SOLVING VARIATIONAL INEQUALITIES INVOLVING NONEXPANSIVE TYPE MAPPINGS

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ABSTRACT. The purpose of this paper is to investigate the asymptotic behavior of algorithms for finding solutions for a certain class of variational inequalities $VI_D(C, I-f)$ involving nonexpansive type mappings in smooth Banach spaces. We study existence of solutions of variational inequalities $VI_D(C, I-f)$ when D is the set of solutions of zeros of accretive operators or the set of fixed points of nonexpansive mappings or the set of fixed points of pseudocontractive mappings. Our convergence analysis covers proximal point algorithm for finding zeros accretive operators as well as functional Helpern algorithm for finding fixed points of nonexpansive mappings in Banach spaces. Our results improve a number of results concerned with viscosity approximation methods in the context of weakly contraction mappings.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$, respectively. Let C be a nonempty closed convex subset of H and $A : C \rightarrow H$ a nonlinear mapping. A is said to be *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0$ for all $x, y \in C$. A monotone operator A is said to be *maximal monotone* if its graph is not properly contained in the graph of any other monotone operator on H.

The variational inequality $VI(C, A)$ is formulated as finding a point $z \in C$ such that

$$\langle Az, z - v \rangle \geq 0 \text{ for all } v \in C.$$

The variational inequalities were initially studied by Stampachhia [10, 12] and ever since have been widely studied. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in H$ satisfying $0 = Au$ and so on.

Existence and approximation of solutions are important aspects of study of variational inequalities. It is well known that if A is Lipschitzian and strongly monotone, then for small $\mu > 0$, the mapping $P_C(I - \mu A)$ is a contraction. In this case, the Banach contraction principle guarantees that $VI(C, A)$ has unique solution x^* and

the sequence of Picard iteration process, given by, $x_{n+1} = P_C(I - \mu A)x_n$, converges strongly to x^*, where P_C is the metric projection from H onto C.

Throughout this paper X is a real Banach space. Let C be a convex subset of a smooth Banach space X, D a nonempty subset of C and $f : C \to C$ a mapping. We consider the following variational inequality $VI_D(C, I - f)$:

to find a $z \in D$ such that $\langle (I - f)z, J(z - v) \rangle \leq 0$ for all $v \in D$,

where J is the duality mapping from X into X^*.

The set of solutions of the variational inequality $VI_D(C, I - f)$ is denoted by $\Omega_D(I - f)$, i.e.,

$$\Omega_D(I - f) = \{u \in C : \langle (I - f)z, J(z - v) \rangle \leq 0 \text{ for all } v \in D\}.$$

We denote $F(T)$ the set of fixed points of mapping $T : C \to C$.

The viscosity approximation method is one of the important methods for existence and approximation solutions of variational inequalities $VI_D(C, I - f)$. The viscosity approximation method was first discussed by Moudafi [16] as below:

Theorem 1.1. *(Theorem 2.1, Moudafi [16]) Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping and $f : C \to C$ a contraction mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by the scheme:*

$$x_n = \frac{1}{1 + \varepsilon_n} T x_n + \frac{\varepsilon_n}{1 + \varepsilon_n} f x_n \text{ for all } \in \mathbb{N},$$

where ε_n is a sequence $(0, 1)$ with $\varepsilon_n \to 0$. Then $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

to find a $\tilde{x} \in D$ such that $\langle (I - f)\tilde{x}, \tilde{x} - x \rangle \leq 0$ for all $x \in F(T)$,

where I is the identity mapping. In other word, $\tilde{x}$ is the unique fixed point of $P_{F(T)}f$.

Xu [24] extended the viscosity approximation method proposed by Moudafi [16] for a nonexpansive mapping in a uniformly smooth Banach space. If Π_C denotes the set of all contractions on C, then he proved the following theorem.

