ON THE BANACH–STONE PROBLEM

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Abstract. Let X and Y be locally compact Hausdorff spaces, let E and F be Banach spaces, and let T be a linear isometry from $C_0(X, E)$ *into* $C_0(Y, F)$. We provide in this paper three new answers to the Banach-Stone problem: (1) T can always be written as a generalized weighted composition operator if and only if F is strictly convex; (2) if T is onto then T can be written as a weighted composition operator in a weak sense; and (3) if T is onto and F does not contain a copy of ℓ_2^∞ then T can be written as a weighted composition operator in the classical sense.

1. Introduction

In [18], Jerison got the first vector-valued version of the Banach–Stone Theorem: Suppose X and Y are compact Hausdorff spaces and E is a Banach space. Jerison proved that if E is strictly convex then every linear isometry T from $C(X, E)$ onto $C(Y, E)$ is a *weighted composition operator* $Tf = h \cdot f \circ \varphi$, that is,

$$Tf(y) = h(y)\left(f(\varphi(y))\right), \quad \forall f \in C(X, E), \forall y \in Y,$$

for some continuous map (in fact, homeomorphism) φ from Y onto X and some continuous operator-valued (in fact, onto isometry-valued) map h from Y into $L(E, E)$. In [19], Lau gave another version: Suppose the Banach dual space E^* of E is strictly convex instead. Then every linear isometry from $C(X, E)$ onto $C(Y, E)$ is also a weighted composition operator.

Recall that a Banach space E is *strictly convex* if every vector in the unit sphere S_E of E is an extreme point of the closed unit ball U_E of E. $C_0(X, E)$ denotes the Banach space of continuous vector-valued functions from the locally compact Hausdorff space X into E vanishing at infinity. We write $C(X, E)$ for $C_0(X, E)$ whenever X is compact, as usual. The norm of f in $C_0(X, E)$ is defined to be $\|f\| = \sup\{\|f(x)\| : x \in X\}$. Moreover, the vector space $L(E, F)$ of bounded linear operators from a Banach space E into a Banach space F is always equipped with the strong operator topology (SOT) in this paper.

Recall that a Banach space E is said to have the *Banach-Stone property* if the existence of a linear isometry T from $C_0(X, E)$ onto $C_0(Y, E)$ ensures X and Y being homeomorphic for all locally compact Hausdorff spaces X and Y. We say that E has the *strong Banach-Stone property* if all such T can be written as a weighted composition operator. It is known that $\ell_2^\infty = \mathbb{R} \oplus_\infty \mathbb{R}$

Date: March 24, 2001; a revision submitted to Studia Math.
1991 *Mathematics Subject Classification.* 46B04, 46E40, 46E15.
Key words and phrases. Banach–Stone Theorem, strict convexity, weighted composition operators.
Partially supported by Taiwan National Science Council, Grant: NSC 86-2115-M110-002, and the Chinese Development Funds, Taiwan, R.O.C.
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does not have the Banach-Stone property, while $\mathbb{R} \oplus_\infty (\mathbb{R} \oplus_2 \mathbb{R})$ has the Banach-Stone property but not the strong Banach-Stone property. In fact, every 3-dimensional Banach space has the Banach-Stone property except for $\mathbb{R} \oplus_\infty \mathbb{R} \oplus_\infty \mathbb{R}$ (see e.g. [3, pp. 142-147]). For another example, put $E = C(Q)$, where $Q = [0,1]^\infty$ is the Hilbert cube. Let $X = [0,1]$ and $Y = \{0\}$. Then the spaces $C(X,E)$ and $C(Y,E)$ are isometric while there is no map from Y onto X. In other words, $C(Q)$ does not have the Banach-Stone property.

Definition 1. We say that a Banach space F *solves the Banach–Stone problem* if every linear isometry from $C_0(X,E)$ onto $C_0(Y,F)$ is a weighted composition operator for all locally compact Hausdorff spaces X and Y and Banach space E.

