A MURRAY-VON NEUMANN TYPE CLASSIFICATION OF C^*-ALGEBRAS

CHI-KEUNG NG AND NGAI-CHING WONG

Abstract. We define type $\mathfrak{A}$, type $\mathfrak{B}$, type $\mathfrak{C}$ as well as C^*-semi-finite C^*-algebras. It is shown that a von Neumann algebra is a type $\mathfrak{A}$, type $\mathfrak{B}$, type $\mathfrak{C}$ or C^*-semi-finite C^*-algebra if and only if it is, respectively, a type I, type II, type III or semi-finite von Neumann algebra.

Moreover, any type I C^*-algebra is of type $\mathfrak{A}$ (actually, type $\mathfrak{A}$ coincides with the discreteness as defined by Peligrad and Zsidó), and any type II C^*-algebra (as defined by Cuntz and Pedersen) is of type $\mathfrak{B}$. Moreover, any type $\mathfrak{C}$ C^*-algebra is of type III (in the sense of Cuntz and Pedersen), any purely infinite C^*-algebra (in the sense of Kirchberg and Rørdam) with real rank zero is of type $\mathfrak{C}$, and any separable purely infinite C^*-algebra with stable rank one is also of type $\mathfrak{C}$.

We also prove that type $\mathfrak{A}$, type $\mathfrak{B}$, type $\mathfrak{C}$ and C^*-semi-finiteness are stable under hereditary C^*-subalgebras, multiplier algebras and strong Morita equivalence. Furthermore, any C^*-algebra A contains a largest type $\mathfrak{A}$ closed ideal $J_{\mathfrak{A}}$, a largest type $\mathfrak{B}$ closed ideal $J_{\mathfrak{B}}$, a largest type $\mathfrak{C}$ closed ideal $J_{\mathfrak{C}}$ as well as a largest C^*-semi-finite closed ideal $J_{\mathfrak{sf}}$. Among them, we have $J_{\mathfrak{A}} + J_{\mathfrak{B}}$ being an essential ideal of $J_{\mathfrak{sf}}$, and $J_{\mathfrak{A}} + J_{\mathfrak{B}} + J_{\mathfrak{C}}$ being an essential ideal of A. On the other hand, $A/J_{\mathfrak{C}}$ is always C^*-semi-finite, and if A is C^*-semi-finite, then $A/J_{\mathfrak{B}}$ is of type $\mathfrak{A}$.

Finally, we show that these results hold if type $\mathfrak{A}$, type $\mathfrak{B}$, type $\mathfrak{C}$ and C^*-semi-finiteness are replaced by discreteness, type II, type III and semi-finiteness (as defined by Cuntz and Pedersen), respectively.

1. Introduction

In their seminal works ([27], see also [26]), Murray and von Neumann defined three types of von Neumann algebras (namely, type I, type II and type III) according to the properties of their projections. They showed that any von Neumann algebra is a sum of a type I, a type II, and a type III von Neumann subalgebras. This classification was shown to be very important and becomes the basic theory for the study of von Neumann algebras (see e.g. [20]).

Since a C^*-algebra needs not have any projection, a similar classification for C^*-algebras seems impossible. There is, however, an interesting classification scheme for C^*-algebras proposed by Cuntz and Pedersen in [14], which captures some features of the classification of Murray and von Neumann.

Date: December 15, 2012.

The authors are supported by National Natural Science Foundation of China (10771106), and Taiwan NSC grant (99-2115-M-110-007-MY3).

The classification theme of C^*-algebras took a drastic turn after an exciting work of Elliott on the classification of AF-algebras through the ordered K-theory, in the sense that two AF-algebras are isomorphic if and only if they have the same ordered K-theory ([16]). Elliott then proposed an invariant consisting of the tracial state space and some K-theory datum of the underlying C^*-algebra (called the *Elliott invariant*) which could be a suitable candidate for a complete invariant for simple separable nuclear C^*-algebras. Although it is known recently that it is not the case (see [37]), this Elliott invariant still works for a very large class of such C^*-algebras (namely, those satisfying certain regularity conditions as described in [18]). Many people are still making progress in this direction in trying to find the biggest class of C^*-algebras that can be classified through the Elliott invariant (see, e.g., [17, 35]). Notice that this classification is very different from the classification in the sense of Murray and von Neumann.

In this article, we reconsider the classification of C^*-algebras through the idea of Murray and von Neumann. Instead of considering projections in a C^*-algebra A, we consider open projections and we twist the definition of the finiteness of projections slightly to obtain our classification scheme.

The notion of open projections was introduced by Akemann (in [1]). A projection p in the universal enveloping von Neumann algebra (i.e. the biduals) A^{**} of a C^*-algebra A (see e.g. [36, §III.2]) is an *open* projection of A if there is an increasing net $\{a_\lambda\}$ of positive elements in A_+ with $\lim_\lambda a_\lambda = p$ in the $\sigma(A^{**}, A^*)$-topology. In the case when A is commutative, open projections of A are exactly characteristic functions of open subsets of the spectrum of A. In general, there is a bijective correspondence between open projections of A and hereditary C^*-subalgebras of A (where the hereditary C^*-subalgebra corresponds to an open projection p is $pA^{**}p \cap A$; see e.g. [30]). Characterizations and further developments of open projections can be found in, e.g., [2, 3, 4, 9, 15, 29, 32]. Since every element in a C^*-algebra is in the closed linear span of its open projections, it is reasonable to believe that the study of open projections will provide fruitful information about the underlying C^*-algebra. Moreover, because of the correspondence between open projections (respectively, central open projections) and hereditary C^*-subalgebras (respectively, closed ideals), the notion of strong Morita equivalence as defined by Rieffel (see [33] and also [11, 34]) is found to be very useful in this scheme.

One might wonder why we do not consider the classification of the universal enveloping von Neumann algebras of C^*-algebras to obtain a classification of C^*-algebras. A reason is that for a C^*-algebra A, its bidual A^{**} always contains many minimum projections (see e.g. [1, II.17]), and hence a reasonable theory of type classification cannot be obtained without serious modifications. Furthermore, A^{**} are usually very far away from A, and information of A might not always be respected very well in A^{**}; for example, c and c_0 have isomorphic biduals, but the structure of their open projections can be used to distinguish them (see e.g. Example 2.1 and also Proposition 2.3(b)).

As in the case of von Neumann algebras, in order to give a classification of C^*-algebras, one needs, first of all, to consider a good equivalence relation among open projections.

After some thoughts and considerations, we end up with the "spatial equivalence" as defined in Section 2, which is weaker than the one defined by Peligrad and Zsidó in [31] and stronger than the ordinary Murray-von Neumann equivalence. One reason for making this choice is that it is precisely the "hereditarily stable version of Murray-von Neumann equivalence" that one might want (see Proposition 2.7(a)(5)), and it also coincides with the "spatial isomorphism" of the hereditary C^*-subalgebras (see Proposition 2.7(a)(2)).

Using the spatial equivalence relation, we introduce in Section 3, the notion of C^*-finite C^*-algebras. It is shown that the sum of all C^*-finite hereditary C^*-subalgebra is a (not necessarily closed) ideal of the given C^*-algebra. In the case when the C^*-algebra is $\mathcal{B}(H)$ or $\mathcal{K}(H)$, this ideal is the ideal of all finite rank operators on H. Moreover, through C^*-finiteness, we define type $\mathfrak{A}$, type $\mathfrak{B}$, type $\mathfrak{C}$ as well as C^*-semi-finite C^*-algebras, and we study some properties of them. In particular, we will show that these properties are stable under taking hereditary C^*-subalgebras, multiplier algebras, unitalization (if the algebra is not unital) as well as strong Morita equivalence. We will also show that the notion of type $\mathfrak{A}$ coincides precisely with the discreteness as defined in [31].

In Section 4, we will compare these notions with some results in the literature and give some examples. In particular, we show that any type I C^*-algebra (see e.g. [30]) is of type $\mathfrak{A}$; any type II C^*-algebra (as defined by Cuntz and Pedersen) is of type $\mathfrak{B}$; any semi-finite C^*-algebras (in the sense of Cuntz and Pedersen) is C^*-semi-finite; any purely infinite C^*-algebra (in the sense of Kirchberg and Rørdam) with real rank zero and any separable purely infinite C^*-algebra with stable rank one are of type $\mathfrak{C}$; and any type $\mathfrak{C}$ C^*-algebra is of type III (as introduced by Cuntz and Pedersen). Using our arguments for these results, we also show that any purely infinite C^*-algebra is of type III. Moreover, a von Neumann algebra M is a type $\mathfrak{A}$, a type $\mathfrak{B}$, a type $\mathfrak{C}$ or a C^*-semi-finite C^*-algebra if and only if M is, respectively, a type I, a type II, a type III, or a semi-finite von Neumann algebra.

In Section 5, we show that any C^*-algebra A contains a largest type $\mathfrak{A}$ closed ideal $J^A_{\mathfrak{A}}$, a largest type $\mathfrak{B}$ closed ideal $J^A_{\mathfrak{B}}$, a largest type $\mathfrak{C}$ closed ideal $J^A_{\mathfrak{C}}$ as well as a largest C^*-semi-finite closed ideal $J^A_{\mathfrak{sf}}$. It is further shown that $J^A_{\mathfrak{A}} + J^A_{\mathfrak{B}}$ is an essential ideal of $J^A_{\mathfrak{sf}}$, and $J^A_{\mathfrak{A}} + J^A_{\mathfrak{B}} + J^A_{\mathfrak{C}}$ is an essential ideal of A. On the other hand, $A/J^A_{\mathfrak{C}}$ is always a C^*-semi-finite C^*-algebra, while $B/J^B_{\mathfrak{B}}$ is always of type $\mathfrak{A}$ if one sets $B := A/J^A_{\mathfrak{C}}$. We also compare $J^{M(A)}_{\mathfrak{A}}$, $J^{M(A)}_{\mathfrak{B}}$, $J^{M(A)}_{\mathfrak{C}}$ and $J^{M(A)}_{\mathfrak{sf}}$ with $J^A_{\mathfrak{A}}$, $J^A_{\mathfrak{B}}$, $J^A_{\mathfrak{C}}$ and $J^A_{\mathfrak{sf}}$, respectively.

In the Appendix, we give a very general classification scheme and observe that most of the results in the main body are actually true in a more general context. In particular, we show that many results in the main body remain valid if one replaces type $\mathfrak{A}$, type $\mathfrak{B}$, type $\mathfrak{C}$ and C^*-semi-finiteness with discreteness, type II, type III and semi-finiteness, respectively.

Notation 1.1. Throughout this paper, A is a non-zero C^*-algebra, $M(A)$ is the multiplier algebra of A, $\mathrm{Z}(A)$ is the center of A, and A^{**} is the bidual of A. Furthermore, $\mathrm{Proj}(A)$ is the set of all projections in A, while $\mathrm{OP}(A) \subseteq \mathrm{Proj}(A^{**})$ is the set of all open projections of A. All ideals in this paper are two-sided ideals (not assumed to be closed unless specified).

If $x, y \in A^{**}$ and E is a subspace of A^{**}, we set $xEy := \{xzy : z \in E\}$, and denote by $\overline{E}$ the norm closure of E. For any $x \in A^{**}$, we set $\mathrm{her}_A(x)$ to be the hereditary C^*-subalgebra $\overline{x^*A^{**}x} \cap A$ of A (note that if $u \in A^{**}$ is a partial isometry, then $\mathrm{her}_A(u) = u^*A^{**}u \cap A = \{x \in A : x = u^*uxu^*u\} = \mathrm{her}_A(u^*u)$). When A is understood, we will use the notation $\mathrm{her}(x)$ instead. Moreover, p_x is the right support projection of a norm one element $x \in A$, i.e. p_x is the $\sigma(A^{**}, A^*)$-limit of $\{(x^*x)^{1/n}\}_{n\in\mathbb{N}}$ and is the smallest open projection in A^{**} with $xp_x = x$.

Acknowledgement: The authors would like to thank Prof. Larry Brown, Prof. Edward Effros and Prof. George Elliott for giving us some comments.

2. Spatial equivalence of open projections

In this section, we will consider a suitable equivalence relation on the set of open projections of a C^*-algebra. Let us start with the following example, which shows that the structure of open projections is rich enough to distinguish c and c_0, while they have isomorphic biduals (see Proposition 2.3(b) below for a more general result).

Example 2.1. The sets of open projections of c_0 and c can be regarded as the collections $\mathfrak{X}$ and $\mathfrak{Y}$, of open subsets of $\mathbb{N}$ and of open subsets of the one point compactification of $\mathbb{N}$, respectively. As ordered sets, $\mathfrak{X}$ and $\mathfrak{Y}$ are not isomorphic. In fact, suppose on the contrary that there is an order isomorphism $\Psi : \mathfrak{Y} \to \mathfrak{X}$. Then $\Psi(\mathbb{N})$ is a proper open subset of $\mathbb{N}$. Let $k \notin \Psi(\mathbb{N})$ and $U \in \mathfrak{Y}$ with $\Psi(U) = \{k\}$. As U is a minimal element, it is a singleton set. Thus, $U \subseteq \mathbb{N}$, which gives the contradiction that $\{k\} \subseteq \Psi(\mathbb{N})$.

Secondly, we give the following well-known remarks which show that open projections and the hereditary C^*-subalgebras they define, are "hereditarily invariant". These will clarify some discussions later on.

Remark 2.2. Let $B \subseteq A$ be a hereditary C^*-subalgebra and $e \in \mathrm{OP}(A)$ be the open projection with $\mathrm{her}_A(e) = B$.

(a) For any $p \in \mathrm{Proj}(B^{**})$, one has $\mathrm{her}_B(p) = \mathrm{her}_A(p)$.

(b) $\mathrm{OP}(B) = \mathrm{OP}(A) \cap B^{**}$. In fact, if $p \in \mathrm{OP}(A) \cap B^{**}$ and $\{a_i\}_{i\in\mathfrak{I}}$ is an approximate unit in $\mathrm{her}_A(p) = \mathrm{her}_B(p)$, then $\{a_i\}_{i\in\mathfrak{I}}$ will $\sigma(B^{**}, B^*)$-converge to p and $p \in \mathrm{OP}(B)$.

(c) If $z \in A$ satisfying $zz^*, z^*z \in B$, then $z \in B$. In fact, as z^*z is dominated by a positive scalar multiple of e, we see that $z^*z \in eA^{**}e \cap A$ (as $(eA^{**}e)_+$ is a hereditary cone of A^{**}_+). Thus, by considering the polar decomposition of z, we see that $ze = z$. Similarly, we have $ez = z$.

Let $j_A : M(A) \to A^{**}$ be the canonical *-monomorphism, i.e. $j_A(x)(f) = \tilde{f}(x)$ $(x \in M(A), f \in A^*)$, where $\tilde{f} \in M(A)^*$ is the unique strictly continuous extension of f. The proposition below can be regarded as a motivation behind the study of C^*-algebras through their open projections. It could be a known result (especially, part (a)). However, since we need it for the equivalence of (1) and (5) in Proposition 2.7(a), we give a proof here for completeness.

Proposition 2.3. *Suppose that A and B are C^*-algebras, and $\Phi : A^{**} \to B^{**}$ is a *-isomorphism.*
(a) If $\Phi\big(j_A(M(A))\big) = j_B(M(B))$, then $\Phi(A) = B$.
(b) If $\Phi(\mathrm{OP}(A)) = \mathrm{OP}(B)$, then $\Phi(A) = B$.