Theorem 1.2. *(Theorem 4.1, Xu [24]) Let C be a nonempty closed convex subset of a uniformly smooth Banach space X, $f \in \Pi_C$ and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then the path $\{x_t : t \in (0, 1)\}$ defined by*

(1.1) $$x_t = t f x_t + (1 - t) T x_t$$

converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \to F(T)$ by

$$Q(f) = \lim_{t \to 0^+} x_t, \quad f \in \Pi_C,$$

then $Q(f)$ solves the variational inequality:

$$\langle (I - f)Q(f), J(Q(f) - v) \rangle \leq 0, \quad f \in \Pi_C \text{ and } v \in F(T).$$

There are already several viscosity-like methods in Hilbert and Banach spaces, and the research is intensively continued which are very useful for approximating to the common element of $F(T)$ and $\Omega_D(I - f)$, when f is a contraction mapping and T is a nonexpansive mapping. In such viscosity methods, the contraction mapping

f not only guarantees the existence of approximating curve $\{x_t\}$ defined by (1.1), but it also ensures the strong convergence of both $\{x_t\}$ and $\{x_n\}$ to an element of $F(T) \cap \Omega_D(I - f)$, where $\{x_n\}$ is a sequence in C generated by the functional Halpern iteration process:

$$(1.2) \qquad x_{n+1} = \alpha_n f x_n + (1 - \alpha_n) T x_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in [0,1] satisfying appropriate conditions. When $fx = u$ for all $x \in C$, then f is a contraction with the Lipschitz constant 0 and (1.2) reduces to $x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n$ which was first studied by Halpern [7].

At this stage, the following natural question arises:

Question 1.3. *Let C be a nonempty closed convex subset of a smooth Banach space X, D a nonempty closed convex subset of C and $f : C \to C$ a nonexpansive mapping. Under what conditions $\Omega_D(I - f) \neq \emptyset$?*

On the other hand, a variety of problems, including convex programming and variational inequalities, can be formulated as finding of zeros of maximal monotone operators. Therefore, one of the most interesting and important problems in the theory of maximal monotone operators is to find an efficient iterative algorithm to compute approximately zeroes of maximal monotone operators. One method for solving zeros of maximal monotone operators is proximal point algorithm. Let T be a maximal monotone operator in a Hilbert space H. The proximal point algorithm generates, for starting $x_1 \in H$, a sequence $\{x_n\}$ in H by

$$(1.3) \qquad x_{n+1} = (I + r_n T)^{-1} x_n \text{ for all } n \in \mathbb{N},$$

where $\{r_n\}$ is a sequence in $(0, \infty)$. Note that (1.3) is equivalent to

$$0 \in \frac{1}{r_n}(x_{n+1} - x_n) + T x_{n+1} \text{ for all } \in \mathbb{N}.$$

This was first introduced by Martinet [15]. If $f : H \to (\infty, \infty]$ is a proper lower semicontinuous convex function, then the algorithm reduces to

$$x_{n+1} = arg\min_{y \in H} \left\{ f(y) + \frac{1}{2r_n} \|x_n - y\|^2 \right\} \text{ for all } n \in \mathbb{N}.$$

Rockafellar [20] studied the proximal point algorithm in the framework of Hilbert space and he proved the following:

Theorem 1.4. *Let H be a Hilbert space, $A \subset H \times H$ a maximal monotone operator and $J_r = (I + rA)^{-1}$ for all $r > 0$. Let $\{x_n\}$ be a sequence in H defined by $x_1 = x \in H$ and*

$$(1.4) \qquad x_{n+1} = J_{r_n} x_n \text{ for all } n \in \mathbb{N},$$

where $\{r_n\}$ is a sequence in $(0, \infty)$ such that $\liminf_{n \to \infty} r_n > 0$. If $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Gulär [4] constructed a counterexample showing that the sequence generated by (1.4) does not converge strongly, in general. This brings us a natural question how to modify the proximal point algorithm so that strongly convergent sequence is guaranteed. Recently, Benavides, Acedo and Xu [2], Kamimura and Takahashi [8], Kim and Xu [9], Mainge [13], Nakajo [17] and Solodov and Svaiter [21] modified proximal point algorithm to generate strongly convergent sequences. Now our concern is the following:

Question 1.5. *Is it possible to modify the proximal point algorithm by noncontraction viscosity method so that it can generate a strongly convergent sequence?*

It is our purpose in this paper to give affirmative answers of Questions 1.3 and 1.5. In Section 2, we will recall the useful definitions and lemmas. Section 3 is devoted to deal with the problem of existence of common element of D and $\Omega_D(I - f)$ when f is a weakly contraction. Section 4 is focused on existence of solutions of variational inequalities $VI_D(C, I - f)$, under the hypothesis of nonexpansiveness or pseudocontractivity of the mapping f. The results of this section are of interest in their own right in the constructive fixed point theory. Section 5 focuses on iterative algorithms for finding solutions of variational inequalities $VI_D(C, I - f)$ when f is a weakly contraction or nonexpansive mapping. Using the established results, we consider the problem of finding a common fixed point of finitely many nonexpansive mappings and the problem of finding a common zero of finitely many accretive operators in Section 6.