Although some authors mainly deal with the case that $E = F$, their arguments can be modified easily to give us solutions of the Banach-Stone problem. In particular, Jerison's result [18] says that strictly convex Banach spaces solve the Banach–Stone problem while Lau's result [19] says that Banach spaces with strictly convex dual also solve the Banach–Stone problem. However, not every Banach space solve the Banach–Stone problem. As a basic counter example, the 2-dimensional Banach space $\ell_2^\infty = \mathbb{R} \oplus_\infty \mathbb{R}$ does not solve the Banach–Stone problem. In fact, the linear isometry T from $C(\{1,2\},\mathbb{R})$ onto $C(\{0\},\ell_2^\infty)$, defined by

$$Tf(0) = (f(1), f(2)),$$

cannot be written as a weighted composition operator. We note that the inverse T^{-1} of T is a weighted composition operation, however. This tells us that the concept of solving the Banach-Stone problem is a non-symmetric generalization of that of having the strong Banach-Stone property. Clearly, every solution of the Banach-Stone problem has the strong Banach-Stone property. We do not know, however, if the reverse implication is always true.

In general, every Banach space containing non-trivial M-summands does not solve the Banach–Stone problem (see, e.g., [3, p. 149]). Recall that a non-trivial closed subspace E_1 of a Banach space E is called an *M-summand* of E if $E = E_1 \oplus_\infty E_2$ for some closed proper subspace E_2 of E. In [10], Cambern proved that a *reflexive* Banach space E solves the Banach–Stone problem if and only if E does not have any non-trivial M-summand. However, a reflexive space with non-trivial M-summand may still have the Banach-Stone property, for example, $\mathbb{R} \oplus_\infty (\mathbb{R} \oplus_2 \mathbb{R})$. In the non-reflexive case, the Banach–Stone problem is still open. Many counter examples were given since then. See, for instance, [8, 9, 3]. Several attempts to attack the Banach–Stone problem have appeared, to name a few, see [1, 20, 2, 3, 6, 5, 12]. Among them are the methods of T-sets of Jerison [18] and M-structures of Behrends (see, e.g., [3]). These results are proved to be very powerful (cf. [4]).

In this paper, without using any technique of T-sets and M-structures we present three new answers to the Banach–Stone problem: Theorem 3 places the strict convexity in the correct position in solving the Banach-Stone problem. It states that every isometry from $C_0(X,E)$ *into* $C_0(Y,F)$ is a generalized weighted composition operator if and only if F is strictly convex. Theorem 4 says that every Banach space does solve the Banach-Stone problem in a *weak* sense. As a corollary, Theorem 6 supplements a well-known result of Behrends ([3, p. 148]; see also [14]) to give that Banach spaces containing no copy of ℓ_2^∞ solve the Banach-Stone problem. The proofs of these results are modeled on that employed in the scalar-version of Holsztyński [13] and Jarosz [15] (cf. [16]). As applications, we shall derive the classical results of Jerison [18] and Lau [19] (see Corollary 8), and a recent result of Hernandez, Beckenstein and Narici [12] (see Corollary 9) as natural consequences of our Theorems 6 and 4, respectively.

We would like to express our deep thanks to Ka-Sing Lau for sharing us with his conjecture which eventually works out as our Theorem 6, and to K. Jarosz for useful comments on a preliminary version of this paper. We are grateful to the Referee for many helpful hints to make our results more accurate.

2. Three new answers to the Banach-Stone problem

In the following, we always assume X and Y are (non-empty) locally compact Hausdorff spaces and E and F are (non-zero) Banach spaces without any additional structure, unless otherwise stated. We first show that the way to write a linear map from $C_0(X,E)$ into $C_0(Y,F)$ as a weighted composition operator is unique.

Proposition 2. *Let T be a linear map from $C_0(X,E)$ into $C_0(Y,F)$. Suppose there exist a map φ from a non-empty subset Y_0 of Y into X and a non-vanishing map h from Y_0 into $L(E,F)$ such that*

$$Tf(y) = h(y)(f(\varphi(y))), \quad \forall y \in Y_0. \tag{1}$$

Then both of φ and h are continuous. Moreover, if (Y_0', φ', h') is another triple satisfying all above conditions then $\varphi(y) = \varphi'(y)$ and $h(y) = h'(y)$ for all y in $Y_0 \cap Y_0'$.

Proof. We prepare the proof in the following three claims.

