $x \in E$ there exists a unique $z \in C$ (denoted by $\Pi_C x$) such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x). \tag{2.4}$$

The mapping Π_C is called the generalized projection from E onto C; see Alber [1], Alber and Reich [2], and Kamimura and Takahashi [14]. The following lemmas are well known; see, for instance, [14].

Lemma 2.4. *Let E be a reflexive, strictly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5. *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq 0\}$.

Let E be a Banach space and let D be a nonempty closed subset of E. A mapping $R : E \to D$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E, \ \forall t \geq 0.$$

A mapping $R : E \to D$ is a *retraction* if $Rx = x$ for all $x \in D$. Let E be a smooth Banach space E and let C be a nonempty subset of E. A mapping $T : C \to C$ is *generalized nonexpansive* [9] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y) \tag{2.5}$$

for all $x \in C$ and $y \in F(T)$. A nonempty subset of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) of E onto D. From [9], we know the following lemmas.

Lemma 2.6 (Ibaraki and Takahashi [9])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Then, a sunny generalized nonexpansive retraction of E onto D is uniquely determined.*

Lemma 2.7 (Ibaraki and Takahashi [9])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction R of E onto D and let $(x, z) \in E \times C$. Then, the following hold:*

1. *$z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0, \ \forall y \in D$;*
2. *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

In 2007, Kohsaka and Takahashi [16] proved the following results.

Lemma 2.8 (Kohsaka and Takahashi [16])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C_* be a nonempty closed convex subset of E^*. Suppose that Π_{C_*} is the generalized projection of E^* onto C_*. Then, R defined by $R = J^{-1}\Pi_{C_*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C_*$.*

Lemma 2.9 (Kohsaka and Takahashi [16])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty subset of E. Then, the following conditions are equivalent*

1. *D is a sunny generalized nonexpansive retract of E;*
2. *D is a generalized nonexpansive retract of E;*
3. *JD is closed and convex.*

In this case, D is closed.

Lemma 2.10 (Kohsaka and Takahashi [16])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction R of E onto D and let $(x, z) \in E \times C$. Then, the following conditions are equivalent*

(1) $z = Rx$;
(2) $\phi(x, z) = \min_{y \in D} \phi(x, y)$.

From Ibaraki and Takahashi [10] we know the following lemma.

Lemma 2.11 (Ibaraki and Takahashi [10]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping of E into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

From Takahashi and Yao [45] we also have the following lemma.

Lemma 2.12 (Takahashi and Yao [45]). *Let E be a Banach space and let C be a closed convex cone of E. Let $T : C \to C$ be a positively homogenuous nonexpansive mapping. Then, for any $x \in C$ and $m \in F(T)$, there exists $j \in Jm$ such that*

$$\langle x - Tx, j \rangle \leq 0,$$

where J is the duality mapping of E into E^*.

Using Lemma 2.12, Takahashi and Yao [45] ontained the following theorem.

Theorem 2.13 (Takahashi and Yao [45]). *Let E be a smooth Banach space and let C be a closed convex cone of E. Let $T : C \to C$ be a positively homogenuous nonexpansive mapping. Then, T is a generalized nonexpansive mapping.*

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a reflexive Banach space E, define s-$\mathrm{Li}_n C_n$ and w-$\mathrm{Ls}_n C_n$ as follows: $x \in$s-$\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$w-$\mathrm{Ls}_n C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies that

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.6}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [24] and we write $C_0 =$M-$\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [24]. We know the following theorem [7].

Lemma 2.14. *Let E be a smooth Banach space and let E^* have a Fréchet differentiable norm. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E. If $C_0 =$M-$\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\Pi_{C_n} x$ converges strongly to $\Pi_{C_0} x$, where Π_{C_n} and Π_{C_0} are the generalized projections of E onto C_n and C_0, respectively.*

3. Semigroups of Positively Homogeneous Nonexpansive Mappings

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. Let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real valued continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $l_s f$ and $r_s f$ as follows:

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s \in S} f(s) \le \mu(f) \le \sup_{s \in S} f(s), \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$. The following theorem is in [34, Theorem 1.4.5].