2. Preliminaries

Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping. T is called a *Lipschitzian* mapping if there exists a constant $L > 0$ such that $\|Tx - Ty\| \le L\|x - y\|$ for all $x, y \in C$ and L is called *Lipschitz constant* of T. A Lipschitzian mapping with Lipschitz constant L is said to be a *contraction* if $L \in [0, 1)$ and a *nonexpansive* if $L = 1$. T is called *pseudocontractive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2 \text{ for all } x, y \in C,$$

where $J : X \to 2^{X^*}$ is the normalized duality mapping which is defined by

$$J(u) = \{j \in X^* : \langle u, j\rangle = \|u\|^2, \|j\|. = \|u\|\}.$$

An operator T with domain $D(T)$ and range $R(T)$ in a Banach space X is said to be *a weakly contraction* if

$$\|Tx - Ty\| \le \|x - y\| - \psi(\|x - y\|) \text{ for all } x, y \in C,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(0) = 0$, $\psi(t) > 0$ for $t > 0$ and $\lim_{t \to \infty} \psi(t) = \infty$.

Remark 2.1. (1) If $\psi(t) = kt$ for all $t \ge 0$, where $k \in (0, 1)$, then T is a contraction with Lipschitz constant $1 - k$.

(2) If domain $D(T)$ of T is bounded, then the hypothesis $\lim_{t \to \infty} \psi(t) = \infty$ is not necessary.

The concept of weakly contraction mappings was introduced by Alber and Guerre-Delabriere [1] in Hilbert space in 1997. They proved that every weakly contraction mapping has a unique fixed point in a Hilbert space. In 2001, Rhoades [19] proved the following very interesting fixed point theorem which is one of generalizations of weakly contraction principle of Alber and Guerre-Delabriere [1] in metric space setting.

Theorem 2.2. *(Theorem 1, Rhoades [19]) Let (X, d) be a complete metric space and $f : X \to X$ a weakly contraction mapping. Then f has a unique fixed point.*

Theorem 2.3. *(Theorem 2, Rhoades [19]) Let C a nonempty closed convex subset of a Banach space X and $f : C \to C$ a weakly contraction mapping with the function ψ. Then the sequence $\{x_n\}$ degenerated by the algorithm:*

$$x_{n+1} = fx_n \ for \ all \ \in \mathbb{N},$$

converges strongly to p, with the following error estimate:

$$\|x_n - p\| \leq \Phi^{-1}(\Phi(\|x_1 - p\|)) - (n-1),$$

where Φ is defined by antiderivative

$$\Phi(t) = \int \frac{dt}{\psi(t)}, \quad \Phi(0) = 0,$$

and Φ^{-1} is the inverse of Φ.

Recall that an operator A with domain $D(A)$ and range $R(A)$ in a Banach space X is said to be *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there is $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$, where J is the duality mapping from X to the dual space X^*. An accretive operator A is said to satisfy the *range condition* if $\overline{D(A)} \subset R(1 + \lambda A)$ for all $\lambda > 0$. If A is accretive, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda : R(1 + \lambda A) \to D(A)$ by $J_\lambda = (I + \lambda A)^{-1}$. It is called the *resolvent* of A. It is well known that for an accretive operator A which satisfies the range condition , $A^{-1}(0) = F(J_\lambda)$ for all $\lambda > 0$. We also define the *Yosida approximation A_r* by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq |Ax| = inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. An accretive operator A is said to be *m-accretive* if $R(I + rA) = X$ for all $r > 0$.

A continuous mapping T with domain $D(T)$ and range $R(T)$ in a Banach space X is said to be *demicompact* at 0 if for any bounded sequence $\{y_n\}$ in $D(T)$ such that $y_n - Ty_n \to 0$ as $n \to \infty$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y \in D(T)$ such that $y_{n_k} \to y$ as $k \to \infty$.

