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AN ABSTRACT CAUCHY PROBLEM FOR HIGHER ORDER FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

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Abstract

The existence results for an abstract Cauchy problem involving a higher order differential inclusion with infinite delay in a Banach space are obtained. We use the concept of the existence family to express the mild solutions and impose the suitable conditions on the nonlinearity via the measure of noncompactness in order to apply the theory of condensing multimaps for the demonstration of our results. An application to some classes of partial differential equations is given.

Keywords: Cauchy problem, functional differential inclusion, infinite delay, higher order, existence family, phase space, fixed point, multivalued map, measure of noncompactness, condensing map.

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1. Introduction

In this work we study the Cauchy problem for higher order differential inclusion of the following form:

$$u^{(N)}(t) + \sum_{i=0}^{N-1} A_i u^{(i)}(t) \in F(t, u(t), u_t),\ t \in [0, T], \tag{1.1}$$

$$u^{(i)}(0) = u_i,\ i = 1, \ldots, N-1, \tag{1.2}$$

$$u(s) = \varphi(s),\ s \in (-\infty, 0], \tag{1.3}$$

where $N \geqslant 1, A_i,\ i = 0, \ldots, N-1$, are linear operators in a Banach space $(X, \|\cdot\|)$ and F is a multivalued map, whose properties will be described in the next section. Here u_t is the history of the state function u up to the time t, that is $u_t(s) = u(t+s)$ for $s \in (-\infty, 0]$.

It should be noted that the higher order differential equations and inclusions of the form (1.1) appear in many realistic models, emerging from mechanics, physics, engineering, control theory etc., in which A_i stands for some partial differential operators. One of the approaches to deal with such problems consists of the reducing them to a first order system in a suitable solution space and to apply the semigroup theory. However, as it was pointed out in [12, 33, 37], this way may be unpractical in the situation when the solution space is difficult to construct or it is complicated for applications. In addition, as it was mentioned in [12, 38], the direct treatment of higher order problems allows to get more general results.

The abstract Cauchy problem for the case $N = 1$ has been studied extensively by the application of powerful methods of the semigroup theory. The description of these methods and comprehensive references can be found, e.g., in [11, 24, 33, 36]. Later on, by using the generalizations for the concept of strongly continuous semigroup, namely, integrated semigroup (see for example, [1, 2, 5, 22, 29, 35]) and regularized semigroup (see [7, 37]), a wide class of first order and second order differential equations and inclusions has been investigated without the assumption that the operator coefficients A_i must be densely defined. We refer to some researches which relate directly to our work, in [7, 8, 17, 20, 26, 31, 32, 38]. In addition to the notions of an integrated semigroup and a regularized semigroup, a general concept of so-called existence family was introduced in [9, 10] and its extension for higher order differential equations was proposed in [39]. In [9], an example was given to demonstrate that there are limitations to both integrated semigroup and regularized semigroup approaches. More precisely, for some equations, the operator A_i generates neither an integrated semigroup nor a regularized semigroup, especially in the case when A_i is formed by a matrix of operators. The reason is that the integrated semigroup requires the operator to have the non-empty resolvent set, while regularized semigroup involves commutative property of the operator entries.

Using the concept of existence family from [39], we prove the solvability of problem (1.1)–(1.3) under some conditions imposed on nonlinear multivalued map F. Our method consists of the employing the fixed point technique for multivalued condensing maps and the typical assumption on F is expressed in the terms of the measure of non-compactness. This technique was developed in [21]. The reader can find also the relevant applications of multivalued analysis to the theory of differential inclusions in [3, 4, 6, 15, 19, 23].

At last, it is worth noting that the study of dynamical systems with unbounded delay attracts the attention of many researchers, see, e.g., [18, 25, 13, 14, 27, 28, 31] and the references therein. Usually, it is assumed that the distributed infinite delay belongs to a special seminormed functional space, whose axioms were introduced by J.K. Hale and J. Kato [16], see also [18].

In the next section, we recall some basic facts concerning the notions of the existence family, the phase space for delay differential equations and inclusions, the measures of non-compactness, and the condensing multivalued maps. Section 3 is devoted to the local and global existence results for problem (1.1)–(1.3). In the last section, we present an application to the existence of solutions for some classes of nonlinear partial differential equations.

2. Preliminaries

2.1. Existence family

For a linear operator A in a Banach space $(X, \|\cdot\|)$, we denote by $\mathcal{D}(A)$ and $\mathcal{R}(A)$, the domain and the range of A, respectively. The notation $[\mathcal{D}(A)]$ stands for the normed space $\mathcal{D}(A)$ endowed with the graph norm

$$\|x\|_{[\mathcal{D}(A)]} = \|x\| + \|Ax\|,\ x \in \mathcal{D}(A).$$

Let $L(X)$ be the space of all bounded linear operators on X. For $B \in L(X)$, by $[\mathcal{R}(B)]$ we denote the Banach space $\mathcal{R}(B)$ with the norm

$$\|x\|_{[\mathcal{R}(B)]} = \inf\{\|y\| : By = x\}.$$

For a positive constant ω, we say that $G \in LT_\omega - L(X)$ if $G : (\omega, \infty) \to L(X)$ and there exists a continuous function $H : [0, \infty) \to L(X), \|H(t)\| = O(e^{\omega t})$ such that for all $\lambda > \omega$, we have

$$G(\lambda)x = \int_0^\infty e^{-\lambda t} H(t)x dt, \text{ for all } x \in X.$$

The characterizations of the space $LT_\omega - L(X)$ can be found in [1, 37]. For $\lambda \in \mathbb{R}$, set

$$P_\lambda = \lambda^N + \sum_{i=0}^{N-1} \lambda^i A_i,\ R_\lambda = P_\lambda^{-1}$$

if the inverse exists.

Let $E_0 \in L(X)$ be injective. We recall the definition of the E_0-existence family from [39].

Definition 2.1. The strongly continuous family of operators $\{E(t)\}_{t \geqslant 0} \subset L(X)$ is said to be an E_0-existence family for the collection of operators

$(A_i)_{i=0}^{N-1}$ if for any $x \in X, t \geqslant 0$, we have $E(\cdot)x \in C^{N-1}((0,\infty);X)$, $E^{(i-1)}(t)x \in \mathcal{D}(A_i)$, $A_iE^{(i-1)}(\cdot)x \in C((0,\infty);X)$, $i = 0,\dots,N-1$, and

$$E(t)x + \sum_{i=0}^{N-1} A_i \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!} E(s)xds = \frac{t^{N-1}}{(N-1)!} E_0 x,$$

where

$$E^{(j)}(t)x = \frac{d^j}{dt^j}(E(t)x),\ j \in \mathbb{N},$$

$$E^{(-j)}(t)x = \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} E(s)xds,\ j \in \mathbb{N}\backslash\{0\}.$$

For an example of the existence family, see [39]. Recall the connection of the existence family to integrated and regularized semigroups in the case $N = 1$ (see [9]).

Let $C \in L(X)$ be injective. Assume that A is a closed linear operator in X such that $CA \subset AC$. Then the C-resolvent set of A is defined by

$$\rho_C(A) = \{\lambda \in \mathbb{C} : (\lambda I - A) \text{ is injective,}$$
$$\mathcal{R}(C) \subset \mathcal{R}(\lambda I - A) \text{ and } (\lambda I - A)^{-1}C \in L(X)\}.$$

Definition 2.2. Let $\omega, r \in \mathbb{R}, r \geqslant 0$. If $(\omega, +\infty) \subset \rho_C(A)$ and there exists $S_r(\cdot) : \mathbb{R}^+ \to L(X)$ satisfying $t \mapsto S_r(t)u \in C(\mathbb{R}^+;X)$ for each $u \in X$ such that

$$\|S_r(t)\|_{L(X)} \leqslant Me^{\omega t},\ t \geqslant 0,\ \ M > 0,$$

and

$$(\lambda I - A)^{-1}Cu = \lambda^r \int_0^{+\infty} e^{-\lambda t} S_r(t)dt,\ \lambda > \omega, u \in X,$$

then we say that A is a subgenerator of an r-times integrated, C-regularized semigroup $\{S_r(t)\}_{t\geqslant 0}$. If $r = 0$ (respectively, $C = I$), then $\{S_r(t)\}_{t\geqslant 0}$ is called a C-regularized (respectively, r-times integrated) semigroup and A is said to be a generator of $\{S_r(t)\}_{t\geqslant 0}$.

