INTO ISOMETRIES OF $C_0(X, E)$'S

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Abstract. Suppose X and Y are locally compact Hausdorff spaces, E and F are Banach spaces and F is strictly convex. We show that every linear isometry T from $C_0(X, E)$ *into* $C_0(Y, F)$ is essentially a weighted composition operator $Tf(y) = h(y)(f(\varphi(y)))$.

Let X and Y be locally compact Hausdorff spaces, E and F Banach spaces and $C_0(X, E)$ and $C_0(Y, F)$ the Banach spaces of continuous E-valued and F-valued functions defined on X and Y vanishing at infinity, respectively. Recall that a Banach space E is said to be *strictly convex* if every norm one element of E is an extreme point of the closed unit ball U_E of E. In [6], Jerison gave a vector version of Banach-Stone Theorem: If X and Y are compact Hausdorff spaces and E is a strictly convex Banach space then every *surjective* isometry T from $C(X, E)$ $(= C_0(X, E))$ onto $C(Y, E)$ $(= C_0(Y, E))$ can be written as a *weighted composition operator*, i.e, $Tf(y) = h(y)(f(\varphi(y)))$, $\forall f \in C(X, E)$, $\forall y \in Y$, where φ is a homeomorphism from Y onto X and h is a continuous map from Y into the space $(B(E, E),\text{SOT})$ of bounded linear operators from E into E equipped with the strong operator topology (SOT) such that $h(y)$ is an isometrically isomorphism from E onto E for all y in Y. After then several generalizations of Banach-Stone Theorem in this direction have appeared (see, for example, [1]). We shall show in this note:

Theorem 1. *Suppose X and Y are locally compact Hausdorff spaces, E and F are Banach spaces, and F is strictly convex. Let T be an* into *linear isometry from $C_0(X, E)$ into $C_0(Y, F)$. Then there exist a continuous function φ from a subset Y_1 of Y onto X and a continuous map h from Y_1 into $(B(E, F),\text{SOT})$ such that for all f in $C_0(X, E)$,*

$$Tf(y) = h(y)(f(\varphi(y))), \quad \forall y \in Y_1.$$

Moreover, $\|h(y)\| = 1$, $\forall y \in Y_1$, and for each e in E and x in X,

$$\sup\{\|h(y)e\| : y \in Y_1 \text{ and } \varphi(y) = x\} = \|e\|.$$

Consequently,

$$\|Tf\| = \|f\| = \|Tf_{|Y_1}\|_\infty \stackrel{\text{def}}{=} \sup_{y_1 \in Y_1} \|Tf(y_1)\|.$$

It is easy to see that Jerison's result [6] is a corollary of Theorem 1. As indicated in [2], there is a counter-example in which the conclusion of Theorem 1 (in fact, even the one of Jerison

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[6]) does not hold while the assumption on strict convexity is not observed. When X and Y are compact Hausdorff spaces, Theorem 1 reduces to a result of Cambern [3]. It is plausible to think that Theorem 1 could be easily obtained from its compact space version [3] by simply extending an into isometry $T : C_0(X,E) \longrightarrow C_0(Y,F)$ to an isometry from $C(X_\infty, E)$ into $C(Y_\infty, F)$ where X_∞ (resp. Y_∞) is the one-point compactification of X (resp. Y). However, an example in [5, Example 9] indicates that even in the simplest case $E = F = \mathbb{R}$ there is an isometry from $C_0(X)$ into $C_0(Y)$ which cannot be extended to an isometry from $C(X_\infty)$ into $C(Y_\infty)$. Thus Theorem 1 cannot be obtained from the statement of the compact space version directly. It is, however, possible to modify the argument in [3] to get a proof of Theorem 1. The key is "$F(x) = 0$ implies $AF(y) = 0$", in Cambern's notation [3, Lemma 2], which allows to define "$A_y(e) = AF(y)$" where F is any function with $F(x) = e$. Instead of going through the reasoning of Cambern once again, we present in the following an alternative approach based on the use of point evaluation type functionals. The technique of the proof we utilize here is influenced by those used in the scalar version as appeared in [7] and [4].