A closed convex subset C of a Banach space X is said to have *normal structure* if for each closed convex bounded subset D of C which contains at least two points, there exists an element x of D which is not a diametral point of D, i.e.,

$$sup\{\|x - y\| : y \in D\} < diam(D),$$

where $diam(D)$ is the diameter of D.

The following theorem related to the existence of fixed points of nonexpansive mappings was proved in Kirk [11].

Theorem 2.4. *(Kirk's fixed point theorem [11]). Let X be a reflexive Banach space and let C be a nonempty closed convex bounded subset of X which has normal structure. Let T be a nonexpansive mapping from C into itself. Then $F(T)$ is nonempty.*

Recall that a Banach space X is said to be *smooth* provided the limit

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in $S = \{x \in X : \|x\| = 1\}$. In this case, the norm of X is said to be *Gâteaux differentiable*. It is said to be *uniformly Gâteaux differentiable*

if for each $y \in S$, this limit is attained uniformly for $x \in S$. It is well known that X is smooth if and only if any duality mapping on X is single-valued. Also if X has a uniformly Gâteaux differentiable norm, then the duality mapping is norm-to-weak* uniformly continuous on bounded sets. It is also well known that every uniformly smooth space (e.g., L_p space, $1 < p < \infty$) has uniformly Gâteaux differentiable norm (see e.g., [3]).

Let C be a convex subset of a Banach space X, D a nonempty subset of C, and P a retraction from C onto D, that is, $Px = x$ for each $x \in D$. A retraction P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. If the sunny retraction P is also nonexpansive, then D is said to be a *sunny nonexpansive retract* of C. The sunny nonexpansive retraction Q_D from C onto D is unique if X is smooth. The following lemmas will be needed in the sequel.

Lemma 2.5. *(Lemma 13.1, Goebel and Reich [5]) Let C be a convex subset of a smooth Banach space X, D a nonempty subset of C and Q_D a retraction from C onto D. Then the following are equivalent:*

(a) Q_D is a sunny and nonexpansive.

(b) $\langle x - Q_D x, J(z - Q_D x) \rangle \leq 0$ for all $x \in C, z \in D$.

(c) $\langle x - y, J(Q_D x - Q_D y) \rangle \geq \|Q_D x - Q_D y\|^2$ for all $x, y \in C$.

Lemma 2.6. *(Cioranescu [3], p.85) Let C be a nonempty closed convex subset of a reflexive strictly convex Banach space X. Then there exists a unique point $x \in C$ such that $\|x\| = \inf\{\|z\| : z \in C\}$.*

Lemma 2.7. *(Proposition 5.3, Goebel and Reich [5]) Let C be a nonempty closed convex subset of a strictly convex Banach space X and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.*

Lemma 2.8. *Let C be a nonempty closed convex subset of a reflexive strictly convex Banach space X and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then there exists a unique point $v \in F(T)$ such that $\|v\| = \inf\{\|z\| : z \in F(T)\}$.*

Proof. It follows from Lemmas 2.6 and 2.7. □

Lemma 2.9. *(Alber and Guerre-Delabriere [1]) Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of nonnegative real numbers such that $\lim_{n \to \infty} \frac{\beta_n}{\alpha_n} = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{\lambda_n\}$ be a sequence of nonnegative real numbers satisfying the recursive inequality:*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \phi(\lambda_n) + \beta_n \text{ for all } n \in \mathbb{N},$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. Then $\{\lambda_n\}$ converges to zero.

Lemma 2.10. *(Lemma 1, Ha and Jung [6]) Let X be a Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X and $\{x_n\}$ a bounded sequence in X. Let LIM be a Banach limit and $y \in C$. Then*

$$LIM_n \|x_n - y\|^2 = \min_{z \in C} LIM_n \|x_n - z\|^2$$

if and only if

$$LIM_n \langle x - y, J(x_n - y) \rangle \leq 0 \; for \; all \; x \in C.$$

Lemma 2.11. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive mapping. Let LIM be a Banach limit and $\{z_n\}$ a bounded sequence in C such that $z_n - Tz_n \to 0$ as $n \to \infty$. Let*

$$M := \{y \in C : LIM_n \|z_n - y\|^2 = \inf_{x \in C} LIM_n \|z_n - x\|^2\}.$$

Suppose that every closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. Then we have the following:

(a) There exists a point u in M such that $u = Tu$.

