Strong Convergence Theorems for Strictly Pseudocontractive Mappings of Browder-Petryshyn Type[1]

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Abstract. Let $q > 1$ and E be a real q-uniformly smooth Banach space, K be a nonempty closed convex subset of E and $T : K \to K$ be a strictly pseudocontractive mapping in the sense of F. E. Browder and W.V. Pstryshyn [1]. Let $\{u_n\}$ be a bounded sequence in K and $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ satisfying some restrictions. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iteration process with errors: $y_n = (1 - \beta_n)x_n + \beta_n T x_n$, $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n, n \geq 1$. Sufficient and necessary conditions for the strong convergence $\{x_n\}$ to a fixed point of T is established.

Keywords: Fixed point, strictly pseudocontractive mapping, Ishikawa iteration process with errors, $q-$uniformly smooth Banach space.

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1. Introduction and Preliminaries

Let E be an arbitrary real Banach space and let $J_q (q > 1)$ denote the generalized duality mapping from E into $2^{E^\star}$ given by

$$J_q(x) = \{f \in E^\star : \langle x, f\rangle = \|x\|^q = \|x\|\|f\|\},$$

where $E^\star$ denote the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and $E^\star$. In particular, J_2 is called the normalized duality mapping and it is usually denote by J. It is well known (see [11]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$ and that if $E^\star$ is strictly convex, then J_q is single-valued.

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is called strictly pseudocontractive [1] if there exists $\lambda > 0$ such that for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \lambda\|(I-T)x - (I-T)y\|^2 \qquad (1.1)$$

where I denotes the identity operator.

Remark 1.1. Without loss of generality we may assume $\lambda \in (0,1)$. In Hilbert spaces, (1.1) is equivalent to the following inequality

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2,\ k = 1-2\lambda < 1. \qquad (1.2)$$

The Mann iterative process (with errors) and the Ishikawa iterative process (with errors) have been extensively applied to approximate the solutions of nonlinear operator equations or fixed points of nonlinear mappings in Hilbert spaces or Banach spaces in the literature. See, e.g., [3-10]. In 1974, Rhoades [9] proved the following convergence theorem using the Mann iterative process.

Theorem 1.1. Let H be a real Hilbert space and K a nonempty compact convex subset of H. Let $T : K \to K$ be a strictly pseudocontractive mapping and let $\{\alpha_n\}$ be a real sequence satisfying the conditions: (i) $\alpha_0 = 1$; (ii) $0 < \alpha_n < 1, n \geq 1$; (iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (iv) $\lim_{n\to\infty} \alpha_n = \alpha < 1$. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by the Mann iterative process

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, n \geq 0$$

converges strongly to a fixed point of T.

Let E be a real Banach space. The modulus of smoothness of E is defined as the function $\rho_E : [0, \infty) \to [0, \infty)$:

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau\}.$$

E is said to be uniformly smooth if and only if $\lim_{\tau\to 0_+}(\rho_E(\tau)/\tau) = 0$. Let $q > 1$. The space E is said to be $q-$uniformly smooth (or to have a modulus of smoothness of power type $q > 1$), if there exists a constant $c_q > 0$ such that $\rho_E(\tau) \leq c_q\tau^q$. It is well known that Hilbert spaces, L_p and l_p spaces, $1 < p < \infty$, as well as the Sobolev spaces, $W_m^p, 1 < p < \infty$, are $q-$uniformly smooth. Hilbert spaces are 2-uniformly smooth while if $1 < p \leq 2$, L_p, l_p and W_m^p are $p-$uniformly smooth. If $p \geq 2$, L_p, l_p and W_m^p are 2-uniformly smooth.

Theorem 1.2 [11]. Let $q > 1$ and E be a real smooth Banach space. Then the following are equivalent:

(1) E is $q-$uniformly smooth.

(2) There exists a constant $c_q > 0$ such that for all $x, y \in E$

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q. \tag{1.3}$$

(3) There exists a constant d_q such that for all $x, y \in E$ and $t \in [0, 1]$

$$\|(1 - t)x + ty\|^q \geq (1 - t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x - y\|^q, \tag{1.4}$$

where $\omega_q(t) = t^q(1 - t) + t(1 - t)^q$.