Claim 1. $\varphi : Y_0 \longrightarrow X$ is continuous.

Suppose otherwise, and $\{y_\lambda\}$ is a net convergent to y in Y_0 such that $\{\varphi(y_\lambda)\}$ does not converge to $\varphi(y)$. By passing to a subnet if necessary, we can assume $\{\varphi(y_\lambda)\}$ converges to some other x in $X_\infty = X \cup \{\infty\}$, the one-point compactification of X. Let U_1 and U_2 be disjoint neighborhoods of x and $\varphi(y)$ in X_∞, respectively. Then $\varphi(y_\lambda) \in U_1$ eventually. Choose an f in $C_0(X,E)$ such that f vanishes outside U_2 and $h(y)(f(\varphi(y))) \neq 0$. We then have $f(\varphi(y_\lambda)) = 0$ and thus $Tf(y_\lambda) = 0$ for all large λ. As a result, $\{Tf(y_\lambda)\}$ cannot converge to $Tf(y) = h(y)(f(\varphi(y))) \neq 0$, a contradiction.

Claim 2. $h : Y_0 \to (L(E,F), \mathrm{SOT})$ is continuous.

Let $\{y_\lambda\}$ be a net convergent to y in Y_0. For each e in E, choose an f in $C_0(X,E)$ such that $f(x) = e$ for all x in a neighborhood of $\varphi(y)$. Since φ is continuous, $f(\varphi(y_\lambda)) = e$ for all large enough λ. Consequently, $\|h(y_\lambda)e - h(y)e\| = \|h(y_\lambda)\Big(f(\varphi(y_\lambda))\Big) - h(y)\Big(f(\varphi(y))\Big)\| = \|Tf(y_\lambda) - Tf(y)\|$ eventually. Since $\{Tf(y_\lambda)\}$ converges to $Tf(y)$, the claim is verified.

Claim 3. $\varphi = \varphi'$ and $h = h'$ on $Y_0 \cap Y_0'$.

Suppose $\varphi(y) \neq \varphi'(y)$ for some y in $Y_0 \cap Y_0'$. Let $x = \varphi(y)$ and $x' = \varphi'(y)$ in X. Let $f \in C_0(X,E)$ such that $f(x) = 0$ and $h'(y)f(x') \neq 0$. Then $Tf(y) = h(y)f(\varphi(y)) = 0$ and

$Tf(y) = h'(y)f(\varphi'(y)) \neq 0$, a contradiction. Hence, φ and φ' agree on $Y_0 \cap Y_0'$. It follows that h and h' also agree on $Y_0 \cap Y_0'$. □

The family of all triples (Y_0, φ, h) which represent a linear isometry T from $C_0(X, E)$ into $C_0(Y, F)$ partially as a weighted composition operator $Tf_{|Y_0} = h \cdot f \circ \varphi$ is direct in the natural ordering induced by set inclusions. Theorem 3 below ensures that this family is non-trivial if, for example, F is strictly convex. Hence, by taking set-theoretical union of all such triples, there exists *the* greatest subset Y_0 of Y on which T can be written as a weighted composition operator. By saying that a linear isometry T from $C_0(X, E)$ into $C_0(Y, F)$ is a *generalized weighted composition operator*, we mean there are a subset Y_1 of Y, a continuous map φ from Y_1 *onto* X and a continuous operator-valued map h from Y_1 into $(L(E, F), SOT)$ such that $Tf_{|Y_1} = h \cdot f \circ \varphi$ and $\|Tf\| = \|Tf_{|Y_1}\| = \sup\{\|Tf(y)\| : y \in Y_1\}$.

Our first theorem places the strict convexity in its correct position in the context of the Banach–Stone problem. In [11], Cambern provided the implication (1) $\Longrightarrow$ (2) of Theorem 3 below when X and Y are compact Hausdorff spaces. In [17], we extended this implication to the locally compact case.

Theorem 3. *Let F be a real Banach space. The following two conditions are equivalent.*

1. *F is strictly convex.*
2. *For all locally compact Hausdorff spaces X and Y and for all real Banach spaces E, every real linear into isometry T from $C_0(X, E)$ into $C_0(Y, F)$ is a generalized weighted composition operator.*

In case the underlying field $\mathbb{K}$ is the field $\mathbb{C}$ of complex numbers, we still have the implication (1) $\Longrightarrow$ (2).