Theorem 3.1 ([34]). *Let S be a commutative semitopological semigroup. Then there exists an invariant mean on $C(S)$, i.e., there exists an element $\mu \in C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let S be a semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of nonexpansive mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as nonexpansive mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and $s \mapsto T_s x$ is continuous for each $x \in C$. We denote by $F(\mathcal{S})$ the set of common fixed points of $T_s,\ s \in S$, i.e.,

$$F(\mathcal{S}) = \cap\{F(T_s) : s \in S\}.$$

The following definition [31] is crucial in the nonlinear ergodic theory of abstract semigroups. Let S be a topological space and Let $C(S)$ be the Banach space of all bounded real valued continuous functions on S with supremum norm. Let E be a reflexive Banach space. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique element z_0 of E such that

$$\mu_s \langle u(s), x^* \rangle = \langle z_0, x^* \rangle, \quad \forall x^* \in E^*.$$

We call such z_0 the *mean vector* of u for μ and denote by $\tau(\mu)u$, i.e., $\tau(\mu)u = z_0$. In particular, if $\mathcal{S} = \{T_s : s \in S\}$ is a continuous representation of S as nonexpansive mappings on C such that $F(\mathcal{S}) \neq \emptyset$. and $u(s) = T_s x$ for all $s \in S$, then there exists $z_0 \in C$ such tat

$$\mu_s \langle T_s x, x^* \rangle = \langle z_0, x^* \rangle, \quad \forall x^* \in E^*.$$

We denote such z_0 by $T_\mu x$. A net $\{\mu < \alpha\}$ of means on $C(S)$ is said to be *asymptotically invariant* if for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0,$$

and it is said to be *strongly asymptotically invariant* if for each $s \in S$,

$$\|l_s^* \mu_\alpha - \mu_\alpha\| \to 0 \quad \text{and} \quad \|r_s^* \mu_\alpha - \mu_\alpha\| \to 0,$$

where l_s^* and r_s^* are the adjoint operators of l_s and r_s, respectively. Such nets were first studied by Day [6]. The following result is in Shioji and Takahashi [30]; see also [19].

Lemma 3.2. *Let S be a commutative semitopological semigroup. Let E be a uniformly convex Banach space, let C be a nonempty, closed and convex subset of E,* *and let B be a bounded subset of C. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{\mu < \alpha\}$ be a strongly asymptotically invariant net of means on $C(S)$. Then for any $t \in S$,*

$$\lim_{\alpha} \sup_{x \in B} \|T_t T_{\mu_\alpha} x - T_{\mu_\alpha} x\| = 0.$$

Using Lemma 2.12, we also the following result.

Lemma 3.3. *Let S be a commutative semitopological semigroup. Let E be a smooth and reflexive Banach space and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as positively homogeneous nonexpansive mappings of E into itself. Let $\{\mu_n\}$ be a mean on $C(S)$ and let $T_\mu x$ be a mean vector of $\{T_s x : s \in S\}$ and μ for every $x \in E$. Then*

$$\phi(T_\mu x, m) \leq \phi(x, m), \quad \forall x \in E, \ m \in F(\mathcal{S}).$$

Proof. Let $x \in E$. Since $F(\mathcal{S})$ is nonempty, $\{T_s x : s \in S\}$ is bounded. Then there exists $T_\mu x \in E$ such that

$$\mu_s \langle T_s x, x^* \rangle = \langle T_\mu x, x^* \rangle, \quad \forall x^* \in E^*.$$

We have that

$$\begin{aligned} \|T_\mu x\| &= \sup\{\ |\langle T_\mu x, z^* \rangle| : \|z^*\| = 1\} \\ &= \sup\{\ |\mu_s \langle T_s x, z^* \rangle| : \|z^*\| = 1\} \\ &\leq \sup\{\ \|\mu\| \cdot \sup_{s \in S} |\langle T_s x, z^* \rangle| : \|z^*\| = 1\} \\ &\leq \sup\{\ \sup_{s \in S} \|T_s x\| \cdot \|z^*\| : \|z^*\| = 1\} \\ &\leq \sup\{\ \sup_{s \in S} \|x\| \cdot \|z^*\| : \|z^*\| = 1\} \\ &= \|x\|. \end{aligned}$$