Furthermore, it was shown in [12, Remark 5] that if E is $q-$uniformly smooth ($q > 1$), then for all $x, y \in E$, there exists a constant $L_\star > 0$ such that

$$\|J_q(x) - J_q(y)\| \leq L_\star\|x - y\|^{q-1}. \tag{1.5}$$

Recently, Osilike and Udomene [13] improved, unified and developed the above Theorem 1.1 and Browder and Petryshyn's corresponding result [1] in two aspects: (i) Hilbert spaces are extended to the setting of $q-$uniformly smooth Banach spaces ($q > 1$); (ii) Mann iterative process is extended to the case of Ishikawa iterative process.

Theorem 1.3 [13, Theorem 2]. Let E be a real $q-$uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and $T : K \to K$ be a strictly pseudocontractive mapping with a nonempty fixed-point set $F(T)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ satisfying the conditions:

(i) $0 < a \leq \alpha_n^{q-1} \leq b < (q\lambda^{q-1}/c_q)(1 - \beta_n), \forall n \geq 1$ and for some constants $a, b \in (0, 1)$;

(ii) $\sum_{n=1}^{\infty} \beta_n^{\tau} < \infty$, where $\tau = \min\{1, (q - 1)\}$.

If $\{x_n\}$ is the sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iterative process

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, n \geq 1. \end{cases}$$

then $\{x_n\}$ converges weakly to a fixed point of T.

Definition 1.2. A mapping T with domain $D(T)$ and range $R(T)$ in E is called demiclosed at a point p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx = p$. Furthermore, T is called demicompact if whenever $\{x_n\}$ is a bounded sequence in $D(T)$ such that $\{x_n - Tx_n\}$ converges strongly, then $\{x_n\}$ has a subsequence which converges strongly.

Theorem 1.4 [13, Corollary 2]. Let E be a real $q-$uniformly smooth Banach space, K be a nonempty closed convex subset of E, $T : K \to K$ be a demicompact strictly pseudocontractive mapping with a nonempty fixed-point set $F(T)$. Let $\{\alpha_n\}$,$\{\beta_n\}$ and $\{x_n\}$ be as in Theorem

1.3. Then $\{x_n\}$ converges strongly to a fixed point of T.

Let E be a real $q-$uniformly smooth Banach space, K be a nonempty closed convex (not necessarily bounded) subset of E with $K+K \subseteq K$, and $T: K \to K$ be a strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K and $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0,1]$ satisfying certain restrictions. Let $\{x_n\}$ be the sequence generated from $x_1 \in K$ by the Ishikawa iterative process with errors:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, n \geq 1. \end{cases} \tag{1.6}$$

In this paper, we establish the sufficient and necessary conditions for the strong convergence of $\{x_n\}$ to a fixed point of T. It is worth noting that comparing with [13,Theorem 2 and Corollary 2] our results have the following features: (i) The uniform convexity assumption on E is removed. (ii) The Ishikawa iterative process is replaced by the Ishikawa iterative process with errors. (iii) Our restrictions imposed on $\{\alpha_n\}$ are much weaker than those in [13, Theorem 2 and Corollary 2]. (iv) We establish the sufficient and necessary conditions on the strong convergence of the Ishikawa iterative process with errors. Furthermore, our results also improve and extend the corresponding results in [1, 9]

Now, we give some preliminaries which will be used in the sequel.

From (1.2) we have

$$\|x-y\| \geq \lambda\|x-y-(Tx-Ty)\| \geq \lambda\|Tx-Ty\| - \lambda\|x-y\|.$$

so that $\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in K$, where $L = (1 + \lambda)/\lambda$. Since

$$\|x - y\| \geq \lambda\|x - y - (Tx - Ty)\|,$$

we have

$$\begin{aligned} \langle x - Tx - (y - Ty), j_q(x - y)\rangle &= \|x - y\|^{q-2}\langle x - Tx - (y - Ty), j(x - y)\rangle \\ &\geq \lambda\|x - y\|^{q-2}\|x - Tx - (y - Ty)\|^2 \\ &\geq \lambda^{q-1}\|x - Tx - (y - Ty)\|^q. \qquad (1.7) \end{aligned}$$

Lemma 1.1 [10]. Let$\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and $a_{n+1} \leq a_n + b_n, \forall n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.