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Proof of Theorem 1. For a Banach space M, we denote by $U_M = \{m \in M : \|m\| \leq 1\}$ the closed unit ball, $S_M = \{m \in M : \|m\| = 1\}$ the unit sphere, and M^* the Banach dual space of M, respectively. For x in X, y in Y, ν in S_{E^*} and μ in S_{F^*}, we set

$$S_{x,\nu} = \{f \in C_0(X,E) : \nu(f(x)) = \|f\| = 1\},$$

$$R_{y,\mu} = \{g \in C_0(Y,F) : \mu(g(y)) = \|g\| = 1\},$$

$$Q_{x,\nu} = \begin{cases} \{y \in Y : T(S_{x,\nu}) \subset R_{y,\mu} \text{ for some } \mu \text{ in } S_{F^*}\}, & \text{if } S_{x,\nu} \neq \emptyset, \\ \emptyset, & \text{if } S_{x,\mu} = \emptyset, \end{cases}$$

and

$$Q_x = \bigcup_{\nu \in S_{E^*}} Q_{x,\nu}.$$

CLAIM 1. $Q_x \neq \emptyset$ for all x in X.

Note that the product space $Y \times U_{F^*}$ is a locally compact Hausdorff space. Define a linear isometry Ψ from $C_0(Y,F)$ into $C_0(Y \times U_{F^*})$ by

$$\Psi(g)(y,\mu) = \mu(g(y)).$$

Fix an e in S_E and then a ν in S_{E^*} such that $\nu(e) = \|e\| = 1$. Then $S_{x,\nu} \neq \emptyset$, $\forall x \in X$. It now suffices to show that

$$\bigcap_{f \in S_{x,\nu}} (\Psi(Tf))^{-1}\{1\} \neq \emptyset, \quad \forall x \in X.$$

For each x in X, consider $f_1, \ldots, f_n$ in $S_{x,\nu}$. Let $h = \sum_{i=1}^n f_i$. We have $\|h\| = n$ and thus there is a y in Y such that $\|Th(y)\| = n$. So a μ in S_{F^*} exists such that $n = \mu(Th(y)) =$

$\sum_{i=1}^n \mu(Tf_i(y))$. It then follows from $\|Tf_i(y)\| \leq 1$, $i = 1,\ldots,n$, that $\mu(Tf_i(y)) = 1$, $i = 1,\ldots,n$, and thus

$$(y,\mu) \in \bigcap_{i=1}^{n}(\Psi(Tf_i)^{-1}(\{1\}) \neq \emptyset.$$

In other words, the family $\{(\Psi(Tf))^{-1}(\{1\}) : f \in S_{x,\nu}\}$ of compact sets has the finite intersection property. Consequently, $Q_x \neq \emptyset$.

Claim 2. $Q_{x_1} \bigcap Q_{x_2} = \emptyset$ if $x_1 \neq x_2$.

Suppose on the contrary the existence of an y in $Q_{x_1} \bigcap Q_{x_2}$. Then there exist ν_1 and ν_2 in S_{E^*} and μ_1 and μ_2 in S_{F^*} such that

$$\mu_1(Tf(y)) = \nu_1(f(x_1)) = 1, \quad \forall f \in S_{x_1,\nu_1}$$

and

$$\mu_2(Tg(y)) = \nu_2(g(x_2)) = 1, \quad \forall g \in S_{x_2,\nu_2}.$$

Let U_1 and U_2 be disjoint neighborhoods of x_1 and x_2, respectively. Choose f_1 in S_{x_1,ν_1} and f_2 in S_{x_2,ν_2} such that f_i is supported in U_i, $i = 1,2$. Then $\|f_1 \pm f_2\| = 1$ implies $\|T(f_1 \pm f_2)(y)\| \leq 1$. In fact, the inequalities $2 = 2\|Tf_1(y)\| = \|T(f_1+f_2)(y) + T(f_1-f_2)(y)\| \leq \|T(f_1+f_2)(y)\| + \|T(f_1-f_2)(y)\|$ ensure that $\|T(f_1 \pm f_2)(y)\| = 1$. By the strict convexity of F, we have $T(f_1+f_2)(y) = T(f_1-f_2)(y)$, and thus a contraction that $Tf_2(y) = 0$!

Let $Y_1 = \bigcup_{x\in X} Q_x$. Define $\varphi : Y_1 \to X$ such that $\varphi(y) = x$ if $y \in Q_x$. For an f in $C_0(X,E)$, we denote $\operatorname{coz} f = \{x \in X : f(x) \neq 0\}$ and $\operatorname{supp} f$ the closure of $\operatorname{coz} f$ in X. An argument similar to that in the proof of Claim 2 will give

Claim 3. For each f in $C_0(X,E)$, $\varphi(y) \notin \operatorname{supp} f$ implies $Tf(y) = 0$.