Using Lemma 2.12, we have that for any $m \in F(\mathcal{S})$,

$$\begin{aligned} \phi(T_\mu x, m) &= \|T_\mu x\|^2 - 2\langle T_\mu x, Jm \rangle + \|m\|^2 \\ &\leq \|x\|^2 - 2\mu_s \langle T_s x, Jm \rangle + \|m\|^2 \\ &\leq \|x\|^2 - 2\mu_s \langle x, Jm \rangle + \|m\|^2 \\ &= \|x\|^2 - 2\langle x, Jm \rangle + \|m\|^2 \\ &= \phi(x, m). \end{aligned}$$

This completes the proof. □

4. Weak convergence theorems

In this section, we prove a weak convergence theorem of Mann's iteration [20] for a commutative family of positively homogenuous nonexpansive mappings in a Banach space. Using Lemma 3.3, we have the following result.

Lemma 4.1. *Let S be a commutative semitopological semigroup. Let E be a smooth and uniformly convex Banach space and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as positively homogeneous nonexpansive mappings of E into itself. If a sequence $\{\mu_n\}$ of means on $C(S)$ is strongly asymptotically invariant.*

Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and let $\{x_n\}$ be a sequence in E generated by $x_1 = x \in E$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N}.$$

If $R_{F(\mathcal{S})}$ is a sunny generalized nonexpansive retraction of E onto $F(\mathcal{S})$, then $\{R_{F(\mathcal{S})} x_n\}$ converges strongly to $z \in F(\mathcal{S})$.

Proof. Let $m \in F(\mathcal{S})$. Using Lemma 3.3, we have that

$$\begin{aligned} \phi(x_{n+1}, m) &= \phi(\alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, m) \\ &\le \alpha_n \phi(x_n, m) + (1-\alpha_n)\phi(T_{\mu_n} x_n, m) \\ &\le \alpha_n \phi(x_n, m) + (1-\alpha_n)\phi(x_n, m) \\ &= \phi(x_n, m). \end{aligned}$$

So, $\lim_{n\to\infty} \phi(x_n, m)$ exists. Since $\{\phi(x_n, m)\}$ is bounded, $\{x_n\}$ and $\{T_{\mu_n} x_n\}$ are bounded. Define $y_n = R_{F(\mathcal{S})} x_n$ for all $n \in \mathbb{N}$. Since $\phi(x_{n+1}, m) \le \phi(x_n, m)$ for all $m \in F(\mathcal{S})$, from $y_n \in F(\mathcal{S})$ we have

$$\phi(x_{n+1}, y_n) \le \phi(x_n, y_n). \tag{4.1}$$

From Lemma 2.7 and (4.1), we have

$$\begin{aligned} \phi(x_{n+1}, y_{n+1}) &= \phi(x_{n+1}, R_{F(\mathcal{S})} x_{n+1}) \\ &\le \phi(x_{n+1}, y_n) - \phi(R_{F(\mathcal{S})} x_{n+1}, y_n) \\ &= \phi(x_{n+1}, y_n) - \phi(y_{n+1}, y_n) \\ &\le \phi(x_{n+1}, y_n) \\ &\le \phi(x_n, y_n). \end{aligned}$$

Then $\phi(x_n, y_n)$ is a convergent sequence. We also have from (4.1) that for all $m \in \mathbb{N}$,

$$\phi(x_{n+m}, y_n) \le \phi(x_n, y_n).$$

From $y_{n+m} = R_{F(\mathcal{S})} x_{n+m}$ and Lemma 2.7, we have

$$\phi(y_{n+m}, y_n) + \phi(x_{n+m}, y_{n+m}) \le \phi(x_{n+m}, y_n) \le \phi(x_n, y_n)$$

and hence

$$\phi(y_{n+m}, y_n) \le \phi(x_n, y_n) - \phi(x_{n+m}, y_{n+m}).$$

Using Lemma 2.5, we have that

$$g(\|y_{n+m} - y_n\|) \le \phi(y_{n+m}, y_n) \le \phi(x_n, y_n) - \phi(x_{n+m}, y_{n+m}),$$

where $g : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing and convex function such that $g(0) = 0$. Then, the properties of g yield that $R_{F(\mathcal{S})} x_n$ converges strongly to an element z of $F(\mathcal{S})$. □

Using Lemma 4.1, we prove the following theorem.