Proof: (a) Let $p_A \in \mathrm{OP}(M(A))$ such that $\mathrm{her}_{M(A)}(p_A) = i_A(A)$. It is not hard to verify that p_A is the support of $\tilde{j}_A$, where $\tilde{j}_A : M(A)^{**} \to A^{**}$ is the *-epimorphism induced by j_A. Consider $\Psi := j_B^{-1} \circ \Phi \circ j_A : M(A) \to M(B)$. Since $j_B \circ \Psi = \Phi \circ j_A$, we see that $\tilde{j}_B \circ \Psi^{**} = \Phi \circ \tilde{j}_A$ (as Φ is automatically weak-*-continuous). Thus, $\tilde{j}_B(\Psi^{**}(p_A)) = 1_{B^{**}}$ which implies $\Psi^{**}(p_A) \geq p_B$. Similarly,

$$(\Psi^{**})^{-1}(p_B) \;=\; (j_A^{-1} \circ \Phi^{-1} \circ j_B)^{**}(p_B) \;\geq\; p_A$$

and we have $\Psi^{**}(p_A) = p_B$. Consequently, $\Psi(\mathrm{her}_{M(A)}(p_A)) = \mathrm{her}_{M(B)}(p_B)$ as required.
(b) If $a \in M(A)_{sa}$ and U is an open subset of $\sigma(a) = \sigma(\Phi(j_A(a)))$, then $\chi_U(\Phi(j_A(a))) = \Phi(\chi_U(j_A(a)))$ is an element of $\mathrm{OP}(B)$ (by [5, Theorem 2.2] and the hypothesis). Thus, by [5, Theorem 2.2] again, we have $\Phi(j_A(a)) \in j_B(M(B))$. A similar argument shows that $\Phi^{-1}(j_B(M(B))) \subseteq j_A(M(A))$. Now, we can apply part (a) to obtain the required conclusion. □

Remark 2.4. Note that if A and B are separable and $\Psi : M(A) \to M(B)$ is a *-isomorphism, then $\Psi(A) = B$, by a result of Brown in [10]. However, the same result is not true if one of them is not separable (e.g. take $A = M(B)$ and $\Psi = \mathrm{id}$, where B is non-unital). Proposition 2.3(a) shows that one has $\Psi(A) = B$ if (and only if) Ψ extends to a *-isomorphism from A^{**} to B^{**}.

We now consider a suitable equivalence relation on $\mathrm{OP}(A)$. A naive choice is to use the original "Murray-von Neumann equivalence" $\sim_{\mathrm{Mv}}$. However, this choice is not good because [23] tells us that two open projections that are Murray-von Neumann equivalent might define non-isomorphic hereditary C^*-subalgebras. On the other hand, one might define $p \sim_{\mathrm{her}} q$ $(p, q \in \mathrm{OP}(A))$ whenever $\mathrm{her}(p) \cong \mathrm{her}(q)$ as C^*-algebras. The problem of this choice is that two distinct open projections of $C([0,1])$ can be equivalent (if they correspond to homeomorphic open subsets of $[0,1]$), which means that the resulting classification, even if possible, will be very different from the Murray-von Neumann classification.

After some thoughts, we end up with an equivalence relation $\sim_{\rm sp}$ on $\mathrm{OP}(A)$: $p \sim_{\rm sp} q$ if there is a partial isometry $u \in A^{**}$ satisfying

$$u^* \operatorname{her}_A(p)u \ = \ \operatorname{her}_A(q) \quad \text{and} \quad u \operatorname{her}_A(q)u^* \ = \ \operatorname{her}_A(p).$$

Note that this relation is precisely the "hereditarily stable version" of the Murray-von Neumann equivalence (see Proposition 2.7(a)(5) below and the discussion following it).

In [31, Definition 1.1], Peligrad and Zsidó introduced another equivalence relation on $\mathrm{Proj}(A^{**})$: $p \sim_{\rm PZ} q$ if there is a partial isometry $u \in A^{**}$ such that

$$p = uu^*, \quad q = u^*u, \quad u^* \operatorname{her}_A(p) \subseteq A \quad \text{and} \quad u \operatorname{her}_A(q) \subseteq A. \tag{2.1}$$

It is not difficult to see that $\sim_{\rm PZ}$ is stronger than $\sim_{\rm sp}$, and a natural description of $\sim_{\rm PZ}$ on the set of range projections of positive elements of A is given in [28, Proposition 4.3]. However, we decide to use $\sim_{\rm sp}$ as it seems to be more natural in the way of using open projections (see Proposition 2.7(a) below). In the Appendix, we will give a brief discussion for the situation when one uses $\sim_{\rm PZ}$ instead of $\sim_{\rm sp}$.

Let us start with an extension of $\sim_{\rm sp}$ to the whole of $\mathrm{Proj}(A^{**})$.

Definition 2.5. We say that $p, q \in \mathrm{Proj}(A^{**})$ are *spatially equivalent with respect to A*, denoted by $p \sim_{\rm sp} q$, if there exists a partial isometry $v \in A^{**}$ satisfying

$$p = vv^*, \quad q = v^*v, \quad v^* \operatorname{her}_A(p)v = \operatorname{her}_A(q) \quad \text{and} \quad v \operatorname{her}_A(q)v^* = \operatorname{her}_A(p). \tag{2.2}$$

In this case, we also say that the hereditary C^*-subalgebras $\operatorname{her}_A(p)$ and $\operatorname{her}_A(q)$ are *spatially isomorphic*.

It might happen that $\operatorname{her}(p) = 0$ but $p \neq 0$ and this is why we need to consider the first two conditions in (2.2). We will see in Proposition 2.7(a) that the first two conditions are redundant if p and q are both open projections.

Obviously, $\sim_{\rm sp}$ is stronger than $\sim_{\rm Mv}$ (for elements in $\mathrm{Proj}(A^{**})$). Moreover, if $p \sim_{\rm sp} q$, then $x \mapsto v^*xv$ is a *-isomorphism from $\operatorname{her}(p)$ to $\operatorname{her}(q)$, which means that $\sim_{\rm sp}$ is stronger than $\sim_{\rm her}$ in the context of open projections.

A good point of the spatial equivalence is that open projections are stable under $\sim_{\rm sp}$, as can be seen in part (b) of the following lemma.

Lemma 2.6. *(a) $\sim_{\rm sp}$ is an equivalence relation in $\mathrm{Proj}(A^{**})$.*

(b) Let $p, q \in \mathrm{Proj}(A^{**})$ and $u \in A^{**}$ be a partial isometry. If p is open, $u^*pu = q$, $\operatorname{her}_A(p) \subseteq u \operatorname{her}_A(q)u^*$ and $\operatorname{her}_A(q) \subseteq u^* \operatorname{her}_A(p)u$, then q is open and $p \sim_{\rm sp} q$. Consequently, if $p \sim_{\rm sp} q$ and p is open, then q is open.

(c) If $B \subseteq A$ is a hereditary C^*-subalgebra and $p, q \in \mathrm{Proj}(B^{**})$, then p and q are spatially equivalent with respect to B if and only if they are spatially equivalent with respect to A.

Proof: (a) It suffices to verify the transitivity. Suppose that p, q and v are as in Definition 2.5. If $w \in A^{**}$ and $r \in \mathrm{Proj}(A^{**})$ satisfy that

$$p = w^*w, \quad r = ww^*, \quad w\,\mathrm{her}_A(p)w^* = \mathrm{her}_A(r) \quad \text{and} \quad w^*\,\mathrm{her}_A(r)w = \mathrm{her}_A(p),$$

then the partial isometry wv gives the equivalence $r \sim_{\mathrm{sp}} q$.

(b) As p is open and $\mathrm{her}_A(p)$ is contained in the weak-*-closed subspace $uA^{**}u^*$, one has $p \leq uu^*$. Let $v := pu$. Then $vv^* = p$ and $v^*v = u^*pu = q$. Moreover, it is clear that $\mathrm{her}_A(p) \subseteq v\,\mathrm{her}_A(q)v^*$ and $\mathrm{her}_A(q) \subseteq v^*\,\mathrm{her}_A(p)v$. Now, it is easy to see that the relations in (2.2) are satisfied. Furthermore, if $\{a_i\}_{i\in\mathfrak{I}}$ is an approximate unit in $\mathrm{her}_A(p)$, then $\{v^*a_iv\}$ is an approximate unit in $\mathrm{her}_A(q)$ that weak-*-converges to $v^*pv = q$, and so q is open. The second statement follows directly from the first one.

(c) Suppose that p and q are spatially equivalent with respect to A and $v \in A^{**}$ satisfies the relations in (2.2). As $vv^*, v^*v \in B^{**}$, Remark 2.2(c) tells us that $v \in B^{**}$. Now the equivalence follows from Remark 2.2(a). □

Proposition 2.7. *(a) If $p, q \in \mathrm{OP}(A)$, the following statements are equivalent.*

(1) $p \sim_{\mathrm{sp}} q$.

(2) $\mathrm{her}(q) = u^*\,\mathrm{her}(p)u$ and $\mathrm{her}(p) = u\,\mathrm{her}(q)u^*$ for a partial isometry $u \in A^{**}$.

(3) $\mathrm{her}(q) \subseteq u^*\,\mathrm{her}(p)u$ and $\mathrm{her}(p) \subseteq u\,\mathrm{her}(q)u^*$ for a partial isometry $u \in A^{**}$.

(4) $q \leq v^*v$ and $v\,\mathrm{her}(q)v^* = \mathrm{her}(p)$ for a partial isometry $v \in A^{**}$.

(5) There is a partial isometry $w \in A^{**}$ such that $p = ww^*$ and

$$\{w^*rw : r \in \mathrm{OP}(A); r \leq p\} \;=\; \{s \in \mathrm{OP}(A) : s \leq q\}.$$

(b) If M is a von Neumann algebra and $p, q \in \mathrm{Proj}(M)$, then $p \sim_{\mathrm{sp}} q$ if and only if $p \sim_{\mathrm{Mv}} q$ as elements in $\mathrm{Proj}(M)$.

Proof: (a) The implications (1) ⇒ (2) ⇒ (3) and (1) ⇒ (4) are clear.

(3) ⇒ (1). Since q is open, one has $q \leq u^*u$. Thus, $(uq)^*uq = q$ and Statement (3) also holds when u is replaced by uq. As p is also open, a similar argument shows that Statement (3) holds if we replace u by $v := puq$ and that $p = vv^*$. Since q is open and $vqv^* = vv^* = p$, Lemma 2.6(b) tells us that $p \sim_{\mathrm{sp}} q$.

(4) ⇒ (2). This follows from $v^*\,\mathrm{her}(p)v = v^*v\,\mathrm{her}(q)v^*v = \mathrm{her}(q)$.

(1) ⇔ (5). As $\mathrm{OP}(\mathrm{her}(p)) = \{r \in \mathrm{OP}(A) : r \leq p\}$ (see Remark 2.2(b)), one knows that $p \sim_{\mathrm{sp}} q$ will imply Statement (5). Conversely, suppose that Statement (5) holds. Then we have $q = w^*pw$, and the map $\Phi : x \mapsto w^*xw$ is a *-isomorphism from $\mathrm{her}(p)^{**}$ to $\mathrm{her}(q)^{**}$. By Proposition 2.3(b), we see that $\Phi(\mathrm{her}(p)) = \mathrm{her}(q)$ and Statement (4) holds.

(b) If $p \sim_{\mathrm{sp}} q$, then $p \sim_{\mathrm{Mv}} q$ as elements in $\mathrm{Proj}(M^{**})$, which implies that $p \sim_{\mathrm{Mv}} q$ as elements in $\mathrm{Proj}(M)$. Conversely, if $v \in M$ satisfying $p = vv^*$ and $q = v^*v$, then clearly $v^*\,\mathrm{her}(p)v = \mathrm{her}(q)$. □

One can reformulate Statement (5) of Proposition 2.7(a) in the following way.

> There is a partial isometry $w \in A^{**}$ that induces Murray-von Neumann equivalences between open subprojections of p (including p) and open subprojections of q (including q).

Therefore, we regard this as a kind of "hereditarily stable version" of the Murray-von Neumann equivalence. Moreover, if $v \in A^{**}$ satisfies the relations in (2.2), then by Lemma 2.6(b), $r \sim_{\rm sp} v^*rv$ for all $r \in {\rm OP}({\rm her}(p))$, which means that spatial equivalence is automatically "hereditarily stable".

Remark 2.8. One might attempt to define $p \precsim_{\rm sp} q$ if there is $q_1 \in {\rm OP}(A)$ with $p \sim_{\rm sp} q_1 \leq q$. However, unlike the Murray-von Neumann equivalence situation, $p \precsim_{\rm sp} q$ and $q \precsim_{\rm sp} p$ does not imply that $p \sim_{\rm sp} q$. This can be shown by using a result of Lin. More precisely, it was shown in [23, Theorem 9] that there exist a separable unital simple C^*-algebra A as well as $p \in {\rm Proj}(A)$ and $u \in A$ such that $uu^* = 1$ and $p_1 = u^*u \leq p$, but ${\rm her}(p)$ and A are not *-isomorphic. In particular, $p \nsim_{\rm sp} 1$. Now, we clearly have $p \precsim_{\rm sp} 1$. On the other hand, as $u \in A$, we have

$$u^*Au \ = \ {\rm her}(p_1) \quad \text{and} \quad u\,{\rm her}(p_1)u^* \ = \ A,$$

which implies that $1 \precsim_{\rm sp} p$.

This example also shows that the same problematic situation appears even if we replace $\sim_{\rm sp}$ with the stronger equivalence relation $\sim_{\rm PZ}$ as defined in (2.1) (because $u \in A$). Nevertheless, it was shown in [31, Theorem 1.13] that a weaker conclusion holds if one adds an extra assumption on either p or q, but we will not recall the details here.

Let us end this section with the following well-known example. We give an explicit argument here for future reference. Note that part (a) of it means that if $a, b \in A_+$ are equivalent in the sense of Blackadar (i.e. there exists $x \in A$ with $a = x^*x$ and $b = xx^*$; see e.g. [28, Definition 2.1]), then their support projections are spatially equivalence (which is a corollary of [28, Proposition 4.3], since $\sim_{\rm PZ}$ is stronger than $\sim_{\rm sp}$).

Example 2.9. Suppose that $x \in A$ with $\|x\| = 1$. Set $a = x^*x$ and $b = xx^*$. Let $x = ua^{1/2}$ be the polar decomposition.

(a) It is easy to see that $\overline{aAa} = u^*(\overline{xAx^*})u$ and $\overline{xAx^*} = u(\overline{aAa})u^*$, i.e. $\overline{xAx^*}$ is spatially isomorphic to $\overline{aAa}$ (by Proposition 2.7(a)).