The properties of r-times integrated, C-regularized semigroup can be found in [9, 37]. Notice that, if $r \in \mathbb{N}, \lambda_0 \in \rho_I(A)$, then A is the generator of an r-times integrated semigroup if and only if A is the generator of a $(\lambda_0 I - A)^{-r}$-regularized semigroup (see for instance, [37, Theorem 1.6.7]). The following assertion shows the relation between an existence family and a C-regularized semigroup.

Theorem 2.1 ([9]). *Suppose that $\{W(t)\}_{t\geqslant 0}$ is a C-regularized semigroup, generated by an extension of A. If $\int_0^t W(s)xds \in \mathcal{D}(A)$ for $t \geqslant 0, x \in X$ then $\{W(t)\}_{t\geqslant 0}$ is a C-existence family for A.*

The condition ensuring that an E_0-existence family for $(A_i)_{i=0}^{N-1}$ exists, is given in the next statement.

Theorem 2.2 ([39]). *Suppose that the operators A_i, $i = 0, \ldots, N-1$, are closed and P_λ is injective for $\lambda > \omega$. Then the collection of operators $(A_i)_{i=0}^{N-1}$ has an E_0-existence family $\{E(t)\}_{t\geqslant 0} \subset L(X)$ satisfying*

$$\|E^{(N-1)}(t)\|, \|A_i E^{(i-1)}(s)\| \leqslant Me^{\omega t},\ i = 0, \ldots, N-1,$$

if and only if $\mathcal{R}(E_0) \subset \mathcal{R}(P_\lambda)$ and

$$\lambda^{N-1} R_\lambda E_0, \lambda^{i-1} A_i R_\lambda E_0 \in LT_\omega - L(X),\ i = 1, \ldots, N-1. \tag{2.1}$$

For $0 \leqslant k \leqslant N-1$, denote

$$\mathbf{D}_k = \left\{ x \in \bigcap_{j=0}^{k} \mathcal{D}(A_j) : A_j x \in \mathcal{R}(E_0) \text{ for all } 0 \leqslant j \leqslant k \right\}. \tag{2.2}$$

For the associated with (1.1)–(1.3) homogeneous problem

$$u^{(N)}(t) + \sum_{i=0}^{N-1} A_i u^{(i)}(t) = 0,\ t \geqslant 0, \tag{2.3}$$

$$u^{(i)}(0) = u_i,\ i = 1, \ldots, N-1,\ \ u(0) = u_0 = \varphi(0) \tag{2.4}$$

we have the following result for the existence of a classical solution, by which is meant the function $u(\cdot) \in C^N((0,\infty); X)$ such that $u^{(i)}(t) \in \mathcal{D}(A_i), t \geqslant 0$, $0 \leqslant i \leqslant N-1$, satisfying (2.3)–(2.4).

Theorem 2.3 ([39]). *Assume that there exists an E_0-existence family $\{E(t)\}_{t\geqslant 0}$ for $(A_i)_{i=0}^{N-1}$, then for $u_0 \in \mathbf{D}_0, \ldots, u_{N-1} \in \mathbf{D}_{N-1}$, problem*

(2.3)–(2.4) *admits a solution given by*

$$u(t) = \sum_{i=0}^{N-1} \left[\frac{t^i}{i!} u_i - \sum_{j=0}^{i} \int_0^t \frac{(t-s)^{i-j}}{(i-j)!} E(s) v_{ij} ds \right],\ t \geqslant 0,$$

where $v_{ij} \in X$ *are such that*

$$A_j u_i = E_0 v_{ij}, 0 \leqslant j \leqslant i, 0 \leqslant i \leqslant N-1,$$

and the solution satisfies, for some locally bounded positive function $R(t)$, *the estimates*

$$\|u^N(t)\|, \|u^{(k)}(t)\|_{[\mathcal{D}(A_k)]} \leqslant R(t) \sum_{i=0}^{N-1} \left(\|u_i\| + \sum_{j=0}^{i} \|A_j u_i\|_{[\mathcal{R}(E_0)]} \right) \tag{2.5}$$

for all $t \geqslant 0$ *and* $0 \leqslant k \leqslant N-1$.

2.2. Phase space

Let $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ be a semi-normed linear space, consisting of functions mapping $(-\infty, 0]$ into a Banach space X. The definition of a phase space $\mathcal{B}$, introduced in [16], is based on the following axioms stating that, if a function $v : (-\infty, T] \to X$ is such that $v|_{[0,T]} \in C([0,T]; X)$ and $v_0 \in \mathcal{B}$, then

($\mathcal{B}1$) $v_t \in \mathcal{B}$ for $t \in [0,T]$;

($\mathcal{B}2$) the function $t \mapsto v_t$ is continuous on $[0,T]$;

($\mathcal{B}3$) $|v_t|_{\mathcal{B}} \leqslant K(t) \sup\{\|v(s)\|_X : 0 \leqslant s \leqslant t\} + M(t)|v_0|_{\mathcal{B}}$, where $K, M : [0,\infty) \to [0,\infty)$, K is continuous, M is locally bounded, and they are independent of v.

Let us give an example of phase space. Suppose that $1 \leqslant p < +\infty$, $0 \leqslant r < +\infty$ and $g : (-\infty, -r] \to \mathbb{R}$ is nonnegative, Borel measurable function on $(-\infty, -r)$. Let CL_g^p is a class of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$ and $g(\theta)\|\varphi(\theta)\|_X^p \in L^1(-\infty, -r)$. A seminorm in CL_g^p is given by

$$|\varphi|_{CL_g^p} = \sup_{-r \leqslant \theta \leqslant 0} \{\|\varphi(\theta)\|_X\} + \left[\int_{-\infty}^{-r} g(\theta) \|\varphi(\theta)\|_X^p d\theta \right]^{\frac{1}{p}}. \tag{2.6}$$

Assume further that

$$\int_s^{-r} g(\theta)d\theta < +\infty, \text{ for every } s \in (-\infty, -r) \text{ and} \tag{2.7}$$

$$g(s+\theta) \leqslant G(s)g(\theta) \text{ for } s \leqslant 0 \text{ and } \theta \in (-\infty, -r), \tag{2.8}$$

where $G : (-\infty, 0] \to \mathbb{R}^+$ is locally bounded. From [18], we know that if (2.7)–(2.8) hold true, then CL_g^p satisfies ($\mathcal{B}1$)–($\mathcal{B}3$). For more examples of phase spaces, see [18].

2.3. Measures of non-compactness and condensing multivalued maps

Recall some basic facts from multivalued analysis, which will be used in this paper. For more details see [3, 4, 6, 15, 19, 21, 23].

Let Y be a Banach space. We denote

- $\mathcal{P}(Y) = \{A \subset Y : A \neq \emptyset\}$,
- $Pv(Y) = \{A \in \mathcal{P}(Y) : \ A \text{ is convex}\}$,
- $K(Y) = \{A \in \mathcal{P}(Y) : \ A \text{ is compact}\}$,
- $Kv(Y) = K(Y) \cap Pv(Y)$,
- $C(Y) = \{A \in \mathcal{P}(Y) : \ A \text{ is closed}\}$,
- $Pb(Y) = \{A \in \mathcal{P}(Y) : \ A \text{ is bounded}\}$.

We will use the following general definition of the measure of non-compactness (see, e.g., [21]).