Theorem 4.2. *Let S be a commutative semitopological semigroup. Let E be a smooth and uniformly convex Banach space and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as positively homogeneous nonexpansive mappings of E into itself. If a sequence $\{\mu_n\}$ of means on $C(S)$ is strongly asymptotically invariant. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le a < 1$ for some $a \in \mathbb{R}$ with $0 < a < 1$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in E$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(\mathcal{S})$. Further, if E has a Fréchet differentiable norm, then $z = \lim_{n\to\infty} R_{F(\mathcal{S})}x_n$, where $R_{F(\mathcal{S})}$ is a sunny generalized nonexpansive retraction of E onto $F(\mathcal{S})$.

Proof. For $x \in E$ and $m \in F(\mathcal{S})$, put $r = \|x - m\|$ and set

$$X = \{u \in E : \|u - m\| \leq r\}.$$

Then, X is a nonempty, bounded, closed and convex suset of E. Furthermore, X is T_s-invariant for every $s \in S$ and contains $x_1 = x$. From Lemma 2.3, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\begin{aligned}
\|x_{n+1} - m\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)T_{\mu_n}x_n - m\|^2 \\
&\leq \alpha_n\|x_n - m\|^2 + (1 - \alpha_n)\|T_{\mu_n}x_n - m\|^2 \\
&\quad - \alpha_n(1 - \alpha_n)g(\|T_{\mu_n}x_n - x_n\|) \\
&\leq \alpha_n\|x_n - m\|^2 + (1 - \alpha_n)\|x_n - m\|^2 \\
&\quad - \alpha_n(1 - \alpha_n)g(\|T_{\mu_n}x_n - x_n\|) \\
&= \|x_n - m\|^2 - \alpha_n(1 - \alpha_n)g(\|T_{\mu_n}x_n - x_n\|) \\
&\leq \|x_n - m\|^2
\end{aligned} \tag{4.2}$$

So, $\lim_{n\to\infty} \|x_n - m\|$ exists. Since $0 \leq \alpha_n \leq a < 1$, we have from (4.2) that

$$\begin{aligned}
\alpha_n(1 - a)g(\|T_{\mu_n}x_n - x_n\|) &\leq \alpha_n(1 - \alpha_n)g(\|T_{\mu_n}x_n - x_n\|) \\
&\leq \|x_n - m\|^2 - \|x_{n+1} - m\|^2.
\end{aligned} \tag{4.3}$$

Since $\lim_{n\to\infty} \|x_n - m\|$ exists, we have from (4.3) that

$$\lim_{n\to\infty} \alpha_n g(\|T_{\mu_n}x_n - x_n\|) = 0. \tag{4.4}$$

From the properties of g and $\{\alpha_n\}$, we have

$$\lim_{n\to\infty} \alpha_n\|T_{\mu_n}x_n - x_n\| = 0. \tag{4.5}$$

In fact, take any subsequence $\{\alpha_{n_i}\|T_{\mu_{n_i}}x_{n_i} - x_{n_i}\|\}$ of $\{\alpha_n\|T_{\mu_n}x_n - x_n\|\}$. If $\lim_{i\to\infty} \alpha_{n_i} = 0$, then $\lim_{i\to\infty} \alpha_{n_i}\|T_{\mu_{n_i}}x_{n_i} - x_{n_i}\| = 0$. If $\lim_{i\to\infty} \alpha_{n_i} \neq 0$, then there exist $\varepsilon > 0$ and a subsequence $\{\alpha_{n_{i_j}}\}$ of $\{\alpha_{n_i}\}$ such that $\alpha_{n_{i_j}} \geq \varepsilon > 0$ for all $j \in \mathbb{N}$. Then we have from (4.4) that $g(\|T_{\mu_{n_{i_j}}}x_{n_{i_j}} - x_{n_{i_j}}\|) = 0$. From the properties of g, we have $\|T_{\mu_{n_{i_j}}}x_{n_{i_j}} - x_{n_{i_j}}\| = 0$ and hence $\alpha_{n_{i_j}}\|T_{\mu_{n_{i_j}}}x_{n_{i_j}} - x_{n_{i_j}}\| = 0$. Therefore