(b) Notice that $u(\overline{aAa})u^* = \overline{xAx^*} \supseteq \overline{xx^*Axx^*} \supseteq \overline{xx^*xAx^*xx^*} \supseteq u\overline{a^{3/2}Aa^{3/2}}u^* = u(\overline{aAa})u^*$, and we have $\overline{xAx^*} = \overline{bAb}$. Similarly, $\overline{x^*Ax} = \overline{aAa}$ and $\overline{x^*A^{**}x} = \overline{aA^{**}a}$, which implies that ${\rm her}(x) = {\rm her}(a)$. On the other hand, as $\overline{aAa}$ is a hereditary C^*-subalgebra of ${\rm her}(a)$ and $\{a^{1/k}\}_{k\in\mathbb{N}}$ is a sequence in $\overline{aAa}$ which is an approximate unit for ${\rm her}(a)$, one has $\overline{aAa} = {\rm her}(a)$. Consequently, ${\rm her}(x) = \overline{x^*Ax}$.

(c) Suppose that $B \subseteq A$ is a hereditary C^*-subalgebra and $x \in B$. Since $\overline{aAa} = \overline{a^2Aa^2}$, we see that $\overline{aBa} = \overline{aAa}$. Therefore, ${\rm her}_B(x) = {\rm her}_A(x)$ by part (b).

3. C^*-semi-finiteness and three types of C^*-algebras

As in the case of von Neumann algebras ([27]), in order to define different "types" of C^*-algebras, we need to define "abelian" and "finite" open projections. "Abelian" open projections are defined in the same way as that of von Neumann algebras. However, in order to define "finite" open projections, we need to use our "hereditarily stable version" of Murray-von Neumann equivalence in Section 2. Note that one cannot go very far with the original Murray-von Neumann equivalence, because there exist $p, q \in \mathrm{OP}(A)$ with $p \sim_{\mathrm{Mv}} q$ but $\mathrm{her}(p)$ and $\mathrm{her}(q)$ are not isomorphic (see [23]). Moreover, one cannot use a direct verbatim translation of the Murray-von Neumann finiteness.

Definition 3.1. (a) Let $q \in \mathrm{OP}(A)$ and $p \in \mathrm{Proj}(qA^{**}q)$. The *closure of* p *in* q, denoted by $\bar{p}^q$, is the smallest closed projection of $\mathrm{her}(q)$ that dominates p.

(b) Let $p, q \in \mathrm{OP}(A)$ with $p \leq q$. The projection p is said to be

i. *dense in* q if $\bar{p}^q = q$;
ii. *abelian* if $\mathrm{her}(p)$ is a commutative C^*-algebra;
iii. C^**-finite* if for any $r, s \in \mathrm{OP}(\mathrm{her}(p))$ with $r \leq s$ and $r \sim_{\mathrm{sp}} s$, one has $\bar{r}^s = s$.

If p is dense in q, we say that $\mathrm{her}(p)$ is *essential* in $\mathrm{her}(q)$. We denote by $\mathrm{OP}_{\mathfrak{C}}(A)$ and $\mathrm{OP}_{\mathcal{F}}(A)$ the set of all abelian open projections and the set of all C^*-finite open projections of A, respectively.

The terminology "p is dense in q" is used in many places (e.g. [31]), while the terminology "essential" comes from [38].

Some people might think that the above definition of C^*-finiteness is not perfect since $\bar{r}^s = s$ does not imply $r \sim_{\mathrm{sp}} s$. In the Appendix, we will consider a variant of this definition which seems more symmetric and is a more direct analogue of the von Neumann algebra finiteness, but such a definition is eventually rejected for some reasons. Other people might wonder why we do not use the finiteness as defined in [14]. The reason is that we want to give a classification scheme for C^*-algebras using open projections (and the definition of finiteness in [14] seems not related to open projections). Nevertheless, in the Appendix, we will also give a brief account for the situation when one uses this finiteness instead.

Remark 3.2. Let $p \in \mathrm{OP}(A)$.

(a) Suppose that p is abelian. If $r, s \in \mathrm{OP}(\mathrm{her}(p))$ satisfying $r \leq s$ and $r \sim_{\mathrm{sp}} s$, then $r = s$. Thus, p is C^*-finite.

(b) If $\mathrm{her}(p)$ is finite dimensional, then p is C^*-finite.

(c) One might ask why we do not define C^*-finiteness of p in the following way: for any $r \in \mathrm{OP}(\mathrm{her}(p))$ with $r \sim_{\mathrm{sp}} p$, one has $\bar{r}^p = p$. The reason is that the stronger condition in Definition 3.1(b) can ensure every open subprojection of a C^*-finite projection being C^*-finite. Such a phenomena is automatic for von Neumann algebras.

(d) A hereditary C^*-subalgebra $B \subseteq A$ is essential in A if and only if for any non-zero hereditary C^*-subalgebra $C \subseteq A$, one has $B \cdot C \neq \{0\}$. Thus, a closed ideal $I \subseteq A$ is essential in the sense of Definition 3.1 if and only it is essential in the usual sense (i.e. any non-zero closed ideal of A intersects I non-trivially).

Definition 3.3. A C^*-algebra A is said to be:

i. *C^*-finite* if $1 \in \mathrm{OP}_{\mathfrak{F}}(A)$;
ii. *C^*-semi-finite* if any element in $\mathrm{OP}(A) \setminus \{0\}$ dominates an element in $\mathrm{OP}_{\mathfrak{F}}(A) \setminus \{0\}$;
iii. *of Type* $\mathfrak{A}$ if any element in $\mathrm{OP}(A) \cap \mathrm{Z}(A^{**}) \setminus \{0\}$ dominates an element in $\mathrm{OP}_{\mathcal{C}}(A) \setminus \{0\}$;
iv. *of Type* $\mathfrak{B}$ if $\mathrm{OP}_{\mathcal{C}}(A) = \{0\}$ but each element in $\mathrm{OP}(A) \cap \mathrm{Z}(A^{**}) \setminus \{0\}$ dominates an element in $\mathrm{OP}_{\mathfrak{F}}(A) \setminus \{0\}$;
v. *of Type* $\mathfrak{C}$ if $\mathrm{OP}_{\mathfrak{F}}(A) = \{0\}$.

Let us give an equivalent form of the above abstract definition through the relation between (respectively, central) open projections and hereditary C^*-subalgebras (respectively, ideals). These relations play very important roles in the discussion in this paper. A C^*-algebra A is

- C^*-finite if and only if for each hereditary C^*-subalgebra $B \subseteq A$, any hereditary C^*-subalgebra of B that is spatially isomorphic to B is essential in B;
- C^*-semi-finite if and only if any non-zero hereditary C^*-subalgebra of A contains a non-zero C^*-finite hereditary C^*-subalgebra;
- of type $\mathfrak{A}$ if and only if every non-zero closed ideal of A contains a non-zero abelian hereditary C^*-subalgebra;
- of type $\mathfrak{B}$ if and only if A does not contain any non-zero abelian hereditary C^*-subalgebra and every non-zero closed ideal of A contains a non-zero C^*-finite hereditary C^*-subalgebra;
- of type $\mathfrak{C}$ if and only if A does not contain any non-zero C^*-finite hereditary C^*-subalgebra.

Remark 3.4. Suppose that A is simple.

(a) A is either of type $\mathfrak{A}$, type $\mathfrak{B}$ or type $\mathfrak{C}$.

(b) We will see in Corollary 4.5 that A is of type $\mathfrak{A}$ if and only if A is of type I (see, e.g., [30, 6.1.1] for its definition). Moreover, if A is of type II (in the sense of [14]), then A is of type $\mathfrak{B}$ (by Proposition 4.7 below), while if A is purely infinite (in the sense of [13]), then A is of type $\mathfrak{C}$ (by Proposition 4.11(a) below and [39, Theroem 1.2(ii)]). However, we do not know if the converse of the last two statements hold.

A positive element $a \in A_+$ is said to be *C^*-finite* if $\mathrm{her}(a)$ (i.e. $\overline{aAa}$) is C^*-finite. Parts (a) and (b) of the following results follow from the argument of [30, Proposition 6.1.7], but since the settings are slightly different, we give a brief account here.

Proposition 3.5. *(a) The sum, $\mathcal{C}(A)$, of all abelian hereditary C^*-subalgebras of A is a (not necessarily closed) ideal of A. If $\mathcal{C}(A)_+ := \mathcal{C}(A) \cap A_+$, then $\mathcal{C}(A)$ is the vector space* $\operatorname{span} \mathcal{C}(A)_+$ *generated by $\mathcal{C}(A)_+$.*

(b) The sum, $\mathcal{F}(A)$, of all C^*-finite hereditary C^*-subalgebras of A is a (not necessarily closed) ideal of A. If $\mathcal{F}(A)_+ := \mathcal{F}(A) \cap A_+$, then $\mathcal{F}(A) = \operatorname{span} \mathcal{F}(A)_+$.

(c) If $B \subseteq A$ is a hereditary C^*-subalgebra, then $\mathcal{C}(B)_+ = \mathcal{C}(A) \cap B_+$ and $\mathcal{F}(B)_+ = \mathcal{F}(A) \cap B_+$.

Proof: (a) This follows directly from the arguments of [30, Proposition 6.1.7] (see also the proof of part (b) below).

(b) Let $F_A \subseteq A_+$ be the set of all C^*-finite elements, and K_A be the smallest hereditary cone containing F_A. By [30, Proposition 1.4.10] and the fact that any positive element dominated by a C^*-finite element is again C^*-finite, we see that elements in K_A are finite sums of elements in F_A, and so, $u^* K_A u = K_A$ for any u in the unitary group $U_{M(A)}$ of $M(A)$. If $L_A := \{x \in A : x^*x \in K_A\}$, then L_A is an ideal of A such that $\operatorname{span} K_A$ coincides with $L_A^* L_A$, which is also an ideal of A. Now, it is clear that $K_A \subseteq \mathcal{F}(A)$ and $\mathcal{F}(A) \subseteq \operatorname{span} K_A$ (because positive elements in a C^*-finite hereditary C^*-subalgebra are C^*-finite).

(c) We will only establish the second equality as the argument for the first one is similar. As K_A is a hereditary cone, the argument of part (b) tells us that $\mathcal{F}(A)_+ = K_A$. It is clear that $\mathcal{F}(B) \subseteq \mathcal{F}(A) \cap B$. Conversely, if $w \in K_A \cap B$ and $w_1, ..., w_n \in F_A$ such that $w = \sum_{i=1}^n w_i$, then $w_i \leq w \in B_+$, which implies that $w_i \in F_A \cap B = F_B$ (see Example 2.9(c)). Consequently, $w \in K_B$ as required. $\square$

Clearly, $\mathcal{C}(A) \subseteq \mathcal{F}(A)$. We will see in Theorem 5.2(d) below that the closed ideal $\overline{\mathcal{C}(A)}$ is of type $\mathfrak{A}$, while $\overline{\mathcal{F}(A)}$ is C^*-semi-finite.

Example 3.6. (a) If A is commutative, then A is of type $\mathfrak{A}$ and is C^*-finite. Moreover, $\mathcal{C}(A) = \mathcal{F}(A) = A$.

(b) Let $p \in \mathrm{OP}(\mathcal{B}(\ell^2)) \subseteq \mathcal{B}(\ell^2)^{**}$ such that $\mathrm{her}(p) = \mathcal{K}(\ell^2)$ (the C^*-algebra of all compact operators). Then $p \neq 1$ but $\mathrm{her}(1-p) = (0)$. In fact, if $T \in \mathrm{her}(1-p)$, we have $pT = 0$ and $ST = SpT = 0$ for any $S \in \mathcal{K}(\ell^2)$, which gives $T = 0$. Moreover, p is dense in 1 because $\mathcal{K}(\ell^2)$ is an essential closed ideal of $\mathcal{B}(\ell^2)$ (see Remark 3.2(d)).

(c) If H is an infinite dimensional Hilbert space, then $\mathcal{K}(H)$ is a C^*-algebra of type $\mathfrak{A}$, which is not C^*-finite but is C^*-semi-finite. In fact, as $\mathcal{K}(H)$ is simple and contains many rank-one projections, it is of type $\mathfrak{A}$. On the other hand, suppose that $e \in \mathrm{Proj}(\mathcal{K}(H))$ is a rank-one projection. Then $1 - e \in \mathrm{OP}(\mathcal{K}(H)) \subseteq \mathcal{B}(H)$ and there is an isometry $v \in \mathcal{B}(H)$ with $vv^* = 1 - e$. Thus,

$$v^* \,\mathrm{her}(1-e) v \;=\; \mathcal{K}(H) \quad \text{and} \quad 1 - e \;\sim_{\mathrm{sp}}\; 1.$$

Moreover, as $e \in \mathrm{Proj}(\mathcal{K}(H))$, we see that $1 - e$ is also a closed projection and hence it is not dense in 1. Finally, as all hereditary C^*-subalgebras of $\mathcal{K}(H)$ are given by projections in $\mathcal{B}(H)$, they are of the form $\mathcal{K}(K)$ for some subspaces $K \subseteq H$. Hence, $\mathcal{K}(H)$ is C^*-semi-finite (see Remark 3.2(b)).

(d) Let H be a Hilbert space. Clearly, $\mathrm{Proj}(\mathcal{K}(H)) \subseteq \mathrm{OP}_{\mathcal{F}}(\mathcal{B}(H))$. Hence, if $\mathfrak{F}(H)$ is the set of all finite rank operators, then $\mathfrak{F}(H) \subseteq \mathcal{F}(\mathcal{B}(H))$. Suppose that $B \subseteq \mathcal{B}(H)$ is a C^*-finite hereditary C^*-subalgebra and $p \in \mathrm{Proj}(B)$. As p is C^*-finite and $pBp = p\mathcal{B}(H)p \cong \mathcal{B}(K)$ for a subspace $K \subseteq H$, we see that K is finite dimensional (see part (c)) and so $p \in \mathcal{K}(H)$. Since $B \subseteq \mathcal{B}(H)$ is a hereditary C^*-subalgebra, B is generated by its projections. Thus, B is a hereditary C^*-subalgebra of $\mathcal{K}(H)$, and $B \cong \mathcal{K}(H')$ for a subspace $H' \subseteq H$. The C^*-finiteness of B again implies that $\dim H' < \infty$, and $B \subseteq \mathfrak{F}(H)$. Consequently,

$$\mathcal{F}(\mathcal{B}(H)) \;=\; \mathfrak{F}(H).$$

On the other hand, since any finite rank projection is a sum of rank-one projections and any rank-one projection belongs to $\mathcal{C}(\mathcal{B}(H))$, we see that $\mathfrak{F}(H) \subseteq \mathcal{C}(\mathcal{B}(H)) \subseteq \mathcal{F}(\mathcal{B}(H))$. Furthermore, by Proposition 3.5(c), we also have $\mathcal{F}(\mathcal{K}(H)) = \mathcal{C}(\mathcal{K}(H)) = \mathfrak{F}(H)$.

Remark 3.7. Let $e \in \mathrm{OP}(A)$ and $z(e)$ be the central support of e.

(a) $z(e) = \sup_{u \in U_{M(A)}} ueu^*$ (see e.g. [30, Lemma 2.6.3]), and $z(e)$ is an open projection with $\mathrm{her}(z(e))$ being the smallest closed ideal containing $\mathrm{her}(e)$.

(b) Recall that $B := \mathrm{her}(e) \subseteq A$ is said to be *full* if $\mathrm{her}(z(e)) = A$. In this case, B is strongly Morita equivalent to A (see e.g. [34]). Consequently, $\mathrm{her}(e)$ is always strongly Morita equivalent to $\mathrm{her}(z(e))$.