Definition 2.3. Let $\mathcal{E}$ be a Banach space and $(\mathcal{A}, \geqslant)$ a partially ordered set. A function $\beta : \mathcal{P}(\mathcal{E}) \to \mathcal{A}$ is called a measure of non-compactness (MNC) in $\mathcal{E}$ if

$$\beta(\overline{co}\,\Omega) = \beta(\Omega) \ \text{ for every } \ \Omega \in \mathcal{P}(\mathcal{E}),$$

where $\overline{co}\,\Omega$ is the closure of convex hull of Ω. A MNC β is called

(i) monotone, if $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$ such that $\Omega_0 \subset \Omega_1$, then $\beta(\Omega_0) \leqslant \beta(\Omega_1)$;

(ii) nonsingular, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in \mathcal{E}, \Omega \in \mathcal{P}(\mathcal{E})$;

(iii) invariant with respect to union with compact set, if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relative compact set $K \subset \mathcal{E}$ and $\Omega \in \mathcal{P}(\mathcal{E})$;

If $\mathcal{A}$ is a cone in a normed space, we say that β is

(iv) algebraically semi-additive, if $\beta(\Omega_0 + \Omega_1) \leqslant \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$;

(v) regular, if $\beta(\Omega) = 0$ is equivalent to the relative compectness of Ω.

An important example of a real-valued MNC is *the Hausdorff MNC*, defined as

$$\chi(\Omega) = \inf\{\varepsilon : \ \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

It should be mentioned that, the Hausdorff MNC satisfies all properties given in above Definition, and, additionally, it has the following features:

- if L is a bounded linear operator in $\mathcal{E}$, then $\chi(L\Omega) \leqslant \|L\|\chi(\Omega)$;
- in separable Banach space $\mathcal{E}$, $\chi(\Omega) = \lim_{m\to\infty} \sup_{x\in\Omega} d(x, \mathcal{E}_m)$, where $\{\mathcal{E}_m\}$ is a sequence of finite dimensional subspaces of $\mathcal{E}$ such that $\mathcal{E}_m \subset \mathcal{E}_{m+1}, m = 1, 2, \dots$ and $\overline{\bigcup_{m=1}^{\infty} \mathcal{E}_m} = \mathcal{E}$.

Definition 2.4. A multi-valued map (multimap) $\mathcal{F} : X \to \mathcal{P}(Y)$ is said to be

(i) upper semi-continuous (u.s.c) if $\mathcal{F}^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$ is an open subset of X for every open set $V \subset Y$;

(ii) compact if its range $\mathcal{F}(X)$ is relatively compact in Y;

(iii) quasicompact if its restriction to any compact subset $A \subset X$ is compact.

In the sequel we will need the following assertion.

Theorem 2.4 ([21]). *Let X and Y be metric spaces and $\mathcal{F} : X \to K(Y)$ a closed quasicompact multimap. Then $\mathcal{F}$ is u.s.c.*

Definition 2.5. A multimap $\mathcal{F} : X \subset \mathcal{E} \to K(\mathcal{E})$ is said to be condensing with respect to a MNC β (β-condensing) if for every bounded set $\Omega \subset X$ that is not relatively compact, we have

$$\beta(\mathcal{F}(\Omega)) \ngeq \beta(\Omega).$$

Suppose that $D \subset \mathcal{E}$ is a nonempty closed convex subset and $\mathcal{U}$ is open set in $\mathcal{E}$ such that $\mathcal{U}_D := \mathcal{U} \cap D \neq \emptyset$. We denote by $\overline{\mathcal{U}}_D$ and $\partial\mathcal{U}_D$, the closure and boundary of $\mathcal{U}_D$ in the relative topology of D, respectively. Let β be a monotone nonsingular MNC in $\mathcal{E}$ and

$$\mathrm{Fix}(\mathcal{F}) := \{x : x \in \mathcal{F}(x)\}$$

the fixed point set of $\mathcal{F}$.

The application of topological degree theory for condensing multimaps (see [21]) yields the following fixed point results.

Theorem 2.5 ([21, Corollary 3.3.1])**.** *Let $\mathcal{M}$ be a bounded convex closed subset of $\mathcal{E}$ and $\mathcal{F} : \mathcal{M} \to Kv(\mathcal{M})$ a u.s.c. β-condensing multimap. Then* $\mathrm{Fix}(\mathcal{F})$ *is a nonempty and compact set.*

Theorem 2.6 ([21, Corollary 3.3.3])**.** *Let $\mathcal{U}$ be a bounded open neighbourhood of $0 \in D$ and $\mathcal{F} : \overline{\mathcal{U}}_D \to Kv(D)$ a u.s.c β-condensing multimap satisfying the boundary condition*

$$u \notin \lambda\mathcal{F}(u)$$

for all $u \in \partial\mathcal{U}_D$ and $0 < \lambda \leqslant 1$. Then $\mathrm{Fix}(\mathcal{F})$ *is a nonempty compact set.*

Let X be a Banach space.

Definition 2.6. Let $G : [0,T] \to K(X)$ be a multifunction.Then G is said to be

- integrable, if it admits a Bochner integrable selection. That is there exists $g : [0,T] \to X$, $g(t) \in G(t)$ for a.e. $t \in [0,T]$ such that $\int_0^T \|g(s)\|_X ds < \infty$;
- integrably bounded, if there exists a function $\xi \in L^1([0,T])$ such that

$$\|G(t)\| := \sup\{\|g\|_X : g \in G(t)\} \leqslant \xi(t) \text{ for a.e. } t \in [0,T].$$

The multifunction G is called measurable if $G^{-1}(V)$ measurable (with respect to the Lebesgue measure in $[0,T]$) for any open subset V of X. We say that G is strongly measurable if there exists a sequence $G_n : [0,T] \to K(X)$,

$n = 1, 2, \dots$ of step multifunctions such that

$$\lim_{n\to\infty} \mathcal{H}(G_n(t), G(t)) = 0 \text{ for a.e. } t \in [0,T],$$

where $\mathcal{H}$ is the Hausdorff metric in $K(X)$.
It is known that, in the case when X is a separable, the definitions of measurable and strongly measurable multifunctions are equivalent and they are equivalent to the assertion that the function $t \mapsto \operatorname{dist}(x, G(t))$ is measurable on $[0,T]$ for each $x \in X$. Furthermore, if G is measurable and integrably bounded, then it is integrable, that is the set of all Bochner integrable selections

$$S^1_G = \{g \in L^1(0,T;X) : \ g(t) \in G(t) \text{ for a.e. } t \in [0,T]\}$$

is non-empty.

Definition 2.7. We say that the multimap $G : [0,T] \times X \times \mathcal{B} \to K(X)$ satisfies the upper Carathéodory conditions if

(1) the multifunction $G(., \eta, \zeta) : [0,T] \to K(X)$ is strongly measurable for each $(\eta, \zeta) \in X \times \mathcal{B}$,

(2) the multimap $G(t, ., .) : X \times \mathcal{B} \to K(X)$ is u.s.c for a.e. $t \in [0,T]$.

The multimap G is said to be locally integrably bounded if for each $r > 0$, there exists the function $\omega_r \in L^1([0,T])$ such that

$$\|G(t,\eta,\zeta)\| = \sup\{\|z\|_X : \ z \in G(t,\eta,\zeta)\} \leqslant \omega_r(t) \quad \text{a.e.} t \in [0,T]$$

for all $(\eta, \zeta) \in X \times \mathcal{B}$ satisfying $\|\eta\|_X + |\zeta|_{\mathcal{B}} \leqslant r$.

Assuming that the multimap $G : [0,T] \times X \times \mathcal{B} \to K(X)$ satisfies the upper Carathéodory conditions and is locally integrably bounded, for a function $u : (-\infty, T] \to X$ such that $u|_{[0,T]} \in C([0,T];X)$ and $u_0 \in \mathcal{B}$, consider the superposition multifunction

$$\Phi : [0,T] \to K(X), \ \ \Phi(t) = G(t, u(t), u_t).$$

By the axioms of phase space, we see that $t \mapsto u_t \in \mathcal{B}$ is a continuous function. Then Φ is integrable. The proof can follow the same way as in [21, Theorem 1.3.5].