(b) If $f : C \to C$ is a weakly contraction mapping with the function ψ and $\langle z_n - f z_n, J(z_n - u)\rangle \le 0$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $\{z_{n_i}\}$ converges strongly to u.

Proof. (a) Define the function $\varphi : C \to \mathbb{R}$ by $\varphi(x) := LIM_n \|z_n - x\|^2, x \in C$. Since X is reflexive, $\varphi(x) \to \infty$ as $\|x\| \to \infty$, and φ is continuous convex function, we have that the set

$$(2.1) \qquad\qquad M := \{y \in C : \varphi(y) = \inf_{x \in C} \varphi(x)\},$$

which is nonempty closed convex and bounded. Furthermore, M is invariant under T. In fact, for each $y \in M$, we have

$$
\begin{aligned}
\varphi(Ty) &= LIM_n \|z_n - Ty\|^2 \\
&\le LIM_n \|Tz_n - Ty\|^2 \\
&\le LIM_n \|z_n - y\|^2 = \varphi(y).
\end{aligned}
$$

So, by the hypothesis, there exists a fixed point u of T in M.

(b) By Lemma 2.10, we have

$$LIM_n \langle z, J(z_n - u)\rangle \le 0 \text{ for all } z \in C.$$

In particular,

$$(2.2) \qquad\qquad LIM_n \langle fu - u, J(z_n - u)\rangle \le 0.$$

Since f is a weakly contraction, we have

$$
\begin{aligned}
\|z_n - u\|^2 &= \langle z_n - f z_n, J(z_n - u)\rangle + \langle f z_n - fu, J(z_n - u)\rangle + \langle fu - u, J(z_n - u)\rangle \\
&\le \langle z_n - f z_n, J(z_n - u)\rangle + \|f z_n - fu\| \, \|z_n - u\| + \langle fu - u, J(z_n - u)\rangle \\
&\le \langle z_n - f z_n, J(z_n - u)\rangle + [\|z_n - u\| - \psi(\|z_n - u\|)]\|z_n - u\| \\
&\quad + \langle fu - u, J(z_n - u)\rangle.
\end{aligned}
$$

Since $\langle z_n - f z_n, J(z_n - u)\rangle \le 0$ for all $n \in \mathbb{N}$, it follows that

$$
\begin{aligned}
\psi(\|z_n - u\|)\|z_n - u\| &\le \langle z_n - f z_n, J(z_n - u)\rangle + \langle fu - u, J(z_n - u)\rangle \\
&\le \langle fu - u, J(z_n - u)\rangle.
\end{aligned}
$$

From (2.2), we obtain

$$LIM_n \psi(\|z_n - u\|)\|z_n - u\| \le 0.$$

Therefore, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \to u$. $\qquad\square$

Lemma 2.12. *Let X be a Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $f : C \to C$ a continuous mapping, $T : C \to C$ a nonexpansive mapping and $\{x_n\}$ a bounded sequence in C such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. Suppose $\{z_t\}$ is a path in C defined by $z_t = tfz_t + (1 - t)Tz_t$, $t \in (0, 1)$ such that $z_t \to z$ as $t \to 0^+$. Then*

$$\limsup_{n \to \infty} \langle fz - z, J(x_n - z) \rangle \leq 0.$$

Proof. Since

$$z_t - x_n = t(fz_t - x_n) + (1 - t)(Tz_t - x_n),$$

we see that

$$
\begin{aligned}
\|z_t - x_n\|^2 &= t\langle fz_t - x_n, J(z_t - x_n) \rangle + (1 - t)\langle Tz_t - x_n, J(z_t - x_n) \rangle \\
&\leq t[\langle fz_t - z_t, J(z_t - x_n) \rangle + \|z_t - x_n\|^2] \\
&\quad + (1 - t)[\langle Tz_t - Tx_n, J(z_t - x_n) \rangle + \|x_n - Tx_n\| \, \|z_t - x_n\|] \\
&\leq t[\langle fz_t - z_t, J(z_t - x_n) \rangle + \|z_t - x_n\|^2] \\
&\quad + (1 - t)[\|z_t - x_n\|^2 + \|x_n - Tx_n\| \, \|z_t - x_n\|].
\end{aligned}
$$