Claim 4. $h(y)$ is well-defined and $\|h(y)\| = 1$ for all y in Y_1.

For each y in Y_1, let

$$J_y = \{f \in C_0(X,E) : \varphi(y) \notin \operatorname{supp} f\}$$

and

$$K_y = \{f \in C_0(X,E) : f(\varphi(y)) = 0\}.$$

It is not hard to see that J_y is dense in K_y. For x in X (resp. y in Y), let δ_x (resp. δ_y) be the point evaluation map $\delta_x(f) = f(x)$ (resp. $\delta_y(g) = g(y)$) of $C_0(X,E)$ (resp. $C_0(Y,F)$). By Claim 3, $J_y \subset \ker(\delta_y \circ T)$ and thus $\ker(\delta_{\varphi(y)}) = K_y \subset \ker(\delta_y \circ T)$. Hence there exists a linear operator $h(y)$ from E into F such that

$$\delta_y \circ T = h(y) \circ \delta_{\varphi(y)}.$$

In other words, for all f in $C_0(X,E)$,

$$Tf(y) = h(y)(f(\varphi(y))).$$

For any e in E, choose an f in $C_0(X,E)$ such that $f(\varphi(y)) = e$ and $\|f\| = \|e\|$. Since $\|h(y)e\| = \|h(y)(f(\varphi(y)))\| = \|Tf(y)\| \leq \|Tf\| = \|f\| = \|e\|$, we conclude that $\|h(y)\| \leq 1$. In fact, it follows from the definition of Y_1 that $\|h(y)\| = 1$, $\forall y \in Y_1$.

The assertion that for each e in E and x in X, $\sup\{\|h(y)e\| : y \in Y_1 \text{ and } \varphi(y) = x\} = \|e\|$ is obvious if we pay attention to functions in the form of $f(w) = g(w)e$ where g is a non-negative continuous function on X vanishing at infinity with maximum value $g(x) = 1$. Consequently, the norm identities $\|Tf\| = \|f\| = \|Tf_{|Y_1}\|_\infty$ are established.

Claim 5. φ is continuous from Y_1 onto X.

Let $\{y_\lambda\}$ be a net convergent to y in Y_1. If $\{\varphi(y_\lambda)\}$ does not converge to $\varphi(y)$, by passing to a subnet if necessary, we assume it converges to an x in $X_\infty = X \bigcup\{\infty\}$, the one-point compactification of X. Let U_1 and U_2 be disjoint neighborhoods of x and $\varphi(y)$ in X_∞, respectively. There exists a λ_0 such that $\varphi(y_\lambda) \in U_1$, $\forall\lambda \geq \lambda_0$, and an f in $C_0(X,E)$ such that $\operatorname{coz} f \subset U_2$ and $\|Tf(y)\| \neq 0$. For $\lambda \geq \lambda_0$, $\varphi(y_\lambda) \notin \operatorname{supp} f$. By Claim 3, $Tf(y_\lambda) = 0$, $\forall y_\lambda \geq \lambda_0$. Thus $\{Tf(y_\lambda)\}$ cannot converge to $Tf(y) \neq 0$, a contradiction. Hence φ is continuous.

Claim 6. $h : Y_1 \to (B(E,F), \mathrm{SOT})$ is continuous.

Let $\{y_\lambda\}$ be a net convergent to y in Y_1. For e in E, f in $C_0(X,E)$ exists such that $f(x) = e$ for all x in a neighborhood of $\varphi(y)$. Since φ is continuous, there is a λ_e such that for all $\lambda \geq \lambda_e$, $\|h(y_\lambda)e - h(y)e\| = \|h(y_\lambda)f(\varphi(y_\lambda)) - h(y)f(\varphi(y))\| = \|Tf(y_\lambda) - Tf(y)\|$. Since $\{Tf(y_\lambda)\}$ converges to $Tf(y)$, the claim is thus verified. The proof is complete. □

To end this note, we would like to remark that Y_1 can be neither open nor closed, and $h(y)$ need not be an isometry in general for y in Y_1 as pointed out by an example in [3].

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