The following is a key result in this paper. An essential ingredient of its proof (in particular, part (b)) is a result of Peligrad and Zsidó in [31].

Proposition 3.8. *Let A and B be two strongly Morita equivalent C^*-algebras.*

(a) A contains a non-zero abelian hereditary C^*-subalgebra if and only if B does.

(b) A contains a non-zero C^*-finite hereditary C^*-subalgebra if and only if B does.

Proof: There exist a C^*-algebra D and $e \in \mathrm{Proj}(M(D))$ such that both A and B are full hereditary C^*-subalgebras of D and we have

$$A \cong eDe \quad \text{and} \quad B \cong (1-e)D(1-e)$$

(see e.g. [8, Theorem II.7.6.9]). Thus, $z(e) = 1 = z(1-e)$.

(a) It suffices to show that A contains a non-zero abelian hereditary C^*-subalgebra whenever D does. Let $p \in \mathrm{OP}_{\mathcal{C}}(D) \setminus \{0\}$. As $pz(e) = p \neq 0$, we see that $pueu^* \neq 0$ for some $u \in U_{M(D)}$. By replacing p with u^*pu, we may assume that $pe \neq 0$, and hence $e\,\mathrm{her}_D(p)e \neq (0)$. If $x, y \in \mathrm{her}_D(p)$ and $\{b_j\}_{j \in \mathfrak{J}}$ is an approximate unit of $\mathrm{her}_D(p)$, then $b_i e b_j \in \mathrm{her}_D(p)$ which implies that

$$xey \;=\; \lim x b_i e b_j y \;=\; \lim y b_i e b_j x \;=\; yex.$$

Consequently, $e\,\mathrm{her}_D(p)e$ is an abelian hereditary C^*-subalgebra of A.

(b) It suffices to show that if D contains a non-zero C^*-finite hereditary C^*-subalgebra, then so does A. Suppose that $p \in \mathrm{OP}_{\mathfrak{F}}(D) \setminus \{0\}$. By [31, Theorem 1.9], there exist $e_0, e_1 \in \mathrm{OP}(\mathrm{her}_D(e))$ and $p_0, p_1 \in \mathrm{OP}(\mathrm{her}_D(p))$ satisfying

$$\overline{e_0 + e_1}^e = e, \quad \overline{p_0 + p_1}^p = p, \quad z(e_0)z(p_0) = 0 \quad \text{and} \quad e_1 \sim_{\mathrm{PZ}} p_1.$$

Suppose that $p_1 = 0$. Then $e_1 = 0$ and $z(e_0)$ is dense in $z(e) = 1$ (by [31, Lemma 1.8]). This implies that $z(p_0) = 0$, and we have a contradiction that $p_0 = 0$ is dense in the non-zero open projection p. Therefore, $p_1 \neq 0$ and is C^*-finite. Since $\mathrm{her}_D(e_1) \cong \mathrm{her}_D(p_1)$ (note that $\sim_{\mathrm{PZ}}$ is stronger than $\sim_{\mathrm{sp}}$), we see that $\mathrm{her}_D(e_1)$ is a non-zero C^*-finite hereditary C^*-subalgebra of $A = \mathrm{her}_D(e)$. □

One may also use the argument of part (b) to obtain part (a), but we keep the alternative argument since it is also interesting.

Suppose that E is a full Hilbert A-module implementing the strong Morita equivalence between A and B, i.e. $B \cong \mathcal{K}_A(E)$ (see e.g. [22]). If I is a closed ideal of A, then EI is a full Hilbert I-module and $\mathcal{K}_I(EI)$ is a closed ideal of B. We also recall from [31, Definition 2.1] that A is said to be *discrete* if any non-zero open projection of A dominates a non-zero abelian open projection.

Theorem 3.9. *(a) Let A and B be two strongly Morita equivalent C^*-algebras. Then A is of type $\mathfrak{A}$ (respectively, type $\mathfrak{B}$ or type $\mathfrak{C}$) if and only if B is of the same type.*

(b) A C^*-algebra A is of type $\mathfrak{A}$ if and only if it is discrete.

Proof: (a) Suppose that A is of type $\mathfrak{B}$. If $\mathrm{OP}_{\mathfrak{C}}(B) \neq \{0\}$, then $\mathrm{OP}_{\mathfrak{C}}(A) \neq \{0\}$ (because of Proposition 3.8(a)), which is a contradiction. Let J be a non-zero closed ideal of B. As in the paragraph above, the strong Morita equivalence of A and B gives a closed ideal J_0 of A that is strongly Morita equivalent to J. As J_0 contains a non-zero C^*-finite hereditary C^*-subalgebra, so is J (by Proposition 3.8(b)). This shows that B is of type $\mathfrak{B}$. The argument for the other two types are similar and easier.

(b) It suffices to show that if A is of type $\mathfrak{A}$, then it is discrete. Let $B \subseteq A$ be a non-zero hereditary C^*-subalgebra and $J \subseteq A$ be the closed ideal generated by B (which is strongly Morita equivalent to B). As J contains a non-zero abelian hereditary C^*-subalgebra, so does B (by Proposition 3.8(a)). □

The following result follows from Proposition 3.8(b) and the argument of Theorem 3.9.

Corollary 3.10. *(a) A is C^*-semi-finite if and only if any non-zero closed ideal of A contains a non-zero C^*-finite hereditary C^*-subalgebra.*

(b) If A is strongly Morita equivalent to a C^*-semi-finite C^*-algebra, then A is also C^*-semi-finite.

(c) A is of type $\mathfrak{B}$ if and only if it is anti-liminary and C^*-semi-finite.

Remark 3.11. As in the case of von Neumann algebra, strong Morita equivalence does not preserve C^*-finiteness. In fact, for any C^*-algebra A, the algebra $A \otimes \mathcal{K}(\ell^2)$ is not C^*-finite (using the same argument as Example 3.6(c); note that $1 \otimes (1 - e)$ is both an open and a closed projection of $A \otimes \mathcal{K}(\ell^2)$). Consequently, any stable C^*-algebra is not C^*-finite.

Recall that a C^*-algebra A has *real rank zero* in the sense of Brown and Pedersen if the set of elements in A_{sa} with finite spectrum is norm dense in A_{sa} (see e.g. [12, Corollary 2.6]). The following result follows from Theorem 3.9(b), Corollary 3.10(c) as well as the fact that any hereditary C^*-subalgebra of a real rank zero C^*-algebra is again of real rank zero (see e.g. [12, Corollary 2.8]).

Corollary 3.12. *Let A be a C^*-algebra with real rank zero.*

(a) A is of type $\mathfrak{A}$ if and only if any projection in $\mathrm{Proj}(A) \setminus \{0\}$ *dominates an abelian projection in* $\mathrm{Proj}(A) \setminus \{0\}$.

(b) A is of type $\mathfrak{B}$ if and only if any projection in $\mathrm{Proj}(A) \setminus \{0\}$ *is non-abelian but dominates a C^*-finite projection in* $\mathrm{Proj}(A) \setminus \{0\}$.

(c) A is of type $\mathfrak{C}$ if and only if A does not contain any non-zero C^*-finite projection.

(d) A is C^*-semi-finite if and only if any projection in $\mathrm{Proj}(A) \setminus \{0\}$ *dominates a C^*-finite projection in* $\mathrm{Proj}(A) \setminus \{0\}$.

Remark 3.13. Suppose that A is a C^*-finite C^*-algebra with real rank zero. If $r, p \in \mathrm{Proj}(A)$ satisfying $r \leq p$ and $r \sim_{\mathrm{Mv}} p$ (as element in A), then $r \sim_{\mathrm{sp}} p$ and so, $r = \bar{r}^p = p$.

Corollary 3.14. *If A is of real rank zero, then the closures of the ideals $\mathcal{C}(A)$ and $\mathcal{F}(A)$ (see Proposition 3.5) are the closed linear spans of abelian projections and of C^*-finite projections in* $\mathrm{Proj}(A)$, *respectively.*

Proof: If $B \subseteq A$ is a C^*-finite hereditary C^*-subalgebra, then B is the closed linear span of $\mathrm{Proj}(B) \cap \mathrm{OP}_{\mathcal{F}}(B)$. Thus, $\mathcal{F}(A)$ lies inside the closed linear span of $\mathrm{Proj}(A) \cap \mathrm{OP}_{\mathcal{F}}(A)$. Conversely, it is clear that $\mathrm{Proj}(A) \cap \mathrm{OP}_{\mathcal{F}}(A) \subseteq \mathcal{F}(A)$. The argument for the statement concerning $\mathcal{C}(A)$ is similar. □

By Remark 3.13 and Corollary 3.14, if M is a von Neumann algebra, $\mathcal{F}(M)$ is dense in the ideal $J(M)$ generated by finite projections (as defined in [19]).

Corollary 3.15. *Let A be of type $\mathfrak{A}$ (respectively, of type $\mathfrak{B}$, of type $\mathfrak{C}$ or C^*-semi-finite).*

(a) If B is a hereditary C^*-subalgebra of A, then B is of type $\mathfrak{A}$ (respectively, of type $\mathfrak{B}$, of type $\mathfrak{C}$ or C^*-semi-finite).

(b) If A is a hereditary C^*-subalgebra of B, the closed ideal $I \subseteq B$ generated by A is of type $\mathfrak{A}$ (respectively, of type $\mathfrak{B}$, of type $\mathfrak{C}$ or C^*-semi-finite).

Proof: (a) As the closed ideal J generated by B is strongly Morita equivalent to B, this result follows directly from Theorem 3.9(a) and Corollary 3.10(b).

(b) This follows from the definitions, and the fact that any hereditary C^*-subalgebra of I intersects A non-trivially. □

Consequently, we have the following result.

Corollary 3.16. *Suppose that A is non-unital, and $\tilde{A}$ is the unitalization of A. Then A is of type $\mathfrak{A}$ (respectively, of type $\mathfrak{B}$, of type $\mathfrak{C}$ or C^*-semi-finite) if and only if $\tilde{A}$ is of type $\mathfrak{A}$ (respectively, of type $\mathfrak{B}$, of type $\mathfrak{C}$ or C^*-semi-finite). The same is true when $\tilde{A}$ is replaced by $M(A)$.*

Our next lemma is probably well-known.

Lemma 3.17. *Let $e, f \in \mathrm{OP}(A)$ and $p, q \in \mathrm{OP}(A) \cap \mathrm{Z}(A^{**})$.*

(a) $ep \in \mathrm{OP}(A)$ and $\mathrm{her}(ep) = \mathrm{her}(e) \cap \mathrm{her}(p)$.

(b) If $e \neq 0$ and $\mathrm{her}(e) \subseteq \mathrm{her}(p) + \mathrm{her}(q)$, then $\mathrm{her}(e) \cap \mathrm{her}(p) \neq (0)$ or $\mathrm{her}(e) \cap \mathrm{her}(q) \neq (0)$.

(c) If $z(e)z(f) = 0$, then $\mathrm{her}(e) + \mathrm{her}(f) = \mathrm{her}(e + f)$.

Proof: Parts (a) and (c) are obvious. To show part (b), note that as $\mathrm{her}(p) + \mathrm{her}(q) \subseteq \mathrm{her}(p + q - pq)$, we have $e \leq p + q - pq$. If $ep = 0 = eq$, one obtains a contradiction that $e = e(p + q - pq) = 0$. Thus, the conclusion follows from part (a). □

Lemma 3.18. *If $\{p_i\}_{i \in \mathfrak{I}}$ is a family in $\mathrm{OP}_{\mathcal{F}}(A)$ with $z(p_i)z(p_j) = 0$ for $i \neq j$, then $p := \sum_{i \in \mathfrak{I}} p_i \in \mathrm{OP}_{\mathcal{F}}(A)$.*

Proof: It is clear that p is an open projection and $z(p) = \sum_{i \in \mathfrak{I}} z(p_i)$. Suppose that $r, q \in \mathrm{OP}(\mathrm{her}(p))$ with $r \leq q$ and $r \sim_{\mathrm{sp}} q$. Let $u \in A^{**}$ with $q = u^*u$ and $u\,\mathrm{her}(q)u^* = \mathrm{her}(r)$. For any $i \in \mathfrak{I}$, we set $q_i := z(p_i)q, r_i := z(p_i)r \in \mathrm{OP}(A)$ and $u_i := z(p_i)u$. It is easy to see that $q = \sum_{i \in \mathfrak{I}} q_i$, $r = \sum_{i \in \mathfrak{I}} r_i$, $q_i = u_i^* u_i$ and $r_i \leq q_i \leq z(p_i)p = p_i$. By Lemma 3.17(c), we see that

$$z(p_i)\,\mathrm{her}(q) \;=\; z(p_i)\big(\mathrm{her}(q_i) + \mathrm{her}\left(\textstyle\sum_{j \in \mathfrak{I} \setminus \{i\}} q_j\right)\big) \;=\; \mathrm{her}(q_i).$$

Similarly, $z(p_i)\,\mathrm{her}(r) = \mathrm{her}(r_i)$ and we have $u_i\,\mathrm{her}(q_i)u_i^* = \mathrm{her}(r_i)$. By Proposition 2.7(a), we know that $r_i \sim_{\mathrm{sp}} q_i$ and the C^*-finiteness of p_i tells us that r_i is dense in q_i. If $e \in \mathrm{OP}(\mathrm{her}(q))$ with $re = 0$, then $e_i := z(p_i)e \in \mathrm{OP}(\mathrm{her}(q_i))$ with $r_i e_i = 0$, which means that $e_i = 0$ (because $\overline{r_i}^{q_i} = q_i$). Consequently, $e = \sum_{i \in \mathfrak{I}} e_i = 0$ and r is dense in q as required. □

Part (a) of the following result is the equivalence of statements (i) and (iii) in [31, Theorem 2.3], while part (b) follows from the proof of [31, Theorem 2.3], Lemma 3.18, Theorem 3.9(a) and Corollary 3.15(b).

Proposition 3.19. *(a) A C^*-algebra A is of type $\mathfrak{A}$ if and only if there is an abelian hereditary C^*-subalgebra of A that generates an essential closed ideal of A.*

(b) A C^*-algebra A is C^*-semi-finite if and only if there is a C^*-finite hereditary C^*-subalgebra of A that generates an essential closed ideal of A.

4. Comparison with existing theories

In this section, we compare our "Murray-von Neumann type classification" with existing results in the literature. Through these comparisons, we obtain many new examples of C^*-algebras of different types. Moreover, we will show that a von Neumann algebra is a type $\mathfrak{A}$, type $\mathfrak{B}$, type $\mathfrak{C}$ or C^*-semi-finite C^*-algebra if and only if it is, respectively, a type I, type II, type III or semi-finiteness von Neumann algebra.

4.1. **Type $\mathfrak{A}$ algebras.**

Recall that a C^*-algebra A is said to be of *type I* if for any irreducible representation (π, H) of A, one has $\mathcal{K}(H) \subseteq \pi(A)$. We have already seen in Theorem 3.9(b) that type $\mathfrak{A}$ is the same as discreteness. Thus, the following result is a direct consequence of [31, Theorem 2.3]. Note that one can also obtain it using Theorem 3.9(a) and [6, Theorems 1.8 and 2.2].

Corollary 4.1. *Any type I C^*-algebra is of type $\mathfrak{A}$.*

The converse of the above is not true even for real rank zero C^*-algebras, as can be seen in the following example.