2. Main Results

Throughout this section, λ denotes the constant appearing in (1.1). L stands for the Lipschitz constant of T, and $c_q, d_q, \omega_q(t)$, and $L_\star$ are the constants appearing in inequalities (1.3)-(1.5), respectively.

Lemma 2.1. Let $q > 1$ and E be a real $q-$uniformly smooth Banach space and K be a non-empty convex subset of E with $K + K \subseteq K$, and $T : K \to K$ be a strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in K,and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$ satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, (ii) $\alpha_n \leq \lambda(q/c_q)^{1/(q-1)}$, and $\sum_{n=1}^{\infty} \beta_n^{\tau} < \infty$ where $\tau = \min\{1, (q - 1)\}$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iterative process (1.6) with errors. Then

(i) $\|x_{n+1} - x^\star\|^q \leq (1 + \delta_n)\|x_n - x^\star\|^q + \theta_n, \forall n \geq 1, \forall x^\star \in F(T)$,

where

$$\delta_n = 2\alpha_n\beta_n\lambda^{q-1}qd_q(1+L)^q + \alpha_n\beta_n^{q-1}qL_\star(1+L)^{q+1} + \alpha_n\beta_nq\lambda^{q-1}(1+L^2)^q$$

and

$$\theta_n = q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q.$$

(ii) There exists a constant $M > 0$ (e.g., $M = e^{\sum_{n=1}^{\infty}\delta_n}$) such that

$$\|x_{n+m} - x^\star\|^q \leq M\|x_n - x^\star\|^q + M\sum_{k=n}^{n+m-1}\theta_k, \forall n, m \geq 1, \forall x^\star \in F(T).$$

Proof. (i) For each $n \geq 1$, from (1.6) we obtain

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_nTy_n + u_n. \tag{2.1}$$

Let $x^\star$ be an arbitrary element in $F(T)$. Then it follows from (1.3) that

$$\begin{aligned}\|x_{n+1} - x^\star\|^q &= \|(1-\alpha_n)x_n + \alpha_nTy_n + u_n - x^\star\|^q \\ &\leq \|(1-\alpha_n)x_n + \alpha_nTy_n - x^\star\|^q + q\langle u_n, j_q(x_{n+1} - u_n - x^\star)\rangle + c_q\|u_n\|^q \\ &\leq \|(1-\alpha_n)x_n + \alpha_nTy_n - x^\star\|^q + q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q. \quad (2.2)\end{aligned}$$

Observe that

$$\begin{aligned}\|(1-\alpha_n)x_n + \alpha_nTy_n - x^\star\|^q &= \|x_n - x^\star - \alpha_n(x_n - Ty_n)\|^q \\ &\leq \|x_n - x^\star\|^q - q\alpha_n\langle x_n - Ty_n, j_q(x_n - x^\star)\rangle \\ &\quad + \alpha_n^qc_q\|x_n - Ty_n\|^q, \quad (2.3)\end{aligned}$$

$$\begin{aligned}\langle x_n - Ty_n, j_q(x_n - x^\star)\rangle &= \langle x_n - y_n, j_q(x_n - x^\star)\rangle + \langle y_n - Ty_n, j_q(x_n - x^\star)\rangle \\ &= \beta_n\langle x_n - Tx_n - (x^\star - Tx^\star), j_q(x_n - x^\star)\rangle \\ &\quad + \langle y_n - Ty_n, j_q(x_n - x^\star)\rangle \\ &\geq \beta_n\lambda^{q-1}\|x_n - Tx_n - (x^\star - Tx^\star)\|^q \\ &\quad + \langle y_n - Ty_n, j_q(x_n - x^\star)\rangle \\ &= \beta_n\lambda^{q-1}\|x_n - Tx_n\|^q + \langle y_n - Ty_n, j_q(x_n - x^\star)\rangle,\end{aligned}$$

and by (1.7)