By boundedness of $\{x_n\}$ and $\{z_t\}$, we have

$$
\begin{aligned}
\langle fz_t - z_t, J(x_n - z_t) \rangle &\leq \frac{1}{t}\|x_n - Tx_n\| \, \|z_t - x_n\| \\
&\leq \frac{1}{t}\|x_n - Tx_n\|K_1
\end{aligned}
$$

for some $K_1 > 0$. Since $x_n - Tx_n \to 0$ as $n \to \infty$, it infers that

$$(2.3) \qquad \limsup_{n \to \infty} \langle fz_t - z_t, J(x_n - z_t) \rangle \leq 0.$$

Further, since $z_t \to z$ as $t \to 0^+$, the set $\{z_t - x_n\}$ is bounded and the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X, it follows that

$$
\begin{aligned}
| \langle fz &- z, J(x_n - z) \rangle - \langle fz_t - z_t, J(x_n - z_t) \rangle | \\
&= | \langle fz - z, J(x_n - z) - J(x_n - z_t) \rangle + \langle fz - z - (fz_t - z_t), J(x_n - z_t) \rangle | \\
&\leq | \langle fz - z, J(x_n - z) - J(x_n - z_t) \rangle | \\
&\quad + \|fz - z - (fz_t - z_t)\| \, \|x_n - z_t\| \to 0 \text{ as } t \to 0^+.
\end{aligned}
$$

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\langle fz - z, J(x_n - z) \rangle < \langle fz_t - z_t, J(x_n - z_t) \rangle + \varepsilon \text{ for all } n \in \mathbb{N} \text{ and } t \in (0, \delta).$$

Using (2.3), we get

$$
\begin{aligned}
\limsup_{n \to \infty} \langle fz - z, J(x_n - z) \rangle &\leq \limsup_{n \to \infty} \langle fz_t - z_t, J(x_n - z_t) \rangle + \varepsilon \\
&\leq \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, we obtain that

$$\limsup_{n \to \infty} \langle fz - z, J(x_n - z) \rangle \leq 0.$$

$\square$

3. VARIATIONAL INEQUALITIES INVOLVING WEAKLY CONTRACTION MAPPINGS

We begin with existence of solutions of $VI_D(C, I - f)$ in a reflexive smooth Banach space when f is a weakly contraction and $D = A^{-1}0$.

Theorem 3.1. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $f : C \to C$ a weakly contraction mapping with the function ψ and $A \subset X \times X$ an accretive operator with resolvent J_t for $t > 0$ such that*

$$(3.1) \qquad \overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$$

and

$$(3.2) \quad E = \{x \in C : x = J_\rho f x \text{ for some } \rho > 0\} \text{ and } f(E) \text{ are bounded.}$$

Suppose that every closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. Then we have the following:

(a) for each $t > 0$, $J_t f$ has a unique fixed point $z_t \in C$;

(b) $\{z_t\}$ converges strongly to $u \in A^{-1}0$ as $t \to \infty$;

(c) u is the unique solution of the variational inequality:

find $\tilde{x} \in A^{-1}0$ such that $\langle (I - f)\tilde{x}, J(\tilde{x} - v) \rangle \leq 0$ for all $v \in A^{-1}0$.

Proof. (a) For each $t > 0$, the mapping $T_t^f : C \to C$ defined by

$$T_t^f x = J_t f x, \ x \in C$$

is a weakly contraction mapping with the function ψ. Indeed, for $x, y \in C$, we have

$$\|T_t^f x - T_t^f y\| = \|J_t f x - J_t f y\| \leq \|fx - fy\| \leq \|x - y\| - \psi(\|x - y\|).$$

By Theorem 2.2, there exists a unique fixed point $z_t \in C$ of T_t^f and hence

$$(3.3) \qquad z_t = J_t f z_t.$$

(b) By (3.2), we see that $\{z_t\}$ is bounded. Since $f(E)$ is bounded, it follows that $\{\|z_t - f z_t\|\}$ is bounded. Hence, for $r > 0$, we have

$$
\begin{aligned}
\|z_t - J_r z_t\| \ &= \ r\|A_r z_t\| \\
&\leq \ r|A z_t| = r|A J_t f z_t| \\
&\leq \ r\|A_t f z_t\| = \frac{r}{t}\|f z_t - J_t f z_t\| = \frac{r}{t}\|z_t - f z_t\| \\
(3.4) \qquad &\leq \ \frac{r}{t}K_2
\end{aligned}
$$