$$\lim_{n\to\infty} \alpha_n\|T_{\mu_n}x_n - x_n\| = 0.$$

Using (4.5) and the definition of $\{x_n\}$, we have that

$$x_{n+1} - T_{\mu_n}x_n = \alpha_n(x_n - T_{\mu_n}x_n) \to 0. \tag{4.6}$$

We have from Lemma 3.2 that for any $s \in S$,

$$\begin{aligned}
\|x_{n+1} - T_s x_{n+1}\| &\leq \|x_{n+1} - T_{\mu_n}x_n\| \\
&\quad + \|T_{\mu_n}x_n - T_s T_{\mu_n}x_n\| + \|T_s T_{\mu_n}x_n - T_s x_{n+1}\| \\
&\leq 2\|x_{n+1} - T_{\mu_n}x_n\| + \|T_{\mu_n}x_n - T_s T_{\mu_n}x_n\| \to 0.
\end{aligned} \tag{4.7}$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in X$. Since E is uniformly convex and

$\lim_{n\to\infty} \|T_s x_n - x_n\| = 0$ for all $s \in S$, we have from Lemma 2.1 that v is a fixed point of T_s. Thus $v \in F(\mathcal{S})$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. We have that $u, v \in F(\mathcal{S})$. As in the proof of Lemma 4.1, we have that for any $m \in F(\mathcal{S})$,

$$\begin{aligned}\phi(x_{n+1}, m) &= \phi(\alpha_n x_n + (1-\alpha_n)T_{\mu_n} x_n, m) \\ &\le \alpha_n \phi(x_n, m) + (1-\alpha_n)\phi(T_{\mu_n} x_n, m) \\ &\le \phi(x_n, m)\end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} \phi(x_n, m)$ exists. Put

$$a = \lim_{n\to\infty} (\phi(x_n, u) - \phi(x_n, v)).$$

Since $\phi(x_n, u) - \phi(x_n, v) = 2\langle x_n, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$, we have

$$a = 2\langle u, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$$

and

$$a = 2\langle v, Jv - Ju\rangle + \|u\|^2 - \|v\|^2.$$

From these equalities, we obtain

$$\langle u - v, Ju - Jv\rangle = 0.$$

Since J is strictly monotone, it follows that $u = v$; see [34]. Therefore, $\{x_n\}$ converges weakly to an element u of $F(\mathcal{S})$. On the other hand, we know from Lemma 4.1 that $\{R_{F(\mathcal{S})} x_n\}$ converges strongly to $z \in F(\mathcal{S})$. From Lemma 2.7, we also have

$$\langle x_n - R_{F(\mathcal{S})} x_n, JR_{F(\mathcal{S})} x_n - Ju\rangle \ge 0.$$

Since E has a Fréchet differentiable norm, the duality mapping J is norm-to-norm continuous. So, we have $\langle u - z, Jz - Ju\rangle \ge 0$. Since J is monotone, we also have $\langle u - z, Jz - Ju\rangle \le 0$. So, we have $\langle u - z, Jz - Ju\rangle = 0$. Since E is strictly convex, we have $z = u$. This completes the proof. □

Using Theorem 4.2, we obtain the following new result for linear contractive mappings of E into itself.

Theorem 4.3. *Let E be a smooth and uniformly convex Banach space and Let $T : E \to E$ be a linear contractive mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in E$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(T)$. Further, if E has a Fréchet differentiable norm, then $z = \lim_{n\to\infty} Rx_n$, where R is a sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. A linear contractive mapping $T : E \to E$ is a positively homogenuous nonexpansive mapping such that $T(0) = 0$. From Theorem 4.2, we get the desired result. □

5. Strong convergence theorems

In this section, we prove a strong convergence theorem by a hybrid method called the shrinking projection method for positively homogenuous nonexpansive mappings in a Banach space.