Proof. Suppose F is strictly convex. No matter the underlying field $\mathbb{K}$ is the real $\mathbb{R}$ or the complex $\mathbb{C}$, we have proved in [17] that every linear isometry T from $C_0(X, E)$ into $C_0(Y, F)$ is a generalized weighted composition operator. For the sake of completeness, we present a sketch of the proof below.

The task is to find a subset Y_1 of Y, a map φ from Y_1 onto X and a map h from Y_1 into $L(E, F)$ such that $Tf_{|Y_1} = h \cdot f \circ \varphi$, $\forall f \in C_0(X, E)$. Denote by S_{E^*} (resp. S_{F^*}) the unit sphere of the dual space of E (resp. F). Let $x \in X$, $y \in Y$, $\mu \in S_{E^*}$ and $\nu \in S_{F^*}$. Consider the sets

$$\begin{aligned} \mathcal{S}_{x,\mu} &= \big\{f \in C_0(X, E) : \mu(f(x)) = \|f\| = 1\big\}, \\ \mathcal{R}_{y,\nu} &= \big\{g \in C_0(Y, F) : \nu(g(y)) = \|g\| = 1\big\}. \end{aligned}$$

$\mathcal{S}_{x,\mu}$ (resp. $\mathcal{R}_{y,\nu}$) can be considered as the norm attaining set of the norm one linear functional $\mu \circ \delta_x$ (resp. $\nu \circ \delta_y$) of $C_0(X, E)$ (resp. $C_0(Y, F)$), where δ_x (resp. δ_y) is the evaluation map at the point x (resp. y). Set

$$\mathcal{Q}_{x,\mu} = \begin{cases} \big\{y \in Y : T(\mathcal{S}_{x,\mu}) \subset \mathcal{R}_{y,\nu} \text{ for some } \nu \text{ in } S_{F^*}\big\}, & \text{if } \mathcal{S}_{x,\mu} \neq \emptyset, \\ \emptyset, & \text{if } \mathcal{S}_{x,\mu} = \emptyset. \end{cases}$$

By a compactness argument, we can show that

$$\mathcal{Q}_{x,\mu} \neq \emptyset \quad \text{whenever} \quad \mathcal{S}_{x,\mu} \neq \emptyset.$$

Since norm attaining linear functionals are dense in the unit sphere S_{E^*} of E^* by the Bishop–Phelps Theorem [7], many $\mathcal{S}_{x,\mu}$ are nonempty. Thus the set

$$\mathcal{Q}_x = \bigcup_{\mu \in S_{E^*}} \mathcal{Q}_{x,\mu} \neq \emptyset$$

for each x in X. Let

$$Y_1 = \bigcup_{x \in X} \mathcal{Q}_x.$$

The strict convexity of F will imply that $\mathcal{Q}_{x_1} \cap \mathcal{Q}_{x_2} = \emptyset$ whenever $x_1 \neq x_2$ in X. This partition defines a map φ from Y_1 onto X such that

$$\varphi(y) = x \quad \text{if} \quad y \in \mathcal{Q}_x.$$

Another key step in the proof is to use the strict convexity of F again to assert that

$$\varphi(y) \notin \operatorname{supp} f \implies Tf(y) = 0, \quad \forall f \in C_0(X, E).$$

From this we have an inclusion $\ker \delta_{\varphi(y)} \subseteq \ker \delta_y \circ T$ by Uryshon's Lemma. It follows that there exists a linear map $h(y)$ from E into F such that $\delta_y \circ T = h(y)\delta_{\varphi(y)}$, or $Tf(y) = h(y)(f(\varphi(y)))$, $\forall f \in C_0(X, E)$, $\forall y \in Y_1$. The continuities of φ and h follow from Proposition 2. It is then easy to see that $\|Tf\| = \|Tf_{|Y_1}\| = \sup\{\|Tf(y)\| : y \in Y_1\}$.