As a consequence, for any $\tau \in (0,T]$, we can define the superposition muti-operator

$$\mathcal{P}_G(u) := S^1_\Phi = \{\phi \in L^1(0,\tau;X) : \ \phi(t) \in G(t,u(t),u_t) \text{ for a.e. } t \in [0,\tau]\}.$$

Let

$$\mathcal{C}_X(-\infty,\tau) = \{u : (-\infty,\tau] \to X \mid u_0 \in \mathcal{B} \ \text{ and } u|_{[0,\tau]} \in C([0,\tau];X)\},$$

be the linear topological space endowed with the seminorm

$$\|u\|_{\mathcal{C}_X(-\infty,\tau)} = |u_0|_{\mathcal{B}} + \|u\|_{C([0,\tau];X)}.$$

We have the following weakly closedness property for $\mathcal{P}_G$, generated by convex-valued multimap G, whose proof can be proceeded as in [21, Lemma 5.1.1].

Lemma 2.7 *Let $G : [0,\tau] \times X \times \mathcal{B} \to Kv(X)$ be a locally integrably bounded, upper Carathéodory multimap and $\{u_n\}$ be a sequence in $\mathcal{C}_X(-\infty,\tau)$ converging to $u_* \in \mathcal{C}_X(-\infty,\tau)$. Suppose that the sequence $\{\phi_n\} \subset L^1(0,\tau;X)$, $\phi_n \in \mathcal{P}_G(u_n)$ weakly converges to a function ϕ_*, then $\phi_* \in \mathcal{P}_G(u_*)$.*

3. Main results

Let $X_0 = [\mathcal{R}(E_0)] \subset X$. Let us consider the multimap $F : [0,T] \times X \times \mathcal{B} \to Kv(X_0)$ arising in problem (1.1)-(1.3). Recalling that the operator E_0 is injective, we define the multimap $F_0 : [0,T] \times X \times \mathcal{B} \to Kv(X)$ as

$$F_0 = E_0^{-1} F. \tag{3.1}$$

We assume that the multimap F_0 satisfies the following hypotheses:

$(F1)$ $F_0 : [0,T] \times X \times \mathcal{B} \to Kv(X)$ is an upper Carathéodory multimap;

$(F2)$ F_0 is locally integrably bounded;

$(F3)$ For any bounded subsets $\mathcal{Q} \subset \mathcal{B}$ and $\Omega \subset X$, we have

$$\chi(F_0(t,\Omega,\mathcal{Q})) \leqslant h(t)\chi(\Omega) + k(t)\psi(\mathcal{Q}) \quad \text{for a.e. } t \in [0,T],$$

where $h,k \in L^1(0,T;X)$ and

$$\psi(\mathcal{Q}) = \sup_{\theta \leqslant 0} \chi(\mathcal{Q}(\theta)) \tag{3.2}$$

is *the modulus of fiber noncompactness* of $\mathcal{Q}$.

Remark 3.1. In the case $X = \mathbb{R}^m$, condition $(F3)$ can be deduced from $(F2)$. Indeed, the local integral boundedness property implies that the set $F_0(t, \Omega, \mathcal{Q})$ is bounded in $\mathbb{R}^m$ for a.e. $t \in [0, T]$ and hence it is precompact. If $\dim(X) = +\infty$, then a particular case of the fulfilling $(F3)$ is the following:

$$F_0(t, ., .) : X \times \mathcal{B} \to Kv(X)$$

is completely continuous for a.e. $t \in [0, T]$, i.e., $F_0(t, ., .)$ maps any bounded set in $X \times \mathcal{B}$ to a relatively compact set in X.

Remark 3.2. If the operator E_0^{-1} is bounded, properties $(F1)$–$(F3)$ for the multimap F_0 easily follow from the corresponding properties for F.

Motivated by [39], we give the definition of a mild solution for (1.1)–(1.3) in the following way.

Definition 3.1. Let $u_i \in \mathbf{D}_i, i = 0, \dots, N-1$ with $u_0 = \varphi(0)$. For a given $\tau \in (0, T]$, a function $u \in \mathcal{C}_X(-\infty, \tau)$ is called a mild solution to problem (1.1)–(1.3) on interval $(-\infty, \tau]$ if it satisfies the integral equation

$$u(t) = \begin{cases} \varphi(t), & \text{for } t \leqslant 0, \\ w(t) + \displaystyle\int_0^t E(t-s)\phi(s)ds & \text{for } t \in [0, \tau], \end{cases}$$

where $\phi \in \mathcal{P}_{F_0}(u)$ and w is the solution of homogeneous problem (2.3)-(2.4) on $[0, \tau]$.

Since the family $\{E(t)\}_{t \geqslant 0}$ is strongly continuous, we are able to define the operator $S : L^1(0, \tau; X) \to C([0, \tau]; X)$ by

$$S(f)(t) = \int_0^t E(t-s)f(s)ds. \tag{3.3}$$

The following assertion can be proved by using the same arguments as in [21, Lemma 4.2.1].

Proposition 3.1. *The operator S has the following properties:*

(S1) *There exists $C_0 > 0$ such that*

$$\|S(f)(t) - S(g)(t)\|_X \leqslant C_0 \int_0^t \|f(s) - g(s)\|_X ds$$

for every $f, g \in L^1(0, \tau; X)$, $t \in [0, \tau]$;

(S2) *for any compact $K \subset X$ and sequence $\{f_n\} \subset L^1(0, \tau; X)$ such that $\{f_n(t)\} \subset K$ for a.e. $t \in [0, \tau]$, the weak convergence $f_n \rightharpoonup f$ implies $S(f_n) \to S(f)$.*

The results of Proposition 3.1 lead to the following assertion, which is similar to [21, Corollary 4.2.4].

Proposition 3.2 *Let $\{\xi_n\} \subset L^1(0, \tau; X)$ be integrably bounded, i.e.,*

$$\|\xi_n(t)\| \leqslant \nu(t), \text{ for a.e. } t \in [0, \tau],$$

where $\nu \in L^1([0, \tau])$. Assume that there exists $q \in L^1([0, \tau])$ such that

$$\chi(\{\xi_n(t)\}) \leqslant q(t), \text{ for a.e. } t \in [0, \tau].$$

Then

$$\chi(\{S(\xi_n)(t)\}) \leqslant 2C_0 \int_0^t q(s) ds$$

for each $t \in [0, \tau]$.

Definition 3.2. A sequence $\{\xi_n\} \subset L^1(0, \tau; X)$ is called semicompact if it is integrably bounded and the set $\{\xi_n(t)\}$ is relatively compact in X for a.e. $t \in [0, \tau]$.

Following [21, Theorem 4.2.1 and Theorem 5.1.1], we have

Proposition 3.3. *If $\{\xi_n\} \subset L^1(0, \tau; X)$ is a semicompact sequence, then $\{\xi_n\}$ is weakly compact in $L^1(0, \tau; X)$ and $\{S(\xi_n)\}$ is relatively compact in $C([0, \tau]; X)$. Moreover, if $\xi_n \rightharpoonup \xi_0$ then $S(\xi_n) \to S(\xi_0)$.*

For any function $v \in C([0,\tau];X)$ belonging to a closed convex subset

$$\mathcal{D}_0 = \{v \in C([0,\tau];X) : v(0) = \varphi(0)\}, \tag{3.4}$$

where φ is the initial function, define a function $v[\varphi] \in \mathcal{C}_X(-\infty,\tau)$ as

$$v[\varphi](t) = \begin{cases} \varphi(t), & \text{if } t \leqslant 0, \\ v(t), & \text{if } t \in [0,\tau]. \end{cases} \tag{3.5}$$

Then it is easy to see that the function $u \in \mathcal{C}_X(-\infty,\tau)$ is a mild solution to problem (1.1)–(1.3) if it can be represented as

$$u = v[\varphi],$$

where $v \in \mathcal{D}_0$ is a fixed point of the multioperator

$$\mathcal{G} : \mathcal{D}_0 \multimap \mathcal{D}_0$$

of the form

$$\mathcal{G}(v) = w + S \circ \mathcal{P}_{F_0}(v[\varphi]),$$

where w is the solution of homogeneous problem (2.3)–(2.4).