Example 4.2. By Example 3.6(c) and Corollary 3.15(b), we know that $\mathcal{B}(\ell^2)$ is of type $\mathfrak{A}$. However, $\mathcal{B}(\ell^2)$ is not a type I C^*-algebra (see e.g. [30, 6.1.2]).

Proposition 4.3. *(a) A is of type I if and only if every primitive quotient of A is of type $\mathfrak{A}$.*

(b) If A is of type $\mathfrak{A}$ and contains no essential primitive ideal, then A is of type I.

Proof: (a) Because of Corollary 4.1 and the fact that quotients of type I C^*-algebras are also of type I, we only need to show the sufficiency. Let $\pi : A \to \mathcal{B}(H)$ be an irreducible representation and B be a non-zero abelian hereditary C^*-subalgebra of $A/\ker\pi$. If $\tilde{\pi} : A/\ker\pi \to \mathcal{B}(H)$ is the induced representation, the restriction $\tilde{\pi}_B : B \to \mathcal{B}(\tilde{\pi}(B)H)$ is non-zero and irreducible. Thus, $\dim \tilde{\pi}(B)H = 1$ and $\tilde{\pi}(b)$ is a rank-one operator (and hence is compact) for any $b \in B \setminus \{0\}$. This shows that $\tilde{\pi}(A/\ker\pi) \cap \mathcal{K}(H) \neq (0)$, and $\pi(A) \supseteq \mathcal{K}(H)$.

(b) Suppose that $\pi : A \to \mathcal{B}(H)$ is an irreducible representation and J is a non-zero closed ideal of A with $J \cap \ker\pi = (0)$. If $B \subseteq J$ is a non-zero abelian hereditary C^*-subalgebra, the restriction $\pi_B : B \to \mathcal{B}(\pi(B)H)$ is non-zero and irreducible. The same argument as in part (a) tells us that $\pi(A) \supseteq \mathcal{K}(H)$. □

Remark 4.4. (a) Proposition 4.3(a) actually shows that A is of type I if and only if any primitive quotient contains a non-zero abelian hereditary C^*-subalgebra, which is likely to be a known fact.

(b) If every quotient of $\mathcal{B}(\ell^2)$ were of type $\mathfrak{A}$, then Proposition 4.3(a) told us that $\mathcal{B}(\ell^2)$ were a type I C^*-algebra, which contradicted [30, 6.1.2]. Consequently, not every quotient of a type $\mathfrak{A}$ C^*-algebra is of type $\mathfrak{A}$.

If A is simple and of type $\mathfrak{A}$, then by Proposition 4.3(b), it is of type I. This, together with Example 3.6(c), gives the following.

Corollary 4.5. *If A is a simple C^*-algebra of type $\mathfrak{A}$, then $A = \mathcal{K}(H)$ for some Hilbert space H. If, in addition, A is C^*-finite, then $A = M_n$ for some positive integer n.*

4.2. **Type $\mathfrak{B}$ algebras and C^*-semi-finite algebras.**

The following is a direct consequence of Remark 3.4(a) and Corollary 4.5.

Corollary 4.6. *Any infinite dimensional C^*-finite simple C^*-algebra is of type $\mathfrak{B}$.*

In the following, we compare type $\mathfrak{B}$ and type $\mathfrak{C}$ with the notions of type *II* and type *III* as introduced by Cuntz and Pedersen in [14]. Let us recall from [14, p.140] that $x \in A_+$ is said to be *finite* if for any sequence $\{z_k\}_{k\in\mathbb{N}}$ in A with $x = \sum_{k=1}^\infty z_k^* z_k$ and $y := \sum_{k=1}^\infty z_k z_k^* \leq x$, one has $y = x$. We also recall that A is said to be *finite* (respectively, *semi-finite*) if every $x \in A_+ \setminus \{0\}$ is finite (respectively, x dominates a non-zero finite element). Furthermore, A is said to be of *type II* if it is anti-liminary and finite, while A is said to be of *type III* if it has no non-zero finite elements (see [14, p.149]).

Let $T_s(A)$ denote the set of all tracial states on A. It follows from [14, Theorem 3.4] that $T_s(A)$ separates points of A_+ if A is finite.

Proposition 4.7. *If $T_s(A)$ separates points of A_+, then A is C^*-finite. Consequently, if A is finite, then A is C^*-finite.*

Proof: Suppose on the contrary that there exist $r, q \in \mathrm{OP}(A)$ with $r \leq q$, $r \sim_{\mathrm{sp}} q$ but $\bar{r}^q \lneq q$. For any $\tau \in T_s(A)$, if $\tilde{\tau}$ is the normal tracial state on A^{**} extending τ, then $\tilde{\tau}(r) = \tilde{\tau}(q)$ (because $r = vv^*$ and $q = v^*v$ for some $v \in A^{**}$). Moreover, if $\{a_i\}_{i\in\mathfrak{I}}$ is an approximate unit in $\mathrm{her}(r)$, one has $\tilde{\tau}(r) = \lim \tau(a_i)$. Since $\bar{r}^q \lneq q$, there exists $s \in \mathrm{OP}(\mathrm{her}(q)) \setminus \{0\}$ with $rs = 0$. If $x \in \mathrm{her}(s)_+$ with $\|x\| = 1$, one can find $\tau_0 \in T_s(A)$ with $\tau_0(x) > 0$. Thus, we have $\tau_0(a_i) + \tau_0(x) \leq \tilde{\tau}_0(q)$ (as $a_i x = 0$ and $a_i + x \leq q$), which gives the contradiction that $\tilde{\tau}_0(r) + \tau_0(x) \leq \tilde{\tau}_0(q)$. □

As in [14], we denote by $\mathscr{F}^A$ the set of all finite elements in A_+. Let $B \subseteq A$ be a hereditary C^*-subalgebra. Then $\mathscr{F}^B = \mathscr{F}^A \cap B$. In fact, it is obvious that $\mathscr{F}^A \cap B \subseteq \mathscr{F}^B$. Conversely, suppose that $x \in \mathscr{F}^B$. Consider $y \in A_+$ and a sequence $\{z_k\}_{k\in\mathbb{N}}$ in A satisfying $y \leq x$, $y = \sum_{k=1}^\infty z_k z_k^*$ and $x = \sum_{k=1}^\infty z_k^* z_k$. Since B_+ is a hereditary cone of A_+, we have $y \in B_+$ and $z_k^* z_k, z_k z_k^* \in B_+$ ($k \in \mathbb{N}$). By Remark 2.2(c), we know that $z_k \in B$ and so, $y = x$ as required.

Corollary 4.8. *(a) A is semi-finite if and only if every non-zero hereditary C^*-subalgebra of A contains a non-zero finite hereditary C^*-subalgebra.*
(b) If A is semi-finite (respectively, of type II), then A is C^*-semi-finite (respectively, of type $\mathfrak{B}$).

Proof: (a) For the necessity, let $B \subseteq A$ be a non-zero hereditary C^*-subalgebra. If $y \in B_+ \setminus \{0\}$, there is $x \in \mathscr{F}^A \setminus \{0\}$ with $x \leq y$. By [14, Lemma 4.1] and [14, Theorem 4.8] as well as their arguments, one can find a non-zero finite hereditary C^*-subalgebra of $\mathrm{her}(x)$. More precisely, let $f \in C(\sigma(x))_+$ such that f vanishes on a neighbourhood of 0 and $f(t) \leq t \leq f(t) + \frac{\|x\|}{2}$ ($t \in \sigma(x)$). There exists $g \in C(\sigma(x))_+$ and $\lambda > 0$ such that $f = fg$ and $g(t) < \lambda t$ ($t \in \sigma(x)$). Then $g(x) \in \mathscr{F}^A$ and $f(x) = f(x)g(x)$, i.e. $f(x) \in \mathscr{F}_0 := \{a \in A_+ : a = ay \text{ for some } y \in \mathscr{F}^A\}$. For any $z \in \mathrm{her}(f(x))_+$, we have $zg(x) = z$ and $z \in \mathscr{F}_0 \cap \mathrm{her}(f(x)) \subseteq \mathscr{F}^A \cap \mathrm{her}(f(x)) = \mathscr{F}^{\mathrm{her}(f(x))}$. Thus, $\mathrm{her}(f(x))$ is a non-zero finite hereditary C^*-subalgebra of $\mathrm{her}(x)$.

For the sufficiency, let $y \in A_+ \setminus \{0\}$ and C be a non-zero finite hereditary C^*-subalgebra of $\mathrm{her}(y)$. Observe that $C_+ = \mathscr{F}^C = \mathscr{F}^A \cap C$. Take any $x \in C_+$ with $\|x\| = 1$. Since $x^{1/2} y x^{1/2} \leq \|y\| x \in \mathscr{F}^A$, we know that $y^{1/2} x y^{1/2} = y^{1/2} x^{1/2} (y^{1/2} x^{1/2})^* \in \mathscr{F}^A$ (because of [14, Lemma 4.1]). As $y^{1/2} x y^{1/2} \leq y$, we see that A is semi-finite.
(b) This follows from part (a), Proposition 4.7 and Corollary 3.10(c). □

Example 4.9. (a) If A is an infinite dimensional simple C^*-algebra with a faithful tracial state, then A is of type $\mathfrak{B}$ (by Corollary 4.6 and Proposition 4.7). In particular, if Γ is an infinite discrete group such that $C_r^*(\Gamma)$ is simple (see e.g. [7] for some examples of such groups), then $C_r^*(\Gamma)$ is of type $\mathfrak{B}$.
(b) Every simple AF algebra which is not of the form $\mathcal{K}(H)$ is of type $\mathfrak{B}$ (because of [14, Proposition 4.11] as well as Corollaries 4.5 and 4.8(b)).

4.3. **Type $\mathfrak{C}$ algebras.**

The following is a consequence of Proposition 4.7 and the argument of the necessity of Corollary 4.8(a).

Corollary 4.10. *If A is of type $\mathfrak{C}$, then it is of type III.*

Next, we compare type $\mathfrak{C}$ with the notion of pure infinity as defined by Cuntz (in the case of simple C^*-algebras) and by Kirchberg and Rørdam (in the general case). Suppose that $a \in M_n(A)$ and $b \in M_m(A)$ $(m, n \in \mathbb{N})$. As in [21, Definition 2.1], we write $a \precsim b$ *relative to* $M_{m,n}(A)$ if there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $M_{m,n}(A)$ such that $\|x_k^* b x_k - a\| \to 0$. Recall that an element $a \in A$ is said to be *properly infinite* if $a \oplus a \precsim a$ relative to $M_{1,2}(A)$. Moreover, A is said to be *purely infinite* if every element in A_+ is properly infinite (see [21, Theorem 4.16]). Note that if A is simple, this notion coincides with the one in [13], namely, every hereditary C^*-subalgebra of A contains a non-zero infinite projection (see e.g. the work of Lin and Zhang in [24]).

Proposition 4.11. *(a) If A has real rank zero and is purely infinite, then it is of type $\mathfrak{C}$.*
(b) If A is a separable purely infinite C^*-algebra with stable rank one, then A is of type $\mathfrak{C}$.

Proof: (a) By [21, Theorem 4.16], any projection in A is properly infinite, and hence is infinite (see e.g. [21, Lemma 3.1]), in the sense that it is Murray-von Neumann equivalent to a proper subprojection. Now, if $p \in \mathrm{Proj}(A)$ and $v \in A$ such that $v^*v = p$ and $q := vv^* \lneq p$, then $p \sim_{\mathrm{sp}} q$ but q is not dense in p (because $p - q \in \mathrm{Proj}(A) \setminus \{0\}$). Therefore, p is not C^*-finite and Corollary 3.12(c) show that A is of type $\mathfrak{C}$.
(b) Suppose on contrary that A contains a non-zero C^*-finite C^*-algebra B and we take any $z \in B_+$ with $\|z\| = 1$. By [21, Theorem 4.16], we see that $z \oplus z \precsim z \oplus 0$ relative to $M_2(A)$, and so, $z \oplus z \precsim z \oplus 0$ relative to $M_2(\mathrm{her}(z))$ (by [21, Lemma 2.2(iii)]). Thus, [28, Proposition 4.13] implies

$$p_z \oplus p_z \ = \ p_{z\oplus z} \ \precsim_{\mathrm{Cu}} \ p_{z\oplus 0} \ = \ p_z \oplus 0$$

(see [28, §3] for the meaning of $\precsim_{\mathrm{Cu}}$). Now, using [28, 6.2(1)'&(2)'], one has $p_z \oplus p_z \sim_{\mathrm{PZ}} p_z \oplus 0$ (clearly, $p_{z\oplus 0} \precsim_{\mathrm{Cu}} p_{z\oplus z}$) and hence $p_z \oplus p_z \sim_{\mathrm{sp}} p_z \oplus 0$. This means that $M_2(\mathrm{her}(z))$ is spatially isomorphic (and hence *-isomorphic) to its hereditary C^*-subalgebra $\mathrm{her}(z) \oplus (0)$, which is not essential in $M_2(\mathrm{her}(z))$ (because $(0) \oplus \mathrm{her}(z)$ is a non-zero hereditary C^*-subalgebra and we have Remark 3.2(d)). As $\mathrm{her}(z)$ is *-isomorphic to $\mathrm{her}(z) \oplus (0)$ and hence to $M_2(\mathrm{her}(z))$, we know that $\mathrm{her}(z)$ is also spatially isomorphic to an inessential hereditary C^*-subalgebra. Consequently, $\mathrm{her}(z)$ is not C^*-finite, which contradicts the fact that B is C^*-finite. □

Let us make the following conjecture. The proposition above tells us that this conjecture holds in some interesting cases.

Conjecture 4.12. *Every purely infinite C^*-algebra is of type $\mathfrak{C}$.*

On the other hand, by Proposition 4.11 and Corollary 4.10, we know that any separable purely infinite C^*-algebra A having real rank zero or stable rank one is of type III. This implication actually holds without these extra assumptions, as can be seen in the following proposition, which gives another evidence for Conjecture 4.12. Note that this proposition also implies [21, Proposition 4.4]. To show this result, let us recall the following notation from [28, p.3476]. For any $\epsilon > 0$, let $f_\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be the function $f_\epsilon(t) = \begin{cases} t/\epsilon & \text{if } t \in [0,\epsilon) \\ 1 & \text{if } t \in [\epsilon,\infty). \end{cases}$ If $\mu \in T_s(A)$ and $a \in A_+$, we define $d_\mu(a) := \sup_{\epsilon>0} \mu(f_\epsilon(a))$ (the definition in [28] is for tracial weights but we only need tracial states here).