Conversely, we assume that F is not strictly convex. In this case, we also assume that the underlying field is $\mathbb{R}$. We want to find a linear isometry T from $C_0(X, E)$ into $C_0(Y, F)$, which cannot be written as a generalized weighted composition operator. To this end, we set $X = Y = \{1, 2\}$ in the discrete topology. Let $E = \mathbb{R}$. Since F is not strictly convex, there are distinct e_1 and e_2 in the unit sphere S_F of F such that $t_0 e_1 + (1 - t_0)e_2 \in S_F$ for some $0 < t_0 < 1$. In fact, $t\, e_1 + (1 - t)\, e_2$ belongs to S_F for all t in $[0, 1]$. Consequently,

$$\|\alpha\, e_1 + \beta\, e_2\| = \alpha + \beta, \text{ for all } \alpha,\ \beta \geq 0. \tag{2}$$

Represent functions f in $C(X)$ as column vectors $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ in which $f(1) = \alpha$ and $f(2) = \beta$. Let $f_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ in $C(X)$. For each $f = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ in $C(X)$, we can write $f = \frac{\alpha+\beta}{2} f_1 + \frac{\alpha-\beta}{2} f_2$. Define a linear map $T : C(X) \to C(Y, F)$ by $Tf_1 = \begin{pmatrix} e_1 \\ -e_1 \end{pmatrix}$ and $Tf_2 = \begin{pmatrix} e_2 \\ e_2 \end{pmatrix}$ in a similar convention. In other words,

$$T \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha + \beta}{2} \begin{pmatrix} e_1 \\ -e_1 \end{pmatrix} + \frac{\alpha - \beta}{2} \begin{pmatrix} e_2 \\ e_2 \end{pmatrix}.$$

Now we show that T is an isometry. First, assume that $|\alpha| \geq |\beta|$. If $\alpha > 0$, then $\frac{\alpha+\beta}{2} \geq 0$ and $\frac{\alpha-\beta}{2} \geq 0$.

$$\|Tf(1)\| = \|\frac{\alpha + \beta}{2} e_1 + \frac{\alpha - \beta}{2} e_2\| = \frac{\alpha + \beta}{2} + \frac{\alpha - \beta}{2} = \alpha,$$

by (2). Moreover,

$$\|Tf(2)\| \leq \frac{\alpha+\beta}{2}\|e_1\| + \frac{\alpha-\beta}{2}\|e_2\| = \alpha.$$

If $\alpha < 0$, then $\frac{\alpha+\beta}{2} \leq 0$ and $\frac{\alpha-\beta}{2} \leq 0$.

$$\|Tf(1)\| = \|\frac{\alpha+\beta}{2}e_1 + \frac{\alpha-\beta}{2}e_2\| = \|\frac{\alpha+\beta}{-2}e_1 + \frac{\alpha-\beta}{-2}e_2\| = \frac{\alpha+\beta}{-2} + \frac{\alpha-\beta}{-2} = -\alpha,$$

by (2) again. On the other hand,

$$\|Tf(2)\| \leq \frac{\alpha+\beta}{-2}\| - e_1\| + \frac{\alpha-\beta}{-2}\|e_2\| = -\alpha.$$

So $\|Tf\| = \|f\| = |\alpha|$ in both cases. When $|\alpha| < |\beta|$, a similar argument applies and also gives $\|Tf\| = \|f\|$. Hence T is an isometry.

Finally, we show that T is not a generalized weighted composition operator. Suppose T were, and there existed a non-empty subset Y_0 of Y, a continuous map φ from Y_0 into X and a linear map $h(y) : \mathbb{R} \to F$ such that $Tf(y) = h(y)\Big(f(\varphi(y))\Big)$, $\forall f \in C(X), \forall y \in Y_0$. For the case $1 \in Y_0$ and $\varphi(1) = 1$, we have $e_1 = Tf_1(1) = h(1)\Big(f_1(1)\Big) = h(1)(1) = h(1)\Big(f_2(1)\Big) = Tf_2(1) = e_2$, a contradiction. Similar contradictions can be derived for other cases. The proof is thus complete. □