$$\begin{aligned}\langle y_n - Ty_n, j_q(x_n - x^\star)\rangle &= \langle y_n - Ty_n - (x^\star - Tx^\star), j_q(x_n - x^\star) - j_q(y_n - x^\star)\rangle \\ &\quad + \langle y_n - Ty_n - (x^\star - Tx^\star), j_q(y_n - x^\star)\rangle \\ &\geq \lambda^{q-1}\|y_n - Ty_n - (x^\star - Tx^\star)\|^q \\ &\quad + \langle y_n - Ty_n - (x^\star - Tx^\star), j_q(x_n - x^\star) - j_q(y_n - x^\star)\rangle \\ &= \lambda^{q-1}\|y_n - Ty_n\|^q \\ &\quad + \langle y_n - Ty_n - (x^\star - Tx^\star), j_q(x_n - x^\star) - j_q(y_n - x^\star)\rangle. \quad (2.4)\end{aligned}$$

Moreover, by using (1.4), we have

$$\begin{aligned}\|y_n - Ty_n\|^q &= \|(1-\beta_n)(x_n - Ty_n) + \beta_n(Tx_n - Ty_n)\|^q \\ &\geq (1-\beta_n)\|x_n - Ty_n\|^q + \beta_n\|Tx_n - Ty_n\|^q - \omega_q(\beta_n)d_q\|x_n - Tx_n\|^q. \quad (2.5)\end{aligned}$$

Then it follows from (2.2)-(2.5) that

$$\begin{aligned}\|x_{n+1} - x^\star\|^q &\leq \|x_n - x^\star\|^q - q\alpha_n\{\beta_n\lambda^{q-1}\|x_n - Tx_n\|^q + \lambda^{q-1}(1-\beta_n)\|x_n - Ty_n\|^q \\ &\quad + \lambda^{q-1}\beta_n\|Tx_n - Ty_n\|^q - \lambda^{q-1}\omega_q(\beta_n)d_q\|x_n - Tx_n\|^q \\ &\quad + \langle y_n - Ty_n, j_q(x_n - x^\star) - j_q(y_n - x^\star)\rangle\} \\ &\quad + \alpha_n^q c_q\|x_n - Ty_n\|^q + q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q \\ &\leq \|x_n - x^\star\|^q - \alpha_n(q\lambda^{q-1}(1-\beta_n) - \alpha_n^{q-1}c_q)\|x_n - Ty_n\|^q \\ &\quad + qd_q\lambda^{q-1}\alpha_n\omega_q(\beta_n)\|x_n - Tx_n\|^q \\ &\quad + q\alpha_n\|y_n - Ty_n\|\|j_q(x_n - x^\star) - j_q(y_n - x^\star)\| \\ &\quad + q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q.\end{aligned}$$

Also, observe that

$$\omega_q(\beta_n) = \beta_n(1-\beta_n)^q + \beta_n^q(1-\beta_n) \leq 2\beta_n,$$

$$\|x_n - Tx_n\| \leq (1+L)\|x_n - x^\star\|,$$

$$\begin{aligned}\|j_q(x_n - x^\star) - j_q(y_n - x^\star)\| &\leq L_\star\beta_n^{q-1}\|x_n - Tx_n\|^{q-1} (\text{using (1.5)}) \\ &\leq L_\star(1+L)^{q-1}\beta_n^{q-1}\|x_n - x^\star\|^{q-1},\end{aligned}$$

and

$$\begin{aligned}\|y_n - Ty_n\| &\leq (1+L)\|y_n - x^\star\| \\ &\leq (1+L)[(1-\beta_n)\|x_n - x^\star\| + \beta_n L\|x_n - x^\star\|] \\ &\leq (1+L)^2\|x_n - x^\star\|.\end{aligned}$$

Consequently, we have

$$\begin{array}{rcl}\|x_{n+1}-x^\star\|^q & \leq & [1+2\alpha_n\beta_n\lambda^{q-1}qd_q(1+L)^q+\alpha_n\beta_n^{q-1}qL_\star(1+L)^{q+1}]\|x_n-x^\star\| \\ & & -\alpha_n[q\lambda^{q-1}(1-\beta_n)-\alpha_n^{q-1}c_q]\|x_n-Ty_n\|^q \\ & & +q\|u_n\|\|x_{n+1}-u_n-x^\star\|^{q-1}+c_q\|u_n\|^q.\end{array}$$