Lemma 3.4. *Assume that F_0 satisfies* (F1)–(F3). *Then $\mathcal{G}$ is a closed multioperator with compact convex values.*

Proof. It is clear that it is sufficient to prove the assertion for the multimap $\widetilde{\mathcal{G}} : \mathcal{D}_0 \multimap C([0,\tau];X)$,

$$\widetilde{\mathcal{G}}(v) = S \circ \mathcal{P}_{F_0}(v[\varphi]).$$

Assume that $\{v_n\} \subset \mathcal{D}_0$ converges to v^* in $\mathcal{D}_0$ and $z_n \in \widetilde{\mathcal{G}}(v_n)$ is such that $z_n \to z^*$ in $C([0,\tau];X)$. We show that $z^* \in \widetilde{\mathcal{G}}(v^*))$. Let $\xi_n \in \mathcal{P}_{F_0}(v_n[\varphi])$ be such that $z_n = S(\xi_n)$. Then we have

$$\xi_n(t) \in F_0(t, v_n(t), v_n[\varphi]_t) \text{ for a.e. } t \in [0,\tau],$$

and by using $(F2)$, we conclude that $\{\xi_n\}$ is integrably bounded. Furthermore, hypothesis $(F3)$ implies that

$$\chi(\{\xi_n(t)\}) \leqslant h(t)\chi(\{v_n(t)\}) + k(t)\psi(\{v_n[\varphi]_t\}) \text{ for a.e. } t \in [0,\tau]. \tag{3.6}$$

The convergence of $\{v_n\}$ in $C([0,\tau];X)$ implies that $\chi(\{v_n(t)\}) = 0$ for all $t \in [0,\tau]$. On the other hand,

$$\psi(\{v_n[\varphi]_t\}) = \sup_{\theta\leqslant 0}\chi(\{v_n[\varphi](t+\theta)\}) = \sup_{s\in[0,t]}\chi(\{v_n(s)\}) = 0. \tag{3.7}$$

Thus taking into account estimate (3.6), we arrive at

$$\chi(\{\xi_n(t)\}) = 0 \ \text{ for a.e. } t \in [0,\tau]$$

and so $\{\xi_n\}$ is a semicompact sequence. It follows from Proposition 3.3 that $\{\xi_n\}$ is weakly compact in $L^1(0,\tau;X)$ and $\{S(\xi_n)\}$ is relatively compact in $C([0,\tau];X)$, that is we may assume, w.l.o.g., that $\xi_n \rightharpoonup \xi^*$ in $L^1(0,\tau;X)$ and $z_n = S(\xi_n) \to S(\xi^*) = z^*$ in $C([0,\tau];X)$. Now applying Lemma 2.7, we have $\xi^* \in \mathcal{P}_{F_0}(v^*[\varphi])$ and then $z^* = S(\xi^*) \in S\circ\mathcal{P}_{F_0}(v^*[\varphi]) = \widetilde{\mathcal{G}}(v^*)$.

It remains to prove that the multimap $\widetilde{\mathcal{G}}$ has compact values. Let $\{z_n\} \subset \widetilde{\mathcal{G}}(v)$ for any $v \in \mathcal{D}_0$. Then there exists $\{\xi_n\} \in \mathcal{P}_{F_0}(v[\varphi])$ such that $z_n = S(\xi_n)$. By using hypotheses $(F2)$–$(F3)$, we obtain that the sequence $\{\xi_n\}$ is semicompact, and so $\{S(\xi_n)\}$ is relatively compact in $C([0,\tau];X)$ by Proposition 3.3. The convexity of values of $\widetilde{\mathcal{G}}$ is obvious. ■

Lemma 3.5. *Under the conditions of Lemma* 3.4, *the multioperator* $\mathcal{G}$ *is u.s.c.*

Proof. Again it is sufficient to prove this assertion for $\widetilde{\mathcal{G}}$. Taking into account Theorem 2.4 and Lemma 3.4, we will show that $\widetilde{\mathcal{G}}$ is a quasicompact multimap. Let $A \subset C([0,\tau];X)$ be a compact set and $\{z_n\} \subset \widetilde{\mathcal{G}}(A)$. Then $z_n = S(\xi_n)$ where $\xi_n \in \mathcal{P}_{F_0}(v_n[\varphi])$ for a certain sequence $\{v_n\} \subset A$. We may assume, w.l.o.g., that $\{v_n\}$ is convergent. By using the same estimates as (3.6)–(3.7), we can see that $\{\xi_n\}$ is a semicompact sequence. Hence $\{S(\xi_n)\}$ is relatively compact in $C([0,\tau];X)$ by Proposition 3.3. ■

We are in a position to demonstrate that $\mathcal{G}$ is a condensing multioperator. We first need a special MNC constructed suitably for our problem. Introduce *the damped modulus of fiber non-compactness* defined as

$$\begin{aligned} &\gamma : \mathcal{P}(C([0,\tau];X)) \to \mathbb{R}_+,\\ &\gamma(\Omega) = \sup_{t\in[0,\tau]} e^{-Lt}\chi(\Omega(t)), \end{aligned} \tag{3.8}$$

where the constant L is chosen such that

$$\ell := \sup_{t\in[0,\tau]} \Big(2C_0 \int_0^t e^{-L(t-s)}[h(s)+k(s)]ds\Big) < 1 \tag{3.9}$$

and

$$\begin{aligned} &mod_C : \mathcal{P}(C([0,\tau];X)) \to \mathbb{R}_+,\\ &mod_C(\Omega) = \lim_{\delta\to 0} \sup_{v\in\Omega} \max_{|t_1-t_2|<\delta} \|v(t_1)-v(t_2)\|, \end{aligned} \tag{3.10}$$

which is called *the modulus of equicontinuity* of Ω in $C([0,\tau];X)$.

Consider the function

$$\begin{aligned} &\nu : \mathcal{P}(C([0,\tau];X)) \to \mathbb{R}^2_+,\\ &\nu(\Omega) = \max{}_{D\in\Delta(\Omega)}(\gamma(D), mod_C(D)), \end{aligned} \tag{3.11}$$

where $\Delta(\Omega)$ is the collection of all countable subsets of Ω and the maximum is taken in the sense of the (partial) ordering in the cone $\mathbb{R}^2_+$. By the same arguments as in [21], one can see that ν is well-defined. That is, the maximum is archived in $\Delta(\Omega)$ and ν is a MNC in the space $C([0,\tau];E)$, which obeys all properties in Definition 2.3 (see [21, Example 2.1.3] for details).

Lemma 3.6 *If the conditions of Lemma* 3.4 *hold true, then the multioperator* $\mathcal{G} : \mathcal{D}_0 \to Kv(\mathcal{D}_0)$ *is* ν*-condensing.*

Proof. Let $\Omega \subset \mathcal{D}_0$ be such that

$$\nu(\mathcal{G}(\Omega)) \geqslant \nu(\Omega). \tag{3.12}$$

Since clearly $\nu(\mathcal{G}(\Omega) = \nu(\widetilde{\mathcal{G}}(\Omega))$, we have

$$\nu(\widetilde{\mathcal{G}}(\Omega)) \geqslant \nu(\Omega). \tag{3.13}$$

Our goal is to demonstrate that Ω is relatively compact in $C([0,\tau];X)$. Indeed, by the definition of ν, there exists a sequence $\{z_n\} \subset \widetilde{\mathcal{G}}(\Omega)$ such that

$$\nu(\widetilde{\mathcal{G}}(\Omega)) = (\gamma(\{z_n\}), mod_C(\{z_n\})).$$

Take two sequences $v_n \in \Omega$, $\xi_n \in \mathcal{P}_{F_0}(v_n[\varphi])$ such that $z_n = S(\xi_n)$. From (3.13) we see that

$$\gamma(\{z_n\}) \geqslant \gamma(\{v_n\}). \tag{3.14}$$

Using $(F3)$, we get

$$\chi(\{\xi_n(s)\}) \leqslant h(s)\chi(\{v_n(s)\}) + k(s)\psi(\{(v_n[\varphi])_s\}) \tag{3.15}$$

for a.e. $s \in [0,\tau]$. By (3.2), we have

$$\psi(\{(v_n[\varphi])_s\}) = \sup_{\theta\leqslant 0} \chi(\{v_n[\varphi](s+\theta)\}) = \sup_{\sigma\in[0,s]} \chi(\{v_n(\sigma)\}).$$