Our second theorem gives a complete answer to the Banach–Stone problem in a *weak* sense. Subject to *no* constraint on X, Y, E, or F, it says that every linear isometry T from $C_0(X,E)$ onto $C_0(Y,F)$ can be written in a weak form of a weighted composition operator. This version of the Banach–Stone Theorem is good enough for many applications. See, for example, Corollaries 8 and 9 below. Before stating it, recall that if $Tf = h \cdot f \circ \varphi$ is a weighted composition operator from $C_0(X,E)$ into $C_0(Y,F)$ then for each bounded linear functional ν of F, we have

$$\nu(Tf(y)) = \nu \circ h(y)(f(\varphi(y))), \quad \forall f \in C_0(X,E), \forall y \in Y.$$

In other words, Tf is again an image of a weighted composition operator when it is thought as a function of y *and* ν in $Y \times F^*$. Note that $\nu \circ h(y) \in E^*$.

In the following, U_{F^*} (resp. S_{F^*}) denotes the closed unit ball (resp. unit sphere) of the dual space F^* of F. Since T is a linear isometry, it dual map T^* sends the set of extreme points of the closed dual ball of the range space onto the set of extreme points of $U_{C_0(X,E)^*}$, which contains exactly all functionals of the form $\delta_x \otimes e^*$. Here, δ_x is evaluation at some x in X and μ is an extreme point of U_{E^*}. Note also that every extreme point of the closed dual ball of the range space of T can be extended to an extreme point of $U_{C_0(Y,F)^*}$. Let A_Y be the set of all such extensions. In particular, we can think of $A_Y \subseteq Y \times U_{F^*}$ and T^*A_Y consists of all $\delta_x \otimes \mu$ with x in X and μ being an extreme point of U_{E^*}. Define $\tilde{\varphi}(y,\nu) = x$ on A_Y if $T^*(\delta_y \otimes \nu) = \delta_x \otimes \mu$ for some μ. In this setting, we have

Theorem 4. *Let T be a linear isometry from $C_0(X,E)$ into $C_0(Y,F)$. Then there exist a continuous map $\tilde{\varphi}$ from A_Y onto X, and a weak* continuous map $\tilde{h}$ from A_Y into E^* such that*

$$\nu(Tf(y)) = \tilde{h}(y,\nu)\left(f(\tilde{\varphi}(y,\nu))\right), \quad \forall f \in C_0(X,E), \forall (y,\nu) \in A_Y.$$

In this case, $\|\tilde{h}(y,\nu)\| \equiv 1$ for all (y,ν) in A_Y and $\|Tf\| = \sup\{|\nu(Tf(y))| : (y,\nu) \in A_Y\}$.

Moreover, if T is onto then the set

$$B_y = \{\nu \in S_{F^*} : (y, \nu) \in A_Y\}$$

contains all extreme points of U_{F^*} for each y in Y.

Theorem 4 can be applied to give some Banach-Stone type theorems in the classical sense. The following lemma is a key.

Lemma 5. *Let T be a linear isometry from $C_0(X,E)$ onto $C_0(Y,F)$. Then T is a weighted composition operator $Tf = h \cdot f \circ \varphi$ if and only if $\widetilde{\varphi}(y,\nu_1) = \widetilde{\varphi}(y,\nu_2)$ for all ν_1, ν_2 in B_y and for all y in Y. In this case, we have $\tilde{h}(y,\nu) = \nu \circ h(y)$ and $\widetilde{\varphi}(y,\nu) = \varphi(y)$, $\forall \nu \in B_y$, $\forall y \in Y$.*

Proof. We verify the sufficiency only. Let $\widetilde{\varphi}(y,\nu_1) = \widetilde{\varphi}(y,\nu_2)$, $\forall \nu_1, \nu_2 \in B_y$ We can define an onto map $\varphi : Y \to X$ by $\varphi(y) = \widetilde{\varphi}(y,\nu)$ for any ν in B_y. If $f(\varphi(y)) = 0$, then $\nu(Tf(y)) = \widetilde{h}(y,\nu)(f(\varphi(y))) = 0$, $\forall \nu \in B_y$. Since B_y is total, $Tf(y) = 0$. As a result, $\ker \delta_{\varphi(y)} \subseteq \ker \delta_y \circ T$. It follows that there exists a linear map $h(y) : E \to F$ such that $Tf(y) = h(y)(f(\varphi(y)))$, $\forall f \in C_0(X,E), \forall y \in Y$. The continuities of φ and h follow from Proposition 2. □