Proposition 4.13. *Any purely infinite C^*-algebra A is of type III.*

Proof: Suppose on the contrary that $\mathscr{F}^A \neq \{0\}$. By the argument of the necessity of Corollary 4.8(a), there is $z \in A_+$ with $\|z\| = 1$ and $\mathrm{her}(z)$ being finite. By the argument of Proposition 4.11(b), one has $z \oplus z \precsim z \oplus 0$ relative to $M_2(\mathrm{her}(z))$. By [28, Remark 2.5], we see that $d_\mu(z \oplus z) \leq d_\mu(z \oplus 0)$ for each $\mu \in T_s(M_2(\mathrm{her}(z)))$. Now, if $\tau \in T_s(\mathrm{her}(z))$, then $\tau \otimes \mathrm{Tr}_2 \in T_s(M_2(\mathrm{her}(z)))$ (where Tr_2 is the canonical tracial state on M_2), and the above tells us that

$$\sup_{\epsilon>0} \tau(f_\epsilon(z)) = \sup_{\epsilon>0} (\tau \otimes \mathrm{Tr}_2)(f_\epsilon(z) \oplus f_\epsilon(z)) \leq \sup_{\epsilon>0} (\tau \otimes \mathrm{Tr}_2)(f_\epsilon(z) \oplus 0) = \sup_{\epsilon>0} \frac{\tau(f_\epsilon(z))}{2},$$

which gives $d_\tau(z) = 0$ and hence $\tau(z) = 0$. This contradicts [14, Theorem 3.4]. □

The following remark gives one more evidence of the above conjecture.

Remark 4.14. Suppose that $a \in A_+$ and there exist $x, y \in \mathrm{her}(a)$ with $x^*x = a = y^*y$ as well as $x^*y = 0$ (note that this condition implies a being properly infinite; see [21, Proposition 3.3(iv)]). By Example 2.9(a)&(b), we see that $\mathrm{her}(a)$ is spatially isomorphic to its hereditary C^*-subalgebra $\mathrm{her}(x^*)$. As $\mathrm{her}(x^*)\,\mathrm{her}(y^*) = (0)$, we see that $\mathrm{her}(x^*)$ is not essential in $\mathrm{her}(a)$. Thus, $\mathrm{her}(a)$ is not C^*-finite. If one can show that the same is true for every properly infinite element $a \in A_+$, then by [21, Theorem 4.16], every purely infinite C^*-algebra is of type $\mathfrak{C}$.

Example 4.15. For any AF-algebra B, the C^*-algebra $\mathcal{O}_2 \otimes B$ is purely infinite (by [21, Proposition 4.5]) and is of real rank zero (by [12, Theorem 3.2]), which means that $\mathcal{O}_2 \otimes B$ is of type $\mathfrak{C}$ (by Proposition 4.11(a)). Note that one may replace $\mathcal{O}_2$ with any unital, simple, separable, purely infinite, nuclear C^*-algebra (which has real rank zero because of [39, Theorem 1.2(ii)]).

4.4. **The case of von Neumann algebras.**

In this subsection, we consider the case of von Neumann algebras. Let us start with the following lemma. Note that one implication of this result follows directly from Proposition 4.7, but we give a longer alternative proof here as this argument is also interesting (see Remark 4.17 below).

Lemma 4.16. *Let M be a von Neumann algebra. Then $p \in \mathrm{Proj}(M)$ is finite as a projection in M if and only if it is C^*-finite.*

Proof: Let $\Lambda_M : M^{**} \to M$ be the canonical *-epimorphism. If $q \in \mathrm{OP}(pMp)$, then $\mathrm{her}_M(q) \subseteq \mathrm{her}_M(\Lambda_M(q))$ and $\Lambda_M(q) \leq p$, which imply that $\Lambda_M(q) = \bar{q}^p$ (as $\bar{q}^p \in pMp$ by e.g. [2, II.1]).

Suppose that $r, q \in \mathrm{OP}(pMp)$ such that $r \leq q$ and $r \sim_{\mathrm{sp}} q$. Consider $w \in M^{**}$ satisfying

$$q = ww^*, \quad r = w^*w, \quad w^* \operatorname{her}(q) w = \operatorname{her}(r) \quad \text{and} \quad w \operatorname{her}(r) w^* = \operatorname{her}(q).$$

Define $v := \Lambda_M(w)$. Then $\Lambda_M(q) = vv^*$ and $\Lambda_M(r) = v^*v$. Since $\Lambda_M(r) \leq \Lambda_M(q) \leq p$, the finiteness of p tells us that $\bar{r}^p = \Lambda_M(r) = \Lambda_M(q) = \bar{q}^p$. If $\bar{r}^q \lneq q$, there is $e \in \mathrm{OP}(\mathrm{her}(q)) \setminus \{0\}$ with $re = 0$. Since $e \in \mathrm{OP}(\mathrm{her}(p))$, we obtain a contradiction that $\bar{r}^p \neq \bar{q}^p$ (as $r \leq p - e$ but $q \nleq p - e$). This shows that p is C^*-finite.

Conversely, suppose that p is C^*-finite and $r \in \mathrm{Proj}(M) \subseteq \mathrm{OP}(M)$ with $r \leq p$ and $r \sim_{\mathrm{Mv}} p$. Then Proposition 2.7(b) implies that $r \sim_{\mathrm{sp}} p$ and so $r = \bar{r}^p = p$. □

Remark 4.17. (a) If $p \in M$ is a (C^*-)finite projection and $r \in \mathrm{OP}(pMp)$ with $r \sim_{\mathrm{sp}} p$. The C^*-finiteness of p gives $\bar{r}^p = p$. Suppose that $w \in M^{**}$ and $v \in M$ are as in the proof of Lemma 4.16 for the case when $q = p$. Then $vv^* = p = \bar{r}^p = v^*v$, which means that v is a unitary in pMp. Moreover, $v^* \operatorname{her}(r) v = \Lambda_M(w^* \operatorname{her}(r) w) = pMp$ and $\operatorname{her}(r) = pMp$. Consequently, $r = p$ (note that one needs $r \in M$ in Remark 3.13).

(b) If A is a C^*-algebra and $p \in \mathrm{OP}(A)$ satisfying $\bar{r}^p = \bar{q}^p$ for any $r, q \in \mathrm{OP}(\mathrm{her}(p))$ with $r \leq q$ and $r \sim_{\mathrm{sp}} q$, then by the argument of Lemma 4.16, we see that p is C^*-finite.

The following is a direct consequence of Lemma 4.16 and Corollary 3.12.

Theorem 4.18. *Let M be a von Neumann algebra.*

(a) M is of type $\mathfrak{A}$ if and only if M is a type I von Neumann algebra.

(b) M is of type $\mathfrak{B}$ if and only if M is a type II von Neumann algebra.

(c) M is of type $\mathfrak{C}$ if and only if M is a type III von Neumann algebra.

(d) M is C^*-semi-finite if and only if M is a semi-finite von Neumann algebra.

5. Factorizations

In this section, we give two factorization type results for general C^*-algebras. Let us first state the following easy lemma. Notice that if A contains a non-zero abelian hereditary C^*-subalgebra B, the closed ideal generated by B is of type $\mathfrak{A}$ (by Corollary 3.15(b) and Remark 3.7(b)), and the same is true for C^*-finite hereditary C^*-subalgebra.

Lemma 5.1. *If A is not of type $\mathfrak{C}$, then A contains a non-zero closed ideal of either type $\mathfrak{A}$ or type $\mathfrak{B}$.*

The following is our first factorization type result, which mimics the corresponding situation for von Neumann algebras.

Theorem 5.2. *Let A be a C^*-algebra.*

(a) There is a largest type $\mathfrak{A}$ (respectively, type $\mathfrak{B}$, type $\mathfrak{C}$ and C^*-semi-finite) hereditary C^*-subalgebra $J_{\mathfrak{A}}$ (respectively, $J_{\mathfrak{B}}$, $J_{\mathfrak{C}}$ and $J_{\mathfrak{sf}}$) of A, which is also an ideal of A.

(b) $J_{\mathfrak{A}}$, $J_{\mathfrak{B}}$ and $J_{\mathfrak{C}}$ are mutually disjoint such that $J_{\mathfrak{A}} + J_{\mathfrak{B}} + J_{\mathfrak{C}}$ is an essential closed ideal of A. If $e_{\mathfrak{A}}, e_{\mathfrak{B}}, e_{\mathfrak{C}} \in \mathrm{OP}(A) \cap Z(A^{**})$ with $J_{\mathfrak{A}} = \mathrm{her}(e_{\mathfrak{A}})$, $J_{\mathfrak{B}} = \mathrm{her}(e_{\mathfrak{B}})$ and $J_{\mathfrak{C}} = \mathrm{her}(e_{\mathfrak{C}})$, then

$$1 = \overline{e_{\mathfrak{A}} + e_{\mathfrak{B}}}^{1} + e_{\mathfrak{C}}.$$

(c) $J_{\mathfrak{A}} + J_{\mathfrak{B}}$ is an essential closed ideal of $J_{\mathfrak{sf}}$. If $e_{\mathfrak{sf}} \in \mathrm{OP}(A)$ with $J_{\mathfrak{sf}} = \mathrm{her}(e_{\mathfrak{sf}})$, then

$$e_{\mathfrak{sf}} = \overline{e_{\mathfrak{A}}}^{e_{\mathfrak{sf}}} + e_{\mathfrak{B}}.$$

(d) The closure of $\mathcal{C}(A)$ and $\mathcal{F}(A)$ (in Proposition 3.5) are essential closed ideals of $J_{\mathfrak{A}}$ and $J_{\mathfrak{sf}}$, respectively.

Proof: (a) We first consider the situation of type $\mathfrak{B}$. Let $\mathfrak{J}_{\mathfrak{B}}$ be the set of all type $\mathfrak{B}$ closed ideals of A. If $\mathfrak{J}_{\mathfrak{B}} = \{(0)\}$, then $J_{\mathfrak{B}} := (0)$ is the largest type $\mathfrak{B}$ hereditary C^*-subalgebra of A (see Corollary 3.15(b) and Remark 3.7(b)). Otherwise, suppose that J_1 and J_2 are distinct elements in $\mathfrak{J}_{\mathfrak{B}}$. If $J_1 + J_2$ contains a non-zero abelian hereditary C^*-algebra B, then by Lemma 3.17(b), one of the two abelian hereditary C^*-subalgebras $B \cap J_1$ and $B \cap J_2$ is non-zero, which contradicts $J_1, J_2 \in \mathfrak{J}_{\mathfrak{B}}$. On the other hand, consider a non-zero closed ideal I of $J_1 + J_2$. Again, by Lemma 3.17(b), we may assume that the closed ideal $I \cap J_1$ is non-zero. Thus, $I \cap J_1$ contains a non-zero C^*-finite hereditary C^*-subalgebra B. This shows that $J_1 + J_2 \in \mathfrak{J}_{\mathfrak{B}}$ and $\mathfrak{J}_{\mathfrak{B}}$ is a directed set.

Let $J_{\mathfrak{B}} := \overline{\sum_{J \in \mathfrak{J}_{\mathfrak{B}}} J}$. Then $e_{\mathfrak{B}} = w^*\text{-}\lim_{J \in \mathfrak{J}_{\mathfrak{B}}} e_J$, where $e_J \in \mathrm{OP}(A) \cap \mathrm{Z}(A^{**})$ with $J = \mathrm{her}(e_J)$. If there is $p \in \mathrm{OP}_{\mathcal{C}}(A) \setminus \{0\}$ such that $\mathrm{her}(p) \subseteq J_{\mathfrak{B}}$, then

$$p \ = \ pe_{\mathfrak{B}} \ = \ pe_{\mathfrak{B}}p \ = \ w^*\text{-}\lim_{J \in \mathfrak{J}_{\mathfrak{B}}} pe_J p,$$

and one can find $J \in \mathfrak{J}_{\mathfrak{B}}$ with the abelian algebra $\mathrm{her}(p) \cap J$ being non-zero (because of Lemma 3.17(a)), which is absurd.

Now, suppose that I is a non-zero closed ideal of $J_{\mathfrak{B}}$. The argument above tells us that $I \cap J \neq (0)$ for some $J \in \mathfrak{J}_{\mathfrak{B}}$, and hence it contains a non-zero C^*-finite hereditary

C^*-subalgebra. Consequently, $J_{\mathfrak{B}} \in \mathfrak{J}_{\mathfrak{B}}$. Finally, if $B \subseteq A$ is a hereditary C^*-subalgebra of type $\mathfrak{B}$, then by Corollary 3.15(b) and Remark 3.7(b), one has $B \subseteq J_{\mathfrak{B}}$.

The arguments for the statements concerning $J_{\mathfrak{A}}$, $J_{\mathfrak{C}}$ and $J_{\mathfrak{sf}}$ are similar and easier.

(b) The first statement follows directly from Lemma 5.1 (any non-type $\mathfrak{C}$ ideal interests either $J_{\mathfrak{A}}$ or $J_{\mathfrak{B}}$). For the second statement, we obviously have $e_{\mathfrak{A}} + e_{\mathfrak{B}} \leq 1 - e_{\mathfrak{C}}$. Suppose that $p \in \mathrm{OP}(A)$ with $e_{\mathfrak{A}} + e_{\mathfrak{B}} \leq 1 - p$. Then $z(p)(e_{\mathfrak{A}} + e_{\mathfrak{B}}) = 0$, and Lemmas 3.17(a) and 5.1 imply that $z(p) \leq e_{\mathfrak{C}}$. Thus, $1 - e_{\mathfrak{C}}$ is the smallest closed projection dominating $e_{\mathfrak{A}} + e_{\mathfrak{B}}$.

(c) This follows from a similar (but easier) argument as part (b).

(d) Clearly, $\mathcal{F}(A) \subseteq J_{\mathfrak{sf}}$ and $\mathcal{C}(A) \subseteq J_{\mathfrak{A}}$. Their closure are both essential because of Proposition 3.19. □

By Proposition 3.19, there is an abelian (respectively, a C^*-finite) hereditary C^*-subalgebra that generates an essential ideal of $J_{\mathfrak{A}}$ (respectively, of $J_{\mathfrak{B}}$). Moreover, by Corollary 4.1(a), Theorem 3.9(b) and [31, Theorem 2.3(vi)], the largest type I closed ideal $A_{postlim}$ of A is an essential ideal of $J_{\mathfrak{A}}$. On the other hand, we recall that A is *anti-liminary* if it does not contain any non-zero commutative hereditary C^*-subalgebra.

Remark 5.3. For any closed ideal J of A, we write $J^\perp$ for the closed ideal $\{a \in A : aJ = (0)\}$. It is easy to see that if J_0 is an essential ideal of J, then $J_0^\perp = J^\perp$.

(a) $J_{\mathfrak{A}}^\perp = A_{postlim}^\perp$ is the largest anti-liminary hereditary C^*-subalgebra of A (note that $\overline{aJ_{\mathfrak{A}}a}$ is a hereditary C^*-subalgebra of $J_{\mathfrak{A}}$ if $a \in A_+$). Furthermore, $J_{\mathfrak{B}} + J_{\mathfrak{C}}$ is an essential ideal of $J_{\mathfrak{A}}^\perp$ (by Theorem 5.2(b) and Lemma 5.1).

(b) $J_{\mathfrak{sf}}^\perp = (J_{\mathfrak{A}} + J_{\mathfrak{B}})^\perp = J_{\mathfrak{C}}$.

(c) $J_{\mathfrak{A}}^\perp \cap J_{\mathfrak{sf}} = J_{\mathfrak{B}}$ (compare with Corollary 3.10(c)).

From now on, we denote by $J_{\mathfrak{A}}^A$, $J_{\mathfrak{B}}^A$, $J_{\mathfrak{C}}^A$ and $J_{\mathfrak{sf}}^A$, respectively, the largest type $\mathfrak{A}$, the largest type $\mathfrak{B}$, the largest type $\mathfrak{C}$ and the largest C^*-semi-finite closed ideals of a C^*-algebra A.