Then (3.15) leads to

$$\begin{aligned}\chi(\{\xi_n(s)\}) &\leqslant h(s)e^{Ls}e^{-Ls}\chi(\{v_n(s)\}) + k(s)e^{Ls} \sup_{\sigma\in[0,s]} e^{-L\sigma}\chi(\{v_n(\sigma)\}) \\ &\leqslant e^{Ls}[h(s)+k(s)]\gamma(\{v_n\}) \end{aligned} \tag{3.16}$$

for a.e. $s \in [0,\tau]$. Therefore Proposition 3.2 yields

$$\chi(\{S(\xi_n)(t)\}) \leqslant 2C_0 \int_0^t e^{Ls}[h(s)+k(s)]ds \cdot \gamma(\{v_n\}).$$

So we have

$$e^{-Lt}\chi(\{z_n(t)\}) \leqslant 2C_0 \int_0^t e^{-L(t-s)}[h(s)+k(s)]ds \cdot \gamma(\{v_n\}). \tag{3.17}$$

Combining (3.14) and (3.17), we arrive at

$$\gamma(\{v_n\}) \leqslant \gamma(\{z_n\}) = \sup_{t\in[0,\tau]} e^{-Lt}\chi(\{z_n(t)\}) \leqslant \ell\gamma(\{v_n\}),$$

where ℓ is defined in (3.9). The last inequality yields $\gamma(\{v_n\}) = 0$. Taking into account (3.16), one observes that $\{\xi_n\}$ is semicompact and once again, Proposition 3.3 guarantees that $\{S(\xi_n)\}$ is relatively compact in $C([0,\tau];X)$.

Hence, $mod_C(\{z_n\}) = 0$ and then

$$\nu(\Omega) = (0, 0).$$

The regularity of ν ensures that Ω is relatively compact in $C([0,\tau];X)$. We have the desired conclusion. ∎

Now we are in position to present the main assertions of this section. The following result is a local existence theorem for problem (1.1)–(1.3).

Theorem 3.7 *Let $u_i \in \mathbf{D}_i$, $i = 0, \dots, N-1$ with $u_0 = \varphi(0)$. Suppose that conditions* (F1)–(F3) *are satisfied and there exists an E_0-existence family for $(A_i)_{i=0}^{N-1}$. Then there exists $\tau \in (0,T]$ such that problem* (1.1)–(1.3) *has at least one mild solution on the interval $(-\infty, \tau]$.*

Proof. Let ρ be a positive number such that

$$\rho > R \sum_{i=0}^{N-1} \left(\|u_i\| + \sum_{j=0}^{i} \|A_j u_i\|_{[\mathcal{R}(E_0)]} \right)$$

where $R = \sup_{t\in[0,T]} R(t)$, $R(t)$ is the function in (2.5). Thus we deduce that $\rho > \|w\|_{C([0,T];X)}$, where w is the solution of homogeneous problem (2.3)–(2.4).

Denote

$$\rho_0 = (K+1)\rho + M|\varphi|_{\mathcal{B}},\ K = \max_{t\in[0,T]} K(t), M = \sup_{t\in[0,T]} M(t),$$
$$C_E^T = \sup_{t\in[0,T]} \|E(t)\|_{L(X)}.$$

By the choice of ρ, one can take $\tau \in (0,T]$ such that

$$\|w\|_{C([0,T];X)} + C_E^T \int_0^t \omega_{\rho_0}(s)ds \leqslant \rho, \tag{3.18}$$

for each $t \in [0,\tau]$, where $\omega_{\rho_0} \in L^1(0,T;X)$ is given in Definition 2.7.

Let $\overline{B}_\rho$ be the closed ball in $C([0,\tau];X)$ centered at 0 with radius ρ. Notice that from $w(0) = \varphi(0)$ and relation (3.18) it follows that

$$\mathcal{D}_\rho := \mathcal{D}_0 \cap \overline{B}_\rho \neq \emptyset.$$

Then for $v \in \mathcal{D}_\rho$ and $z \in \mathcal{G}(v) = w + S \circ \mathcal{P}_{F_0}(v[\varphi])$, we have the estimates

$$\begin{aligned} \|z(t)\|_X &\leqslant \|w\|_{C([0,T];X)} + \int_0^t \|E(t-s)F_0(s, v(s), v[\varphi]_s)\|_X ds \\ &\leqslant \|w\|_{C([0,T];X)} + C_E^T \int_0^t \omega_{\rho_0}(s)ds \leqslant \rho, \end{aligned} \tag{3.19}$$

for all $t \in [0, \tau]$. Here we have used assumption $(F2)$, taking into account that

$$\begin{aligned} \|v(s)\|_X + |v[\varphi]_s|_{\mathcal{B}} &\leqslant \|v\|_{C([0,\tau];X)} + K(s)\|v\|_{C([0,s];X)} + M(s)|\varphi|_{\mathcal{B}} \\ &\leqslant (K+1)\|v\|_{C([0,\tau];X)} + M|\varphi|_{\mathcal{B}} = \rho_0, \end{aligned}$$

for all $s \in (0, \tau]$.

Evidently, (3.19) implies

$$\|z\|_{C([0,\tau];X)} \leqslant \rho$$

and so $\mathcal{G}$ maps $\mathcal{D}_\rho$ into itself. By applying Theorem 2.5, we conclude that $\mathcal{G}$ has a fixed point $v_* \in \mathcal{D}_\rho$, which induces the desired solution $u_* = v_*[\varphi]$. ■

In order to get the global existence result, we need to replace condition $(F2)$ with a stronger one. Indeed, we impose the following assumption

$(F2')$ There exists a function $\kappa \in L^1([0,T])$ such that

$$\|F_0(t, \eta, \zeta)\| := \sup\{\|f\|_X : f \in F_0(t, \eta, \zeta)\} \leqslant \kappa(t)(\|\eta\|_X + |\zeta|_{\mathcal{B}}),$$

for all $\eta \in X$ and $\zeta \in \mathcal{B}$.

In addition, we need the following version of the generalized Gronwall-Bellman inequality (see, e.g., [34]).

Lemma 3.8 *Assume that $f(\cdot), g(\cdot)$ and $y(\cdot)$ are non-negative integrable functions on $[0,T]$ satisfying the integral inequality*

$$y(t) \leqslant g(t) + \int_0^t f(s)y(s)ds, \ t \in [0,T].$$

Then

$$y(t) \leqslant g(t) + \int_0^t \exp\Big\{ \int_s^t f(\theta)d\theta \Big\} f(s)g(s)ds, \ t \in [0,T].$$

Theorem 3.9 *Let $u_i \in \mathbf{D}_i$, $i = 0, \dots, N-1$ with $u_0 = \varphi(0)$. Assume that there exists an E_0-existence family for $(A_i)_{i=0}^{N-1}$. If assumptions $(F1)$, $(F2')$ and $(F3)$ hold, then the set of solutions to problem* (1.1)–(1.3) *is nonempty and compact.*

Proof. Using Theorem 2.6 and Lemmas 3.4, 3.5 and 3.6, we have to prove that the set of all functions $v \in C([0,T];X)$ satisfying the family of inclusions

$$v \in \lambda \mathcal{G}(v) = \lambda w + \lambda S \circ \mathcal{P}_{F_0}(v[\varphi])$$

for $\lambda \in (0,1]$ is a priori bounded.

Applying condition $(F2')$, we observe that

$$\begin{aligned} \|v(t)\|_X &\leqslant \lambda \|w(t)\|_X + \lambda \sup_{t\in[0,T]} \|E(t)\|_{L(X)} \int_0^t \|F_0(s, v(s), v[\varphi]_s)\|_X ds \\ &\leqslant \|w\|_{C([0,T];X)} + C_E^T \int_0^t \kappa(s)(\|v(s)\|_X + |v[\varphi]_s|_{\mathcal{B}}) ds, \qquad (3.20) \end{aligned}$$

where $C_E^T = \sup_{t\in[0,T]} \|E(t)\|_{L(X)}$. By $(\mathcal{B}3)$, we have the following estimate

$$\|v(s)\|_X + |v[\varphi]_s|_{\mathcal{B}} \leqslant (K+1)\|v(s)\|_X + M|\varphi|_{\mathcal{B}}, \qquad (3.21)$$

for $s \in [0,t], 0 < t \leqslant T$, $K = \max_{t\in[0,T]} K(t), M = \sup_{t\in[0,T]} M(t)$.