We are now ready to provide an answer to the Banach–Stone problem in the classical sense. Recall that $\ell_2^\infty = \mathbb{R} \oplus_\infty \mathbb{R}$ does not solve the Banach–Stone problem. We say that a (real or complex) Banach space F *does not contain a copy of* ℓ_2^∞ if there is no *real* linear isometric embedding of ℓ_2^∞ into F. It is easy to see that $\ell_2^\infty = \mathbb{R} \oplus_\infty \mathbb{R}$ is real linear isometrically isomorphic to $\ell_2^1 = \mathbb{R} \oplus_1 \mathbb{R}$ since their unit balls are both squares. Consequently, F does not contain a copy of ℓ_2^∞ if and only if at least one of the norms $\|e_1 \pm e_2\| < 2$ whenever $\|e_1\| = \|e_2\| = 1$; for else the linear span of $\{e_1, e_2\}$ will be a copy of ℓ_2^1 ($\cong \ell_2^\infty$). For comparison, F is strictly convex if and only if *both* of the norms $\|e_1 \pm e_2\| < 2$ whenever $\|e_1\| = \|e_2\| = 1$.

Theorem 6. *Let X and Y be locally compact Hausdorff spaces and let E and F be Banach spaces. Suppose F does not contain a copy of ℓ_2^∞. Then every linear isometry T from $C_0(X,E)$ onto $C_0(Y,F)$ is a weighted composition operator*

$$Tf(y) = h(y)\left(f(\varphi(y))\right), \quad \forall f \in C_0(X,E), \forall y \in Y,$$

for some continuous map φ from Y onto X and continuous map h from Y into $(L(E,F), SOT)$.

Proof. We have to verify the condition stated in Lemma 5. Suppose on the contrary that there existed ν_1 and ν_2 in S_{F^*} such that $\widetilde{\varphi}(y,\nu_1) = x_1 \neq x_2 = \widetilde{\varphi}(y,\nu_2)$. By the definition of $\widetilde{\varphi}$, there exist extreme points μ_1 and μ_2 of U_{E^*} such that $T^*(\delta_y \otimes \nu_1) = \delta_{x_1} \otimes \mu_1$ and $T^*(\delta_y \otimes \nu_2) = \delta_{x_2} \otimes \mu_2$. Let U_1 and U_2 be disjoint neighborhoods of x_1 and x_2, respectively. Choose f_i in $C_0(X,E)$ such that f_i is supported by U_i and $\mu_i(f_i(x_i)) = \|f_i\| = 1$ for $i = 1, 2$. Consequently,

$$\|Tf_1(y)\| = \|Tf_2(y)\| = 1. \tag{3}$$

Moreover, $\|f_1 \pm f_2\| = 1$ implies $\|T(f_1 \pm f_2)(y)\| \leq 1$. In fact, the inequalities

$$2 = 2\|Tf_1(y)\| = \|T(f_1 + f_2)(y) + T(f_1 - f_2)(y)\| \leq \|T(f_1 + f_2)(y)\| + \|T(f_1 - f_2)(y)\| \leq 2$$

ensure that $\|T(f_1 \pm f_2)(y)\| = 1$. Since F does not contain a copy of ℓ_2^∞, we must have at least one of the norms $\|T(f_1 + f_2)(y) \pm T(f_1 - f_2)(y)\| < 2$. But this conflicts with (3). □

Remark 7. When neither E or F contains l_2^∞, Theorem 6 implies that every linear surjective isometry T from $C_0(X,E)$ onto $C_0(Y,F)$ is a weighted composition operator $Tf = h\cdot f\circ\varphi$ such that φ is a homeomorphism from Y onto X. However, a more general statement is known: it is enough to assume that the set of centralizers of E and F are both trivial (see e.g. [3, pp. 147-148]). In fact, even spaces with just non-trivial multiplier spaces contain ℓ_2^∞. See K. Jarosz [14] for details.