The following is a direct application of Theorem 4.18.

Corollary 5.4. *Let M be a von Neumann algebra. If M_I, M_{II} and M_{III} are respectively the type I summand, the type II summand and the type III summand of M, then $J_{\mathfrak{A}}^M = M_I$, $J_{\mathfrak{B}}^M = M_{II}$ and $J_{\mathfrak{C}}^M = M_{III}$.*

Our next theorem is the second factorization type result, which seems to be more interesting for C^*-algebra (c.f. [14, Proposition 4.13]).

Theorem 5.5. *Let A be a C^*-algebra.*

(a) $A/J_{\mathfrak{C}}^A$ is C^*-semi-finite and $A/(J_{\mathfrak{A}}^A)^\perp$ is of type $\mathfrak{A}$.

(b) If A is C^*-semi-finite, then $A/J_{\mathfrak{B}}^A$ is of type $\mathfrak{A}$.

Proof: (a) If $A/J^A_{\mathfrak{C}} = (0)$, then it is C^*-semi-finite by the definition. We assume $A/J^A_{\mathfrak{C}} \neq (0)$, and consider $Q : A \to A/J^A_{\mathfrak{C}}$ to be the canonical map. Let I be a non-zero closed ideal of $A/J^A_{\mathfrak{C}}$ and $J := Q^{-1}(I)$. Since $J \supsetneq J^A_{\mathfrak{C}}$, one knows that J contains a non-zero C^*-finite hereditary C^*-subalgebra B. Since $B \cap J^A_{\mathfrak{C}} = (0)$, the *-homomorphism Q restricts to an injection on B. Thus, $Q(B) \subseteq I$ is also a non-zero C^*-finite hereditary C^*-subalgebra, and $A/J^A_{\mathfrak{C}}$ is C^*-semi-finite (by Corollary 3.10(a)). The proof of the second statement is similar.

(b) This follows from part (a) and Remark 5.3(c). □

Remark 5.6. (a) If a statement is true for all C^*-algebras of both type $\mathfrak{A}$ and type $\mathfrak{B}$ and is stable under extensions of C^*-algebras, then this statement is true for all C^*-semi-finite C^*-algebras. If, in addition, this statement is true for all type $\mathfrak{C}$ C^*-algebras, then it is true for all C^*-algebras.

(b) If a statement is true for all discrete C^*-algebras as well as all anti-liminary C^*-algebras and is stable under extensions of C^*-algebras, then this statement is true for all C^*-algebras.

The following results follows from Theorem 3.9(a).

Corollary 5.7. *If A and B are strongly Morita equivalent, then the closed ideal of B that corresponds to $J^A_{\mathfrak{A}}$ (respectively, $J^A_{\mathfrak{B}}$, $J^A_{\mathfrak{C}}$ and $J^A_{\mathfrak{sf}}$) under the strong Morita equivalence (see the paragraph preceding Theorem 3.9) is precisely $J^B_{\mathfrak{A}}$ (respectively, $J^B_{\mathfrak{B}}$, $J^B_{\mathfrak{C}}$ and $J^B_{\mathfrak{sf}}$).*

Remark 5.8. It is natural to ask if the closure $\overline{\mathcal{C}(\cdot)}$ of $\mathcal{C}(\cdot)$ (see Proposition 3.5) is also stable under strong Morita equivalence. Unfortunately, it is not the case. Suppose that A is any type I C^*-algebra. Then by [6, Theorems 1.8 and 2.2], there is a commutative C^*-algebra B that is strongly Morita equivalent to A. Notice that $\mathcal{C}(B) = B$ and $\overline{\mathcal{C}(A)}$ is of type I_0 (by [30, Proposition 6.1.7]). Thus, if $\overline{\mathcal{C}(\cdot)}$ is stable under strong Morita equivalence, then any type I C^*-algebra A will coincide with $\overline{\mathcal{C}(A)}$ and hence is liminary (see e.g. [30, Corollary 6.1.6]), which is absurd.

In the remainder of the section, we compare J^A_* with $J^{M(A)}_*$.

Proposition 5.9. *(a) If $B \subseteq A$ is a hereditary C^*-subalgebra, then $J^B_{\mathfrak{A}} = J^A_{\mathfrak{A}} \cap B$, $J^B_{\mathfrak{B}} = J^A_{\mathfrak{B}} \cap B$, $J^B_{\mathfrak{C}} = J^A_{\mathfrak{C}} \cap B$ and $J^B_{\mathfrak{sf}} = J^A_{\mathfrak{sf}} \cap B$.*

(b) $J^{M(A)}_{\mathfrak{A}} = \{x \in M(A) : xA \subseteq J^A_{\mathfrak{A}}\}$. Similar statements hold for $J_{\mathfrak{B}}$, $J_{\mathfrak{C}}$ and $J_{\mathfrak{sf}}$.

(c) $J^{M(A)}_{\mathfrak{B}} = \{x \in M(A) : xJ^A_{\mathfrak{A}} = (0)$ and $xA \subseteq J^A_{\mathfrak{sf}}\}$

(d) $J^{M(A)}_{\mathfrak{C}} = \{x \in M(A) : xJ^A_{\mathfrak{sf}} = (0)\} = \{x \in M(A) : xJ^A_{\mathfrak{A}} = \{0\}$ and $xJ^A_{\mathfrak{B}} = (0)\}$.

Proof: (a) Note that $J^B_{\mathfrak{A}} \subseteq B \cap J^A_{\mathfrak{A}}$ by Theorem 5.2(a). Conversely, since $B \cap J^A_{\mathfrak{A}}$ is a type $\mathfrak{A}$ closed ideal of B (by Corollary 3.15(a)), we have $B \cap J^A_{\mathfrak{A}} \subseteq J^B_{\mathfrak{A}}$. The other cases follow from similar arguments.

(b) We will only consider the case of $J_{\mathfrak{B}}$ (since the other cases follow from similar and easier arguments). Notice that $J^{M(A)}_{\mathfrak{B}} \cdot A = J^{M(A)}_{\mathfrak{B}} \cap A = J^A_{\mathfrak{B}}$ (by part (a)) and

$$J^{M(A)}_{\mathfrak{B}} \ \subseteq \ J_0 \ := \ \{x \in M(A) : xA \subseteq J^A_{\mathfrak{B}}\}.$$

Suppose that the closed ideal $J_0 \subseteq M(A)$ contains a non-zero abelian hereditary C^*-subalgebra B. The abelian hereditary C^*-subalgebra $B \cap A = B \cdot A \cdot B$ is contained in $J^A_{\mathfrak{B}}$ and so, $B \cdot A = (0)$, which contradicts the fact that A is essential in $M(A)$ (see Remark 3.2(d)). Furthermore, let I be a non-zero closed ideal of J_0. Then $I \cdot A = I \cap A \neq (0)$ and is a closed ideal of $J^A_{\mathfrak{B}}$. Thus, $I \cap A$ contains a non-zero C^*-finite hereditary C^*-subalgebra. Consequently, J_0 is of type $\mathfrak{B}$ and is a subset of $J^{M(A)}_{\mathfrak{B}}$.

(c) By part (b), we know that $xJ^A_{\mathfrak{A}} = (0)$ if and only if $xJ^{M(A)}_{\mathfrak{A}} = (0)$. Thus, this part follows from part (b) and Remark 5.3(c).

(d) The first equality follows from a similar argument as part (c) and the second one follows from Remark 5.3(b). □

Appendix A. Some remarks on classification schemes

In this appendix, we consider other possible classification schemes for C^*-algebras.

A property $\mathcal{P}$ concerning C^*-algebras is said to be *hereditarily stable* if for any C^*-algebra A satisfying $\mathcal{P}$, all hereditary C^*-subalgebras of A will also satisfy $\mathcal{P}$. A sequence $\{\mathcal{P}_1, ..., \mathcal{P}_n\}$ of hereditarily stable properties is said to be *compatible* if $\mathcal{P}_{i-1}$ is stronger than $\mathcal{P}_i$ for $i = 1, ..., n$, where $\mathcal{P}_0$ means "the C^*-algebra is zero".

Let $\{\mathcal{P}_i\}_{i=1,...n}$ be a sequence of compatible hereditarily stable properties. We set $\mathcal{P}_{n+1}$ to be the property: "the C^*-algebra contains zero" (i.e. a tautology), and say that a C^*-algebra is of *type* $\mathcal{T}^{\mathcal{P}}_i$ $(i = 1, ..., n + 1)$ if

> A does not contain a non-zero hereditary C^*-subalgebra with property $\mathcal{P}_{i-1}$ and any non-zero closed ideal of A contains a non-zero hereditary C^*-algebra with property $\mathcal{P}_i$.

Moreover, we set $\mathrm{OP}^{\mathcal{P}}_i(A) := \{e \in \mathrm{OP}(A) : \mathrm{her}(e) \text{ has Property } \mathcal{P}_i\}$.

The arguments for the corresponding results in the main body of this article give the following (note that for part (c), one needs the argument of Proposition 3.8(b)).

Theorem A.1. *Let $\{\mathcal{P}_i\}_{i=1,...n}$ be a sequence of compatible hereditarily stable properties concerning C^*-algebras. Suppose that A is a C^*-algebra and $i \in \{1, ..., n + 1\}$.*

(a) *The sum, $\mathcal{I}_i(A)$, of all hereditary C^*-subalgebras of A with property $\mathcal{P}_i$ is an ideal of A and is the linear span of its cone $\mathcal{I}_i(A) \cap A_+$. If $B \subseteq A$ is a hereditary C^*-subalgebra, then $\mathcal{I}_i(B)_+ = \mathcal{I}_i(A)_+ \cap B$.*
(b) *If A is simple, then A is of type $\mathfrak{T}_j^{\mathcal{P}}$ for exactly one $j = 1, ..., n+1$.*
(c) *If A is strongly Morita equivalent to a C^*-algebra of type $\mathfrak{T}_i^{\mathcal{P}}$, then A is of type $\mathfrak{T}_i^{\mathcal{P}}$.*
(d) *If every non-zero closed ideal of A contains a non-zero hereditary C^*-subalgebra with property $\mathcal{P}_i$, then every non-zero hereditary C^*-subalgebra of A contains a non-zero hereditary C^*-algebra with property $\mathcal{P}_i$.*
(e) *If A is a hereditary C^*-subalgebra of a C^*-algebra of type $\mathfrak{T}_i^{\mathcal{P}}$, then A is of type $\mathfrak{T}_i^{\mathcal{P}}$.*
(f) *If A contains an essential hereditary C^*-subalgebra of type $\mathfrak{T}_i^{\mathcal{P}}$, then A is of type $\mathfrak{T}_i^{\mathcal{P}}$. Consequently, A is of type $\mathfrak{T}_i^{\mathcal{P}}$ if and only if $M(A)$ is of type $\mathfrak{T}_i^{\mathcal{P}}$ (equivalently, the unital C^*-subalgebra of $M(A)$ generated by A is of type $\mathfrak{T}_i^{\mathcal{P}}$).*
(g) *If A has real rank zero, then A is of $\mathfrak{T}_i^{\mathcal{P}}$ if and only if $\mathrm{OP}_{i-1}^{\mathcal{P}}(A) \cap \mathrm{Proj}(A) = \{0\}$ and any element in $\mathrm{Proj}(A) \setminus \{0\}$ dominates an element in $\mathrm{OP}_i^{\mathcal{P}}(A) \cap \mathrm{Proj}(A) \setminus \{0\}$.*
(h) *There is a largest type $\mathfrak{T}_i^{\mathcal{P}}$ hereditary C^*-subalgebra $J^A_{\mathfrak{T}_i^{\mathcal{P}}} \subseteq A$, which is an ideal of A. Furthermore, $J^A_{\mathfrak{T}_1^{\mathcal{P}}}, ..., J^A_{\mathfrak{T}_{n+1}^{\mathcal{P}}}$ are mutually disjoint.*
(i) *If $e_{\mathfrak{T}_i^{\mathcal{P}}} \in \mathrm{OP}(A)$ with $J^A_{\mathfrak{T}_i^{\mathcal{P}}} = \mathrm{her}(e_{\mathfrak{T}_i^{\mathcal{P}}})$, then $\overline{\sum_{i=1}^n e_{\mathfrak{T}_i^{\mathcal{P}}}}^1 + e_{\mathfrak{T}_{n+1}^{\mathcal{P}}} = 1$, and $J^A_{\mathfrak{T}_1^{\mathcal{P}}} + ... + J^A_{\mathfrak{T}_{n+1}^{\mathcal{P}}}$ is an essential closed ideal of A.*
(j) *Strong Morita equivalence respects $J^A_{\mathfrak{T}_i^{\mathcal{P}}}$.*
(k) *If every non-zero closed ideal of A contains a non-zero hereditary C^*-subalgebra having property $\mathcal{P}_i$, then every non-zero closed ideal of $A/J^A_{\mathfrak{T}_i^{\mathcal{P}}}$ contains a non-zero hereditary C^*-subalgebra having property $\mathcal{P}_{i-1}$.*

The above provides many classification schemes for C^*-algebras (with appropriate choices of properties) that could be very different from the one in the main body of this paper.

Note, however, that different choices of properties might give rise to the same classification. We give a brief consideration of this in the following result. We say that two sequences of properties $\{\mathcal{P}_i\}_{i=1,...,n}$ and $\{\mathcal{P}'_i\}_{i=1,...,n}$ are *hereditarily equivalent* if any non-zero C^*-algebra satisfying $\mathcal{P}_i$ contains a non-zero hereditary C^*-subalgebra satisfying $\mathcal{P}'_i$ and vice versa (for each $i \in \{1, ..., n\}$).

Proposition A.2. *Let $\{\mathcal{P}_i\}_{i=1,...,n}$ and $\{\mathcal{P}'_i\}_{i=1,...,n}$ be two sequences of compatible hereditarily stable properties concerning C^*-algebras. Then $\{\mathcal{P}_i\}_{i=1,...,n}$ and $\{\mathcal{P}'_i\}_{i=1,...,n}$ are hereditarily equivalent if and only if type $\mathfrak{T}_i^{\mathcal{P}}$ coincides with type $\mathfrak{T}_i^{\mathcal{P}'}$ for every $i = 1, ..., n+1$.*

Proof: The sufficiency follows from the definitions and we will only establish the necessity. Suppose that type $\mathfrak{T}_i^{\mathcal{P}}$ coincides with type $\mathfrak{T}_i^{\mathcal{P}'}$, for all $i \in \{1, ..., n\}$. If $A \neq (0)$ satisfies $\mathcal{P}_i$, then $J^A_{\mathfrak{T}_k^{\mathcal{P}}} = (0)$ for $k > i$, and there is $j \in \{1, ..., i\}$ with $J^A_{\mathfrak{T}_j^{\mathcal{P}'}} = J^A_{\mathfrak{T}_j^{\mathcal{P}}} \neq (0)$ (by Theorem A.1(i)). Thus, one obtain a non-zero hereditary C^*-subalgebra $B \subseteq A$ satisfying

$\mathcal{P}'_j$ and hence $\mathcal{P}'_i$. By symmetry, $\{\mathcal{P}_i\}_{i=1,\dots,n}$ and $\{\mathcal{P}'_i\}_{i=1,\dots,n}$ are hereditarily equivalent. $\square$

If we keep $\mathcal{P}_1$ as in the main body of this paper (namely, it stands for "the C^*-algebra is commutative") and twist the definition of C^*-finiteness (or $\mathcal{P}_2$), we will obtain another classification, which might or might not be the same as the one in the main body. We will discuss two such variants in the following. Notice that the first one is weaker than C^*-finiteness while the second one is stronger than C^*-finiteness.