Theorem 5.1. *Let S be a commutative semitopological semigroup. Let E be a uniformly convex Banach space which has a Fréchet differentiable norm and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as positively homogeneous nonexpansive mappings of E into itself. If a sequence $\{\mu_n\}$ of means on $C(S)$ is strongly asymptotically invariant. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le a < 1$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, $C_1 = E$ and*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \le \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1}. Then, $\{x_n\}$ converges strongly to $z = R_{F(\mathcal{S})}x$, where $R_{F(\mathcal{S})}$ is the sunny generalized nonexpansive retraction of E onto $F(\mathcal{S})$.

Proof. Since $T_s : E \to E$ is a generalized nonexpansive mapping for every $s \in S$, we have from Lemma 2.11 that $F(\mathcal{S})$ is a sunny generalized nonexpansive retract of E. We shall show that JC_n are closed and convex and $F(\mathcal{S}) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from the assumption that $JC_1 = JE = E^*$ is closed and convex, and $F(\mathcal{S}) \subset C_1$. Suppose that JC_k is closed and convex and $F(\mathcal{S}) \subset C_k$ for some $k \in \mathbb{N}$. From the definition of ϕ, we have that for $z \in C_k$,

$$\begin{aligned} &\phi(u_k, z) \le \phi(x_k, z) \\ \Longleftrightarrow &\|u_k\|^2 - \|x_k\|^2 - 2\langle u_k - x_k, Jz\rangle \le 0. \end{aligned}$$

So, JC_{k+1} is closed and convex. If $z \in F(\mathcal{S}) \subset C_k$, then we have

$$\begin{aligned} \phi(u_n, z) &= \phi(\alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, z) \\ &\le \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(T_{\mu_n} x_n, z) \\ &\le \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(x_n, z) \\ &= \phi(x_n, z). \end{aligned}$$

Hence, we have $z \in C_{k+1}$. By induction, we have that JC_n are closed and convex and $F(\mathcal{S}) \subset C_n$ for all $n \in \mathbb{N}$. Since JC_n is closed and convex, from Lemma 2.6 there exists a unique sunny generalized nonexpansive retraction R_{C_n} of E onto C_n. We also know from Lemma 2.8 that such R_{C_n} is denoted by $J^{-1}\Pi_{JC_n}J$, where J is the duality mapping of E and Π_{JC_n} is the generalized projection of E onto JC_n. Thus, $\{x_n\}$ is well-defined.

Since $\{JC_n\}$ is a nonincreasing sequence of nonempty, closed and convex subsets of E^* with respect to inclusion, it follows that

$$\emptyset \ne JF(\mathcal{S}) \subset \text{M-}\lim_{n\to\infty} JC_n = \cap_{n=1}^{\infty} JC_n. \tag{5.1}$$

Put $C_0^* = \cap_{n=1}^{\infty} JC_n$. Then, by Theorem 2.14 we have that $\{\Pi_{JC_{n+1}}Jx\}$ converges strongly to $x_0^* = \Pi_{C_0^*}Jx$. Since E^* is a Fréchet differencial norm, J^{-1} is continuous. So, we have

$$x_{n+1} = R_{n+1}x = J^{-1}\Pi_{JC_{n+1}}Jx \to J^{-1}x_0^*.$$

To complete the proof, it is sufficient to show that $J^{-1}x_0^* = R_{F(\mathcal{S})}x$.