It follows from (3.20)–(3.21) that

$$\begin{aligned} \|v(t)\|_X \leqslant \|w\|_{C([0,T];X)} &+ C_E^T M |\varphi|_{\mathcal{B}} \int_0^T \kappa(s) ds \\ &+ C_E^T (K+1) \int_0^t \kappa(s) \|v(s)\|_X ds, \end{aligned}$$

for all $t \in [0,T]$. Applying Lemma 3.8 with

$$\begin{aligned} &g(t) = g_0 := \|w\|_{C([0,T];X)} + C_E^T M |\varphi|_{\mathcal{B}} \int_0^T \kappa(s) ds, \\ &f(t) = C_E^T \kappa(t),\ y(t) = \|v(t)\|_X, \end{aligned}$$

for $t \in [0,T]$, we obtain

$$\|v\|_{C([0,T];X)} \leqslant R_0,$$

where

$$R_0 = g_0\Big[1 + C_E^T \exp\Big\{C_E^T \int_0^T \kappa(t)dt\Big\} \int_0^T \kappa(t)dt\Big].$$

Finally, taking $\mathcal{U} = B(0,R)$ in $C([0,T];X)$ with $R > R_0$, we consider the multioperator $\mathcal{G}$ as a map from $\overline{\mathcal{U}}_{\mathcal{D}_0} \neq \emptyset$ to $Kv(\mathcal{D}_0)$, where $\mathcal{D}_0$ is defined in (3.4) with $\tau = T$. It satisfies the hypotheses of Theorem 2.6 and so $\mathrm{Fix}(\mathcal{G})$ is a nonempty compact set, obviously yielding the desired result. ∎

4. Application

Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m,\ m \geqslant 1$. Denote

$$|\alpha| = \sum_{i=1}^m \alpha_i,\ D^\alpha = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x_m}\Big)^{\alpha_m}.$$

Given a complex polynomial of degree k in $\mathbb{R}^m$

$$P(x) = \sum_{|\alpha|\leqslant k} a_\alpha (ix)^\alpha, \tag{4.1}$$

we define

$$P(D) = \sum_{|\alpha|\leqslant k} a_\alpha D^\alpha.$$

In this section, taking $X = L^p(\mathbb{R}^m), 1 < p < \infty$, and

$$\mathcal{D}(P(D)) = \{f \in L^p(\mathbb{R}^m) : P(D)f \in L^p(\mathbb{R}^m)\},$$

we consider the following Cauchy problem in $L^p(\mathbb{R}^m)$

$$\begin{aligned} &\frac{\partial^2 u(t,x)}{\partial t^2} + P(D)\frac{\partial u(t,x)}{\partial t} + Q(D)u(t,x) \\ &\quad = \int_{-\infty}^t \int_{\mathbb{R}^m} \mathcal{K}(x,y)\xi(s-t,y) f(t,u(t,y),u(s-t,y))dyds, \\ &\qquad\qquad x \in \mathbb{R}^m,\ t \in [0,T], \end{aligned} \tag{4.2}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \tag{4.3}$$

$$u(s,x) = \varphi(s,x),\ s \in (-\infty, 0], \tag{4.4}$$

where P and Q are polynomials as in (4.1) with degrees k and ℓ, respectively. We assume that

$$\mathcal{K} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R},$$

is a smooth function and

$$\xi :\ (-\infty, 0] \times \mathbb{R}^m \to \mathbb{R},$$

is a continuous function satisfying

$$|\xi(\theta, y)| \leqslant C_\xi e^{h_0\theta} \text{ for all } (\theta, y) \in (-\infty, 0] \times \mathbb{R}^m, \tag{4.5}$$

where C_ξ and h_0 are positive constants. In addition, assume that the function

$$f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$$

is such that $f(\cdot, u, v)$ is measurable and $f(t, \cdot, \cdot)$ obeys the Lipschitz type property:

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leqslant \zeta(t)|u_1 - u_2| + \mu(t)|v_1 - v_2| \tag{4.6}$$

for all $t \in [0,T]$ and $u_j, v_j \in \mathbb{R},\ j = 1, 2$, where $\zeta, \mu \in L^1(0,T)$.

It is worth noting that the homogeneous form of (4.2)–(4.4) was presented in [38], where the nonlinearity and delay term were absent. For our problem, we use the phase space $\mathcal{B} = CL^p_g$ defined by (2.6) with $g(\theta) = e^{h\theta}$, $h \in (0, h_0]$. It is obvious that g satisfies conditions (2.7)–(2.8).

We first show that, under suitable hypotheses, there is an existence family for $(P(D), Q(D))$. To this end, we recall some definitions and results from [38].

Let $C \in L(X)$ be injective, A_0 and A_1 be closed linear operators on X.

Definition 4.1. A pair $\{S_0(t), S_1(t)\}_{t\geqslant 0}$ of strongly continuous families of bounded operators on X is called strong C-propagation family for (A_0, A_1) if

(i) C commutes with $S_0(t), S_1(t)$ for each $t \geqslant 0$;

(ii) for each $x \in X$, $S_1(\cdot)x \in C^1([0,\infty); X)$, $S_1(t)X \subset \mathcal{D}(A_1), (t \geqslant 0)$ and $A_1S_1(\cdot)x \in C([0,\infty); X)$;

(iii) for each $x \in X$ and $t \geqslant 0$, $\int_0^t S_1(s)xds \in \mathcal{D}(A_0)$ and

$$A_0 \int_0^t S_1(s)xds = Cx - S_1'(t)x - A_1 S_1(t)x, \quad S_1(0) = 0,$$

where

$$S_1'(t)x = \frac{d}{dt} S_1(t)x;$$

(iv) there exist constants $M, \omega > 0$ such that

$$\|S_0(t)\|, \|A_1 S_1(t)\|, \|S_1'(t)\| \leqslant Me^{\omega t}, \quad t \geqslant 0;$$

(v) any solution $u(\cdot)$ of the problem

$$u'' + A_1 u' + A_0 u = 0, \tag{4.7}$$

$$u(0) = u_0, u'(0) = u_1, \tag{4.8}$$

with initial values $u_0, u_1 \in \mathcal{R}(C)$ can be expressed in the form

$$u(t) = S_0(t)C^{-1}u_0 + S_1(t)C^{-1}u_1, \ t \geqslant 0.$$

Cauchy problem (4.7)–(4.8) is said to be strongly C-well-posed if there exists a strong C-propagation for (A_0, A_1).

Proposition 4.1 ([38, Proposition 1.6])**.** *If Cauchy problem* (4.7)–(4.8) *is strongly C-well-posed then*

$$\lambda R_\lambda Cx, \ A_1 R_\lambda Cx \in LT_w - L(X), \ x \in X.$$

Using Theorem 2.2 and Proposition 4.1, we see that if problem (4.7)–(4.8) is strongly C-well-posed, then there exists a C-existence family for (A_0, A_1). The following assertion gives a sufficient condition for the strong C-well-posedness of (4.7)–(4.8).