Since $\alpha_n \leq \lambda(q/c_q)^{1/(q-1)}$, and

$$\begin{array}{rcl}\|x_n-Ty_n\| & \leq & \|x_n-x^\star\|+L\|x^\star-y_n\| \\ & \leq & \|x_n-x^\star\|+L[(1-\beta_n)\|x_n-x^\star\|+\beta_nL\|x_n-x^\star\|] \\ & \leq & (1+L^2)\|x_n-x^\star\|,\end{array}$$

we conclude that (i) is valid.

(ii) It follows from conclusion (i) that for all $n,m\geq 1$ and $x^\star\in F(T)$

$$\begin{array}{rcl}\|x_{n+m}-x^\star\|^q & \leq & (1+\delta_{n+m-1})\|x_{n+m-1}-x^\star\|^q+\theta_{n+m-1} \\ & \leq & (1+\delta_{n+m-1})(1+\delta_{n+m-2})\|x_{n+m-2}-x^\star\|^q \\ & & +(1+\delta_{n+m-1})\theta_{n+m-2}+\theta_{n+m-1} \\ & \leq & (1+\delta_{n+m-1})(1+\delta_{n+m-2})(1+\delta_{n+m-3})\|x_{n+m-3}-x^\star\|^q \\ & & +(1+\delta_{n+m-1})(1+\delta_{n+m-2})\theta_{n+m-3}+(1+\delta_{n+m-1})\theta_{n+m-2}+\theta_{n+m-1} \\ & \leq & \ldots \\ & \leq & e^{\sum_{k=n}^{n+m-1}\delta_k}\|x_n-x^\star\|^q+e^{\sum_{k=n}^{n+m-1}\delta_k}\sum_{k=n}^{n+m+1}\theta_k \\ & \leq & M\|x_n-x^\star\|^q+M\sum_{k=n}^{n+m+1}\theta_k,\end{array}$$

where $M=e^{\sum_{k=1}^{\infty}\delta_k}$. This shows that conclusion (ii) is also valid.

Theorem 2.1. Let $q>1$ and E be a real $q-$uniformly smooth Banach space, K be a nonempty closed convex subset of E with $K+K\subseteq K$, and $T:K\to K$ be a strictly pseudocontractive mapping with $F(T)\neq\emptyset$. Let $\{u_n\}$ be a bounded sequence in K. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty}\|u_n\|<\infty$;

(ii) $\alpha_n\leq\lambda(q/c_q)^{1/(q-1)}$, and $\sum_{n=1}^{\infty}\beta_n^{\tau}<\infty$, where $\tau=\min\{1,(q-1)\}$.

Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iterative process (1.6) with errors. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\{x_n\}$ is bounded and

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0$$

where $d(x_n, F(T))$ is the distance of x_n to set $F(T)$, i.e., $d(x_n, F(T)) = \inf_{u^\star \in F(T)} d(x_n, u^\star)$.

Proof "Necessity". Suppose that $\{x_n\}$ converges strongly to a fixed point of T, say, $y^\star \in F(T)$. Then it is clear that $\{x_n\}$ is bounded. Note that

$$d(x_n, F(T)) = \inf_{u^\star \in F(T)} d(x_n, u^\star) \leq d(x_n, y^\star) \quad \text{as } n \to \infty.$$

Therefore,

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0$$

"Sufficiency". Suppose that $\{x_n\}$ is bounded and that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. First, from Lemma 2.1(i), we obtain

$$\|x_{n+1} - x^\star\|^q \leq (1+\delta_n)\|x_n - x^\star\|^q + \theta_n, \quad n \geq 1, x^\star \in F(T),$$

where

$$\delta_n = 2\alpha_n\beta_n\lambda^{q-1}qd_q(1+L)^q + \alpha_n\beta_n^{q-1}qL_\star(1+L)^{q+1} + \alpha_n\beta_nq\lambda^{q-1}(1+L)^{2q},$$

and

$$\theta_n = q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q, \quad \forall n \geq 1.$$