We remark that Theorem 6 is still not optimum for the Banach-Stone problem. For example, the Banach space $F = \mathbb{R}\oplus_1(\mathbb{R}\oplus_2\mathbb{R})$ does contain a copy of ℓ_2^1 ($\cong \ell_2^\infty$). Since F is reflexive and contains no non-trivial M-summand, by a theorem of Cambern [10], F solves the Banach–Stone problem. Nevertheless, Theorem 6 does include some famous solutions of the Banach-Stone problem.

Corollary 8 (Jerison [18] and Lau [19])**.** *Let X and Y be locally compact Hausdorff spaces and let E and F be Banach spaces. Suppose F or its Banach dual F^* is strictly convex. Then every linear isometry T from $C_0(X,E)$ onto $C_0(Y,F)$ is a weighted composition operator $Tf = h\cdot f\circ\varphi$. In case E or its Banach dual E^* is also strictly convex, φ is a homeomorphism from Y onto X and $h(y)$ is a linear isometry from E onto F for all y in Y.*

Proof. We claim that a Banach space F does not contain a copy of ℓ_2^∞ whenever F or its dual F^* is not strictly convex. In fact, suppose F contains a copy of ℓ_2^∞. Then it is plain that F cannot be strictly convex. At the same time, the Banach dual F^* of F contains a copy of ℓ_2^1. Thus F^* cannot be strictly convex, either. The desired assertions follow from Theorem 6. □

Hernandez, Beckenstein and Narici derived Corollary 8 as a consequence of their results in [12]. Recall that the *cozero* of an f in $C_0(X,E)$ is the set $\{x\in X : f(x)\neq 0\}$. A linear map T from $C_0(X,E)$ into $C_0(Y,F)$ is said to be *separating* if Tf and Tg have disjoint cozeroes whenever f and g have disjoint cozeroes. They showed that if T is a linear onto isometry such that both T and its inverse T^{-1} are separating then T must be a weighted composition operator. They also verified that a surjective linear isometry T must be separating if E and F are both strictly convex. The same also holds if E^* and F^* are both strictly convex, instead. From these, they get Corollary 8. We find out that parts of their results can also be obtained by our approach. We present a new proof of the following

Corollary 9 (Hernandez, Beckenstein and Narici [12])**.** *Let X and Y be locally compact Hausdorff spaces. Let E and F be Banach spaces. Every separating linear isometry T from $C_0(X,E)$ onto $C_0(Y,F)$ is a weighted composition operator.*

Proof. By Theorem 4, we write

$$\nu(Tf(y)) = \tilde{h}(y,\nu)f(\tilde{\varphi}(y,\nu)),\quad \forall (y,\nu)\in A_Y.$$

It suffices to verify the conditions stated in Lemma 5. Suppose, on the contrary, that $\tilde{\varphi}(y,\nu_1)\neq\tilde{\varphi}(y,\nu_2)$ for some y in Y and ν_1, ν_2 in B_y. Let U_1 and U_2 be disjoint open neighborhoods of $x_1 = \tilde{\varphi}(y,\nu_1)$ and $x_2 = \tilde{\varphi}(y,\nu_2)$ in X, respectively. Choose f_i in $C_0(X,E)$ such that f_i is supported by U_i and $\tilde{h}(y,\nu_i)f_i(x_i) \neq 0$, $i = 1,2$. Then f_1 and f_2 have disjoint cozeroes. Since T is assumed to be separating, Tf_1 and Tf_2 have disjoint cozeroes, too. However,

$$\nu_1(Tf_1(y)) = \tilde{h}(y,\nu_1)f_1(\tilde{\varphi}(y,\nu_1)) = \tilde{h}(y,\nu_1)f_1(x_1) \neq 0$$

and

$$\nu_2(Tf_2(y)) = \tilde{h}(y,\nu_2)f_2(\tilde{\varphi}(y,\nu_2)) = \tilde{h}(y,\nu_2)f_2(x_2) \neq 0,$$

a contradiction. Hence, we have $\tilde{\varphi}(y,\nu_1) = \tilde{\varphi}(y,\nu_2)$, $\forall \nu_1,\nu_2 \in B_y$, $\forall y \in Y$, as asserted. □

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