Since $x_n = R_{C_n}x$ and $x_{n+1} = R_{C_{n+1}}x \in C_{n+1} \subset C_n$, we have from Lemma 2.7 and (2.2) that

$$\begin{aligned} 0 &\le 2\langle x - x_n, Jx_n - Jx_{n+1}\rangle \\ &= \phi(x, x_{n+1}) - \phi(x, x_n) - \phi(x_n, x_{n+1}) \\ &\le \phi(x, x_{n+1}) - \phi(x, x_n). \end{aligned}$$

Thus we get that

$$\phi(x, x_n) \le \phi(x, x_{n+1}). \tag{5.2}$$

Furthermore, since $x_n = R_{C_n}x$ and $z \in F(T) \subset C_n$, from Lemma 2.10 we have

$$\phi(x, x_n) \le \phi(x, z). \tag{5.3}$$

Then we have that $\lim_{n\to\infty} \phi(x, x_n)$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{u_n\}$ and $\{T_{\mu_n}x_n\}$ are also bounded. From

$$\begin{aligned} \phi(x_n, x_{n+1}) &= \phi(R_{C_n}x, x_{n+1}) \\ &= \phi(x, x_{n+1}) - \phi(x, R_{C_n}x) \\ &= \phi(x, x_{n+1}) - \phi(x, x_n) \to 0, \end{aligned}$$

we have that

$$\phi(x_n, x_{n+1}) \to 0. \tag{5.4}$$

From $x_{n+1} \in C_{n+1}$, we have that $\phi(u_n, x_{n+1}) \le \phi(x_n, x_{n+1})$. So, we get that $\phi(u_n, x_{n+1}) \to 0$. Using Lemma 2.4, we have

$$\lim_{n\to\infty} \|u_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

So, we have

$$\|u_n - x_n\| \le \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{5.5}$$

Since $\|x_n - u_n\| = \|x_n - \alpha_n x_n - (1 - \alpha_n)T_{\mu_n}x_n\| = (1 - \alpha_n)\|x_n - T_{\mu_n}x_n\|$ and $0 \le \alpha_n \le a < 1$, we have that

$$\|T_{\mu_n}x_n - x_n\| \to 0. \tag{5.6}$$

We have Lemma 3.2 that for any $s \in S$,

$$\begin{aligned} \|x_n - T_s x_n\| &\le \|x_n - T_{\mu_n}x_n\| + \|T_{\mu_n}x_n - T_s T_{\mu_n}x_n\| + \|T_s T_{\mu_n}x_n - T_s x_n\| \\ &\le 2\|x_n - T_{\mu_n}x_n\| + \|T_{\mu_n}x_n - T_s T_{\mu_n}x_n\| \to 0. \end{aligned}$$

Since $x_{n+1} \to J^{-1}x_0^*$ and T_s is continuous, we have $J^{-1}x_0^* \in F(T_s)$. Therefore, we have $J^{-1}x_0^* \in F(\mathcal{S})$.

Put $z_0 = R_{F(\mathcal{S})}x$. Since $z_0 = R_{F(\mathcal{S})}x \subset C_{n+1}$ and $x_{n+1} = R_{C_{n+1}}x$, we have that

$$\phi(x, x_{n+1}) \le \phi(x, z_0). \tag{5.7}$$

So, we have that

$$\begin{aligned} \phi(x, J^{-1}x_0^*) &= \|x\|^2 - 2\langle x, x_0^*\rangle + \|J^{-1}x_0^*\|^2 \\ &= \lim_{n\to\infty} (\|x\|^2 - 2\langle x, Jx_n\rangle + \|x_n\|^2) \\ &= \lim_{n\to\infty} \phi(x, x_n) \\ &\le \phi(x, z_0). \end{aligned}$$

Then we get $z_0 = J^{-1}x_0^*$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. $\square$

Using Theorem 5.1, we prove a strong convergence theorem for linear contractive mappings in a Banach space.

Theorem 5.2. *Let S be a commutative semitopological semigroup. Let E be a uniformly convex Banach space which has a Fréchet differentiable norm and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive mappings of E into itself. If a sequence $\{\mu_n\}$ of means on $C(S)$ is strongly asymptotically invariant. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le a < 1$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, $C_1 = E$ and*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \le \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1}. Then, $\{x_n\}$ converges strongly to $z = R_{F(\mathcal{S})}x$, where $R_{F(\mathcal{S})}$ is the sunny generalized nonexpansive retraction of E onto $F(\mathcal{S})$.

Proof. A linear contractive mapping $T_s : E \to E$ is positively homogenuous and nonexpansive. So, using Theorem 5.1, we obtain the desired result. $\square$

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