Since $\sum_{n=1}^{\infty}\|u_n\| < \infty$, $\sum_{n=1}^{\infty}\|u_n\|^q < \infty$. Note that $\{x_n\}$ and $\{u_n\}$ are both bounded. Thus, there is a number $\tilde{M} > 0$ such that $\|x_{n+1} - u_n - x^\star\| \le \tilde{M}$, and $\|x_n - x^\star\| \le \tilde{M}, \forall n \ge 1$. Hence,

$$\sum_{n=1}^{\infty}\theta_n = \sum_{n=1}^{\infty}(q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q)$$
$$\le q\tilde{M}^{q-1}\sum_{n=1}^{\infty}\|u_n\| + c_q\sum_{n=1}^{\infty}\|u_n\|^q < \infty.$$

On the other hand, it follows from condition (ii) that $\sum_{n=1}^{\infty}\delta_n\tilde{M}^q < \infty$. Also, observe that

$$\|x_{n+1} - x^\star\|^q \le (1+\delta_n)\|x_n - x^\star\|^q + \theta_n \le \|x_n - x^\star\|^q + \delta_n\tilde{M}^q + \theta_n. \quad (2.6)$$

This implies that

$$(d(x_{n+1}, F(T)))^q \le [d(x_n, F(T))]^q + \delta_n\tilde{M}^q + \theta_n.$$

By Lemma 1.1, we infer that $\lim_{n\to\infty}(d(x_n, F(T))^q$ exists, that is, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Since $\lim_{n\to\infty}\inf d(x_n, F(T)) = 0$, we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Now we claim that $\{x_n\}$ is a Cauchy sequence. Indeed, according to Lemma 2.1(ii), we deduce that there exists a constant $M > 0$ such that

$$\|x_{n+m} - x^\star\| \le M\|x_n - x^\star\|^q + M\sum_{k=n}^{n+m+1}\theta_k, \forall n, m \ge 1, x^\star \in F(T).$$

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty}\theta_n < \infty$, for an arbitrary $\varepsilon > 0$, there exists an integer $N_1 \ge 1$ such that for all $n \ge N_1$

$$d(x_n, F(T)) < (\frac{\varepsilon}{3M})^{1/q}\cdot\frac{1}{2^{(q-1)/q}}, \text{ and } \sum_{k=n}^{\infty}\theta_k < \frac{\varepsilon}{6M}\cdot\frac{1}{2^{q-1}}.$$

Hence, $d(x_{N_1}, F(T)) < (\frac{\varepsilon}{3M})^{1/q} \cdot \frac{1}{2^{(q-1)/q}}$. This implies that there exists an $x_1^\star \in F(T)$ such that

$$d(x_{N_1}, x_1^\star) < (\frac{\varepsilon}{3M})^{1/q} \cdot \frac{1}{2^{(q-1)/q}}.$$

In view of Jensen's Inequality [2, p.183], we conclude that

$$\|x_{n+m} - x_n\|^q \leq 2^{q-1}(\|x_n - x_1^\star\|^q + \|x_{n+m} - x_1^\star\|^q). \tag{2.7}$$

Since for all $n \geq N_1$, we have

$$\begin{array}{rcl} \|x_n - x_1^\star\|^q & \leq & M\|x_{N_1} - x_1^\star\|^q + M\sum_{k=N_1}^{n}\theta_k \\ & \leq & M\|x_{N_1} - x_1^\star\|^q + M\sum_{k=N_1}^{\infty}\theta_k \\ & \leq & M\frac{\varepsilon}{3M} \cdot \frac{1}{2^{(q-1)}} + M\frac{\varepsilon}{6M} \cdot \frac{1}{2^{q-1}} \\ & = & \frac{\varepsilon}{2} \cdot \frac{1}{2^{q-1}}, \end{array}$$