(I) The first variant is given by replacing our spatial equivalence $\sim_{\rm sp}$ with the stronger equivalence relation as defined in [31, Definition 1.1] (see Relations (2.1)).

More precisely, we say that a C^*-algebra A satisfies $\mathcal{P}_2^{(1)}$ if for any $q \in \mathrm{OP}(A)$ and $r \in \mathrm{OP}(\mathrm{her}(q))$ with $r \sim_{\rm PZ} q$, one has $\bar{r}^q = q$. In order for $\mathcal{P}_2^{(1)}$ to be unambiguous, we need to show that $\sim_{\rm PZ}$ is "hereditarily invariant", in the sense that if $B \subseteq A$ is a hereditary C^*-subalgebra and $p, q \in \mathrm{OP}(B)$, then $p \sim_{\rm PZ} q$ as elements in $\mathrm{OP}(B)$ if and only if $p \sim_{\rm PZ} q$ as elements in $\mathrm{OP}(A)$.

In fact, it is easy to see that the sufficiency of the above holds (because of Remark 2.2(a)). Conversely, suppose that $p \sim_{\rm PZ} q$ as elements in $\mathrm{OP}(A)$, and $u \in A^{**}$ satisfying Relations (2.1). Let $e \in \mathrm{OP}(A)$ with $B = \mathrm{her}(e)$. Since $u^*u, uu^* \in B^{**}$, Remark 2.2(c) tells us that $u \in B^{**} = eA^{**}e$. Thus, if $x \in \mathrm{her}(p)$, then $u^*x \in eA^{**}e \cap A = B$. Similarly, $u\,\mathrm{her}(q) \subseteq B$.

On the other hand, it is not hard to check that everything in the main body of this paper remains valid if one uses $\sim_{\rm PZ}$ instead of $\sim_{\rm sp}$. Furthermore, we have the following "elementwise description" for $\mathcal{P}_2^{(1)}$ in the case of separable C^*-algebras.

Proposition A.3. *(a) If A satisfies $\mathcal{P}_2^{(1)}$, then for any $a \in A_+$ and $x \in \mathrm{her}(a)$ with $x^*x = a$, the right ideal $R := \{y \in \mathrm{her}(a) : x^*y = 0\}$ is zero.*

(b) If A is separable, the converse of the above also holds.

Proof: (a) The statement is clear if $a = 0$ and we assume that $\|a\| = 1$. Let $q, r \in \mathrm{OP}(A)$ satisfying $\mathrm{her}(q) = \mathrm{her}(a)$ and $\mathrm{her}(r) = \mathrm{her}(x^*) \subseteq \mathrm{her}(a)$. If $x = ua^{1/2}$ is the polar decomposition, then $u^*u = q$, $u\,\mathrm{her}(q) \subseteq \overline{xAa^{1/2}} \subseteq \mathrm{her}(q)$ and $u^*\,\mathrm{her}(r) \subseteq \overline{a^{1/2}Ax^*} \subseteq \mathrm{her}(q)$ (see Example 2.9). Moreover, $\mathrm{her}(r) = u\,\mathrm{her}(q)u^*$ and $\mathrm{her}(q) = u^*\,\mathrm{her}(r)u$ (again by Example 2.9). Consequently, $r \sim_{\rm PZ} q$. If $R \neq (0)$, then $B := R \cap R^*$ is a non-zero hereditary C^*-subalgebra of $\mathrm{her}(a)$ with $\mathrm{her}(x^*) \cdot B = \{0\}$. Thus, r is not dense in q which contradicts the hypothesis.

(b) Suppose on contrary that there exist $q \in \mathrm{OP}(A)$ and $r \in \mathrm{OP}(\mathrm{her}(q))$ with $r \sim_{\rm PZ} q$ but $\bar{r}^q \lneq q$. The separability of A gives $a, b \in \mathrm{her}(q)_+$ such that $\|a\| = \|b\| = 1$, $\mathrm{her}(q) = \mathrm{her}(a)$ and $\mathrm{her}(r) = \mathrm{her}(b)$ (see e.g. [25, Theorem 3.2.5]). Since $r \sim_{\rm PZ} q$ as elements in $\mathrm{OP}(\mathrm{her}(a))$, there exists $x \in \mathrm{her}(a)$ with $x^*x = a$ and $xx^* = b$ (by [28, Proposition 4.3]). As $\bar{r}^q \lneq q$, there is a non-zero hereditary C^*-subalgebra

$B \subseteq \mathrm{her}(q)$ with $\mathrm{her}(r) \cdot B = \{0\}$. Thus, if $y \in B \setminus \{0\}$, then $b^{1/2}y = 0$, which implies that $x^*y = 0$. Consequently, we have a contradiction that $R \neq (0)$. □

As a finial remark of this first variant, one might also replaces our spatial equivalence $\sim_{\mathrm{sp}}$ with the "Cuntz equivalence" $\sim_{\mathrm{Cu}}$ as defined by Ortega, Rørdam and Thiel (see [28, Definition 3.9]). Note that by [28, Corollary 5.9], the resulting property is stronger than $\mathcal{P}_2^{(1)}$.

(II) The second variant is that $\mathcal{P}_2^{(2)}$ stands for "the C^*-algebra is finite (in the sense of [14])". Notice that any abelian C^*-algebra is finite, and any C^*-subalgebra of a finite C^*-algebra is again finite. Thus, if $\mathcal{P}_1^{(2)}$ coincides with $\mathcal{P}_1$ as in the above, then $\{\mathcal{P}_1^{(2)}, \mathcal{P}_2^{(2)}\}$ is a compatible sequence of hereditarily stable properties concerning C^*-algebras.

By Corollary 4.8(a) and Theorem A.1(d), we see that $\mathcal{T}_2^{\mathcal{P}^{(2)}}$ is the same as type II (in the sense of Cuntz and Pedersen). Moreover, by the argument of Corollary 4.8(a), we know that $\mathcal{T}_3^{\mathcal{P}^{(2)}}$ is the same as type III (in the sense of Cuntz and Pedersen). Note, however, that type $\mathcal{T}_1^{\mathcal{P}^{(2)}}$ coincides with discreteness (in the sense of Peligrad and Zsidó) instead of type I (see Example 4.2). Furthermore, Corollary 4.8(b) tells us that type $\mathcal{T}_2^{\mathcal{P}^{(2)}}$ is stronger than type $\mathfrak{B}$.

On the other hand, it is clear from [14, Theorem 3.4] that a von Neumann algebra is finite as a von Neumann algebra if and only if it is finite in the sense of [14]. Let us restate this, together with some statements in Proposition A.1, in the following result.

Corollary A.4. *Suppose that A is a C^*-algebra and M is a von Neumann algebra.*
(a) M is a type I (respectively, type II, type III or semi-finite) von Neumann algebra if and only if M is a discrete (respectively, type II, type III or semi-finite) C^*-algebra.
(b) If A is strongly Morita equivalent to a discrete (respectively, type II, type III or semi-finite) C^*-algebra, then A has the same property.
(c) If A is a hereditary C^*-subalgebra of a discrete (respectively, type II, type III or semi-finite) C^*-algebra, then A also has the same property.
(d) If A contains an essential hereditary C^*-subalgebra that is discrete (respectively, of type II, of type III or semi-finite), then A also has the same property. Consequently, A is discrete (respectively, of type II, of type III or semi-finite) if and only if $M(A)$ has the same property.
(e) If A has real rank zero, then A is of type II (respectively, of type III) if and only if for each $p \in \mathrm{Proj}(A) \setminus \{0\}$, the C^*-algebra pAp is not commutative but p dominates a projection $q \in \mathrm{Proj}(A) \setminus \{0\}$ with qAq being finite (respectively, pAp is not finite).

(f) *The sum of the largest discrete closed ideal, the largest type II closed ideal and the largest type III closed ideal of A (all of them exist) is essential in A.*
(g) *The quotient of A by its largest type III closed ideal is semi-finite. Moreover, if A is semi-finite, then the quotient of A by its largest type II closed ideal is discrete.*

Let us end this appendix with the following questions:

Q1. Is every C^*-algebra satisfying $\mathcal{P}_2^{(1)}$ contains a non-zero finite element?

Q2. Is every C^*-finite C^*-algebra contains a non-zero finite element?

Clearly, a positive answer to Q1 will give a positive answer to Q2. Moreover, by Proposition A.2 and the argument of Corollary 4.8(a), a positive answer to Q1 will imply "type $\mathcal{T}_2^{\mathcal{P}^{(1)}}$ = type $\mathfrak{B}$ = type II" and "type $\mathcal{T}_3^{\mathcal{P}^{(1)}}$ = type $\mathfrak{C}$ = type III". On the other hand, a positive answer to Q2 will imply "type $\mathfrak{B}$ = type II" and "type $\mathfrak{C}$ = type III". In any of these cases, Proposition 4.13 tells us that Conjecture 4.12 holds.

References

[1] C. A. Akemann, The General Stone-Weierstrass problem, J. Funct. Anal., **4** (1969), 277-294.
[2] C. A. Akemann, Left ideal structure of C^*-algebras, J. Funct. Anal., **6** (1970), 305-317.
[3] C. A. Akemann and S. Eilers, Regularity of projections revisited, J. Oper. Theory, **48** (2002), 515-534.
[4] C. A. Akemann and G. K. Pedersen, Complications of semicontinuity in C^*-algebra theory, Duke Math. J., **40** (1973), 785-795.
[5] C. A. Akemann, G. K. Pedersen and J. Tomiyama, Multipliers of C^*-algebras, J. Funct. Anal., **13** (1973), 277-301.
[6] W. Beer, On Morita equivalence of nuclear C^*-algebras, J. Pure Appl. Alg., **26** (1982), 249-267.
[7] M. Bekka, M. Cowling and P. de la Harpe, Some groups whose reduced C^*-aglebra is simple, Publ. Math. I.H.E.S., **80** (1994), 117-134.
[8] B. Blackadar, Operator algebras - theory of C^*-algebras and von Neumann algebras, in *Operator Algebras and Non-commutative Geometry III*, Encyc. Math. Sci., **122**, Springer-Verlag, Berlin (2006).
[9] L. G. Brown, Semicontinuity and multipliers of C^*-algebras, Canad. J. Math., **XL, 4** (1988), 865-988.
[10] L. G. Brown, Determination of A from $M(A)$ and related matters, C. R. Math. Rep. Acad. Sci. Canada, **10** (1988), 273-278.
[11] L. G. Brown and M. A. Rieffel, Stable isomorphism and strong Morita equivalence of C^*-algebras, Pacific J. Math., **71** (1977), 349-363.
[12] L. G. Brown and G. K. Pedersen, C^*-algebras of real rank zero, J. Funct. Anal., **99** (1991), 131-149.
[13] J. Cuntz, K-theory for certain C^*-algebras, Ann. of Math., **113** (1981), 181-197.
[14] J. Cuntz and G. K. Pedersen, Equivalence and traces on C^*-algebras, J. Funct. Anal., **33** (1979), 135-164.
[15] E. G. Effros, Order ideals in C^*-algebras and its dual, Duke Math. J., **30** (1963), 391-412.
[16] G. A. Elliott, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, J. Alg. **38** (1976), 29-44.
[17] G. A. Elliott, On the classification of C^*-algebras of real rank zero, J. Reine Angew. Math., **443** (1993), 179-219.
[18] G. A. Elliott and A. S. Toms, Regularity properties in the classification program for separable amenable C^*-algebras, Bull. Amer. Math. Soc., **45** (2008), 229 - 245.

[19] H. Halpern, V. Kaftal, P. W. Ng and S. Zhang, Finite sums of projections in von Neumann algebras, preprint (arXiv:1007.4679v1).

[20] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras Vol. II: Advanced theory*, Pure and Applied Mathematics **100**, Academic Press (1986).

[21] E. Kirchberg and M. Rørdam, Non-simple purely infinite C^*-algebras, Amer. J. Math., **122** (2000), 637-666.

[22] E. C. Lance, *Hilbert C^*-modules - A toolkit for operator algebraists*, Lond. Math. Soc. Lect. Note Ser. **210**, Camb. Univ. Press (1995).

[23] H. Lin, Equivalent open projections and corresponding hereditary C^*-subalgebras, J. Lond. Math. Soc., **41** (1990), 295-301.

[24] H. Lin and S. Zhang, On infinite simple C^*-algebras, J. Funct. Anal., **100** (1991), 221-231.

[25] G.J. Murphy, *C^*-algebras and operator theory*, Academic Press (1990).

[26] F. J. Murray, The rings of operators papers, in *The legacy of John von Neumann* (Hempstead, NY, 1988), 57-60, Proc. Sympos. Pure Math. **50**, Amer. Math. Soc., Providence, R.I. (1990).

[27] F. J. Murray and J. Von Neumann, On rings of operators, Ann. of Math. (2), **37** (1936), 116-229.

[28] E. Ortega, M. Rørdam and H. Thiel, The Cuntz semigroup and comparison of open projections, J. Funct. Anal., **260** (2011), 3474-3493.

[29] G. K. Pedersen, Applications of weak-*-semicontinuity in C^*-algebra theory, Duke Math. J., **39** (1972), 431-450.

[30] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press (1979).

[31] C. Peligrad and L. Zsidó, Open projections of C^*-algebras: Comparison and Regularity, *Operator Theoretical Methods, 17th Int. Conf. on Operator Theory, Timisoara (Romania), June 23-26, 1998*, Theta Found. Bucharest (2000), 285-300

[32] R. T. Prosser, On the ideal structure of operator algebras, Memoirs Amer. Math. Soc., **45** (1963).

[33] M. A. Rieffel, Morita equivalence for C^*-algebras and W^*-algebras, J. Pure Appl. Alg., **5** (1974), 51-96.

[34] M. A. Rieffel, Morita equivalence for operator algebras, in *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, Proc. Sympos. Pure Math. **38**, Amer. Math. Soc., Providence, R.I. (1982), 285 - 298.

[35] M. Rørdam, Classification of nuclear, simple C^*-algebras, in *Classification of nuclear C^*-algebras, Entropy in operator algebras*, Encyclopaedia Math. Sci. **126**, Springer, Berlin (2002), 1-145.

[36] M. Takesaki, *Theory of Operator algebras I*, Springer-Verlag New York (1979).

[37] A. S. Toms, On the classification problem for nuclear C^*-algebras, Ann. of Math. (2), **167** (2008), 1029-1044.

[38] S. Zhang, Stable isomorphism of hereditary C^*-subalgebras and stable equivalence of open projections, Proc. Amer. Math. Soc., **105** (1989), 677-682.

[39] S. Zhang, Certain C^*-algebras with real rank zero and their corona and multiplier algebras part I, Pac. J. Math. **155** (1992), 169 - 197.

(Chi-Keung Ng) Chern Institute of Mathematics, Nankai University, Tianjin 300071, China.

E-mail address: ckng@nankai.edu.cn

(Ngai-Ching Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan.

E-mail address: wong@math.nsysu.edu.tw