Theorem 4.2 ([38])**.** *Let $P(x)$, $Q(x)$ be complex polynomials of degrees k, ℓ respectively. Assume that*

$$\sup_{x \in \mathbb{R}^m} Re\Big(- P(x) + \sqrt{P^2(x) - 4Q(x)}\Big) < \infty. \tag{4.9}$$

Let $A_1 = P(D)$, $A_0 = Q(D)$. *Then Cauchy problem* (4.7)–(4.8) *is strongly* $(I-\Delta)^{-\gamma}$*-well-posed for*

$$\gamma \geqslant \frac{1}{4}(n_p+1)d_M \tag{4.10}$$

where $n_p = n|\frac{1}{2}-\frac{1}{p}|$ *and* $d_M = \max\{2k,\ell\}$. *In addition, if there exists* $r \in (0, d_M]$ *such that*

$$|P^2(x) - 4Q(x)| \geqslant C_0|x|^r,\ |x| \geqslant L_0 \tag{4.11}$$

for some $C_0, L_0 > 0$ *then* γ *can be improved as*

$$\gamma \geqslant \frac{1}{4}(n_p d_M + d_M - r). \tag{4.12}$$

Denoting by $W^{\kappa,p}(\mathbb{R}^m)$ the usual Sobolev space, we obtain the solvability result for problem (4.2)–(4.4) in the following way.

Theorem 4.3 *Assume that the hypotheses of Theorem* 4.2 *hold. If we have*

$$(I-\Delta_x)^{\gamma}\mathcal{K}(x,y) \in L^p(\mathbb{R}^m; L^{p'}(\mathbb{R}^m)),$$

where p' *is the conjugate of* p, *and conditions* (4.5)–(4.6) *hold true, then problem* (4.2)–(4.4) *has at least one mild solution with initial data*

$$u_0 \in W^{2\gamma+\ell,p}(\mathbb{R}^m), u_1 \in W^{2\gamma+\max\{k,\ell\},p}(\mathbb{R}^m).$$

Proof. By Theorem 4.2, one can see that $(P(D), Q(D))$ has an $(I-\Delta)^{-\gamma}$-existence family in $L^p(\mathbb{R}^m)$. For problem (4.2)–(4.4), set

$$F(t,u,u_t)(x) = \int_{-\infty}^{t}\int_{\mathbb{R}^m} \mathcal{K}(x,y)\xi(s-t,y)f(t,u(t,y),u(s-t,y))dyds.$$

Then F may be considered as the map

$$F:\ [0,T]\times L^p(\mathbb{R}^m)\times\mathcal{B} \to L^p(\mathbb{R}^m),$$
$$F(t,\eta,\phi)(x) = \int_{-\infty}^{0}\int_{\mathbb{R}^m} \mathcal{K}(x,y)\xi(\theta,y)f(t,\eta(y),\phi(\theta,y))dyd\theta.$$

Thus in this case,

$$F_0(t,\eta,\phi)(x) = \int_{-\infty}^0 \int_{\mathbb{R}^m} \mathcal{K}_0(x,y)\xi(\theta,y)f(t,\eta(y),\phi(\theta,y))dyd\theta$$

where $\mathcal{K}_0(x,y) = (I-\Delta_x)^\gamma \mathcal{K}(x,y)$. We have the following estimates by using (4.5), (4.6) and the Hölder inequality:

$$\begin{aligned}
&|F_0(t,\eta_1,\phi_1)(x) - F_0(t,\eta_2,\phi_2)(x)| \\
&\leqslant \int_{-\infty}^0 \int_{\mathbb{R}^m} C_\xi |\mathcal{K}_0(x,y)| e^{h_0\theta}\big[\zeta(t)|\eta_1(y)-\eta_2(y)| + \mu(t)|\phi_1(\theta,y)-\phi_2(\theta,y)|\big] dyd\theta \\
&\leqslant C_\xi\zeta(t)\int_{-\infty}^0 e^{h_0\theta}d\theta \int_{\mathbb{R}^m} |\mathcal{K}_0(x,y)||\eta_1(y)-\eta_2(y)|dy \\
&\quad + C_\xi\mu(t)\int_{-\infty}^0 e^{h_0\theta}\int_{\mathbb{R}^m} |\mathcal{K}_0(x,y)||\phi_1(\theta,y)-\phi_2(\theta,y)|dyd\theta \\
&\leqslant \frac{1}{h_0}C_\xi\zeta(t)\|\eta_1-\eta_2\|_p\Big(\int_{\mathbb{R}^m} |\mathcal{K}_0(x,y)|^{p'}dy\Big)^{\frac{1}{p'}} \\
&\quad + C_\xi\mu(t)\Big(\int_{-\infty}^0 e^{h_0\theta}\|\phi_1(\theta)-\phi_2(\theta)\|_p d\theta\Big)\Big(\int_{\mathbb{R}^m} |\mathcal{K}_0(x,y)|^{p'}dy\Big)^{\frac{1}{p'}},
\end{aligned}$$

where $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R}^m)}$. Then

$$\begin{aligned}
&\|F_0(t,\eta_1,\phi_1) - F_0(t,\eta_2,\phi_2)\|_p \\
&\leqslant C_\xi C_{\mathcal{K}}\Big[\frac{1}{h_0}\zeta(t)\|\eta_1-\eta_2\|_p + \mu(t)\int_{-\infty}^0 e^{h_0\theta}\|\phi_1(\theta)-\phi_2(\theta)\|_p d\theta\Big],
\end{aligned} \tag{4.13}$$

where

$$C_{\mathcal{K}} = \Big[\int_{\mathbb{R}^m}\Big(\int_{\mathbb{R}^m} |\mathcal{K}_0(x,y)|^{p'}dy\Big)^{p/p'}dx\Big]^{\frac{1}{p}}.$$

Notice that, by the Hölder inequality, we get

$$\int_{-\infty}^{0} e^{h_0\theta}\|\phi_1(\theta)-\phi_2(\theta)\|_p d\theta \leqslant \Big[\frac{p-1}{ph_0-h}\Big]^{\frac{p-1}{p}} \Big(\int_{-\infty}^{0} e^{h\theta}\|\phi_1(\theta)-\phi_2(\theta)\|_p^p\Big)^{\frac{1}{p}}$$
$$\leqslant \Big[\frac{p-1}{ph_0-h}\Big]^{\frac{p-1}{p}} |\phi_1-\phi_2|_{\mathcal{B}},$$

for $0 < h \leqslant h_0$. Putting this together with (4.13), we obtain

$$\|F_0(t,\eta_1,\phi_1)(x) - F_0(t,\eta_2,\phi_2)\|_p \leqslant \zeta_0(t)\|\eta_1-\eta_2\|_p + \mu_0(t)|\phi_1-\phi_2|_{\mathcal{B}}, \tag{4.14}$$

where

$$\zeta_0(t) = \frac{1}{h_0} C_\xi C_{\mathcal{K}} \zeta(t),\ \mu_0(t) = C_\xi C_{\mathcal{K}}\Big[\frac{p-1}{ph_0-h}\Big]^{\frac{p-1}{p}} \mu(t).$$

One can check that inequality (4.14) produces conditions $(F1)$, $(F2')$ and $(F3)$. Finally, we have

$$\mathcal{R}((I-\Delta)^{-\gamma}) = W^{2\gamma,p}(\mathbb{R}^m),$$

and therefore

$$\mathbf{D}_0 = W^{2\gamma+\ell,p}(\mathbb{R}^m),$$
$$\mathbf{D}_1 = W^{2\gamma+\max\{k,\ell\},p}(\mathbb{R}^m)$$

due to (2.2). The proof is completed. ■

Let us give some examples to the hypotheses of Theorem 4.2. Taking

$$P(D) = 0,\ Q(D) = -\Delta,$$

we see that (4.2) turns into a semilinear wave equation. In this case we have $Q(x) = |x|^2$ and

$$-P(x) + \sqrt{P^2(x) - 4Q(x)} = 2i|x|.$$

Hence relation (4.9) is obvious. Moreover, (4.11) is satisfied with $r = 2$, then following (4.12) we can choose $\gamma \geqslant \frac{1}{2}n_p$. If, in addition, $p = 2$ then $\gamma = 0$ is suitable and we have an I-existence family for $(P(D), Q(D))$.

As the second example, we can take

$$P(D) = i\Delta,\ Q(D) = I - \Delta,$$

then

$$-P(x) + \sqrt{P^2(x) - 4Q(x)} = 2i(|x|^2 + 1).$$

This ensures (4.9). Evidently, (4.11) is also fulfilled with $r = 2$ and then $(P(D), Q(D))$ has an $(I - \Delta)^{-\gamma}$-existence family with $\gamma \geqslant n_p$.

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