and

$$\begin{array}{rcl} \|x_{n+m} - x_1^\star\|^q & \leq & M\|x_{N_1} - x_1^\star\|^q + M\sum_{k=N_1}^{n+m-1}\theta_k \\ & \leq & M\|x_{N_1} - x_1^\star\|^q + M\sum_{k=N_1}^{\infty}\theta_k \\ & \leq & M\frac{\varepsilon}{3M} \cdot \frac{1}{2q-1} + M\frac{\varepsilon}{6M} \cdot \frac{1}{2^{q-1}} \\ & = & \frac{\varepsilon}{2} \cdot \frac{1}{2^{q-1}}, \end{array}$$

so, from (2.7), we get

$$\|x_{n+m} - x_n\|^q \leq 2^{q-1}(\frac{\varepsilon}{2} \cdot \frac{1}{2^{q-1}} + \frac{\varepsilon}{2} \cdot \frac{1}{2^{q-1}}) = \varepsilon, \quad \forall n \geq N_1, m \geq 1.$$

This shows that $\{x_n\}$ is Cauchy sequence. Since the space E is complete, $\lim_{n\to\infty} x_n$ exists. Thus, we may assume that $\lim_{n\to\infty} x_n = u^\star$.

Next, we claim that $u^\star$ is a fixed point of T, i.e., $u^\star \in F(T)$. Indeed, since $d(u^\star, F(T)) = 0$, for any $\varepsilon > 0$, there is $z^\star \in F(T)$ such that $\|u^\star - z^\star\| < \varepsilon$. Then we have

$$\begin{array}{rcl} \|Tu^\star - u^\star\| & \leq & \|Tu^\star - Tz^\star\| + \|u^\star - z^\star\| \\ & \leq & (1+L)\varepsilon. \end{array}$$

As ε is arbitrary, we conclude that $Tu^\star = u^\star$ and hence $u^\star$ is a fixed point of T. This completes the proof of Theorem 2.1.

Theorem 2.2. Let $q > 1$ and E be a real $q-$uniformly smooth Banach space, K be a nonempty closed convex subset of E with $K + K \subseteq K$, and $T : K \to K$ be a strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K, and $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \|u_n\| < \infty$;

(ii) $\alpha_n \leq \lambda(q/c_q)^{1/(q-1)}$, and $\sum_{n=1}^{\infty} \beta_n^\tau < \infty$, where $\tau = \min\{1, (q-1)\}$.

Let $\{x_n\}$ be the sequence generated from arbitrary $x_1 \in K$ by the Ishikawa iterative process (1.6) with errors. Then $\{x_n\}$ converges strongly to a fixed point $u^\star$ of T if and only if $\{x_n\}$ is bounded and $\{x_n\}$ has a subsequence which is strongly convergent to a fixed point $u^\star$ of T.

Proof. The conclusion of Theorem 2.2 follows immediately from Theorem 2.1.

Remark 2.1. If we take $\beta_n = 0, \forall n \geq 1$ in Theorems 2.1 and 2.2, then we can obtain the corresponding results on the strong convergence of the Mann iterative process with errors

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n + u_n, \quad \forall n \geq 1.$$

In addition, if we take $u_n = 0, \forall n \geq 1$ in (1.6), then Theorems 2.1 and 2.2 are still valid under the lack of the assumption that $\{x_n\}$ is bounded. Indeed, when $u_n = 0, \forall n \geq 1$, it follows from Lemma 2.1(i) that

$$\|x_{n+1} - x^\star\|^q \le (1+\delta_n)\|x_n - x^\star\|^q \le e^{\sum_{j=1}^{n}\delta_j}\|x_1 - x^\star\|^q \le e^{\sum_{j=1}^{\infty}\delta_j}\|x_1 - x^\star\|^q < \infty.$$

This shows that $\{x_n\}$ is bounded.

Remark 2.2. Recall that Ishikawa iterative process with errors introduced by Liu [3] is stated as follows: Let K be a nonempty convex subset of E with $K + K \subseteq K$. For any given $x_1 \in K$, the sequence $\{x_n\}$ is defined by the iterative scheme

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, n \ge 1, \end{cases}$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K, and $\{\alpha_n\}$ as well as $\{\beta_n\}$ are real sequences in $[0,1]$. Naturally, one may ask the following open question.

Open Question: Are Theorems 2.1 and 2.2 extendable to the case of the Ishikawa iterative process with errors in the sense of Liu [3]?

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