for some $K_2 > 0$. We may assume that $\{t_n\}$ is a sequence in $(0, \infty)$ such that $\lim\limits_{n \to \infty} t_n = \infty$ and $\{z_{t_n}\}$ is bounded. Set $z_n := z_{t_n}$. For $r > 0$, we see from (3.4) that

$$\|z_n - J_r z_n\| \leq \frac{r}{t_n}K_2 \to 0 \text{ as } n \to \infty.$$

By Lemma 2.11 (a), there exists an element $u \in M \cap F(J_r)$. Thus, $A^{-1}0 \neq \emptyset$. Since A is accretive, we obtain from (3.3) that

$$(3.5) \qquad \frac{1}{t}(f z_t - z_t) = \frac{1}{t}(I - J_t)f z_t = A_t f z_t \in A J_t f z_t = A z_t,$$

which implies that

$$\frac{1}{t}\langle fz_t - J_t fz_t, J(z_t - u)\rangle = \langle A_t fz_t, J(z_t - u)\rangle \geq 0.$$

Thus, we have

(3.6) $$\langle z_t - fz_t, J(z_t - u)\rangle \leq 0.$$

It is easy to see from Lemma 2.11 (b) that there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $\{z_{n_i}\}$ converges strongly to u.

In order to prove that $\{z_n\}$ converges strongly to an element of $A^{-1}0$, assume that there is another subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \to \tilde{u}$. Since $z_n - J_r z_n \to 0$, it follows from the continuity of T and the fact $z_{n_i} \to z$ that $\tilde{u} \in F(J_r)$. Using (3.6), we have that

(3.7) $$\langle z_t - fz_t, J(z_t - v)\rangle \leq 0 \text{ for all } v \in F(J_r).$$

By norm to weak* uniform continuity of J, we obtain

$$\langle u - fu, J(u - \tilde{u})\rangle \leq 0$$

and

$$\langle \tilde{u} - f\tilde{u}, J(\tilde{u} - u)\rangle \leq 0.$$

Adding these two inequalities yields that

$$\langle u - \tilde{u} + f\tilde{u} - fu, J(u - \tilde{u})\rangle \leq 0,$$

from which it follows that

$$\begin{aligned}\|u - \tilde{u}\|^2 &\leq& \|fu - f\tilde{u}\|\,\|u - \tilde{u}\| \\ &\leq& [\|u - \tilde{u}\| - \psi(\|u - \tilde{u}\|)]\|u - \tilde{u}\|.\end{aligned}$$

Thus, we see that $u = \tilde{u}$ and hence $\{z_n\}$ converges strongly to u.

We finally prove that the path $\{z_t\}$ converges strongly. Towards this end, we assume that $\{t_{n'}\}$ is another subsequence in $(0, \infty)$ such that $z_{t_{n'}} \to u'$ as $t_{n'} \to \infty$. By (3.4), we obtain $u' \in F(J_r)$. From (3.7), we have that

$$\langle u - fu, J(u - u')\rangle \leq 0 \text{ and } \langle u' - fu', J(u' - u)\rangle \leq 0.$$

We must have $u = u'$. Therefore, $\{z_t\}$ converges strongly to $u \in A^{-1}0$.

(c) Since $z_t \to u \in A^{-1}0$ as $t \to \infty$, it follows from (3.7) that

$$\langle u - fu, J(u - v)\rangle \leq 0 \text{ for all } v \in A^{-1}0.$$

$\square$

Remark 3.2. Similar results can be found in Reich [18], Takahashi and Ueda [23] and Xu [25] when f is constant.

Remark 3.3. If $A^{-1}0$ is nonempty, then the path $\{z_t\}$ defined by (3.3) is bounded. Indeed, for $v \in A^{-1}0$ and $t > 0$, we have

$$\begin{aligned}\|z_t - v\| &\leq& \|J_t fz_t - J_t v\| \leq \|fz_t - v\| \\ &\leq& \|fz_t - fv\| + \|fv - v\| \\ &\leq& \|z_t - v\| - \psi(\|z_t - v\|) + \|fv - v\|,\end{aligned}$$

which implies that

(3.8) $$\psi(\|z_t - v\|) \leq \|fv - v\|.$$

Suppose $\{z_t - v\}$ is not bounded. Then there exists a sequence $\{t_k\}$ in $(0, \infty)$ with $t_k \to \infty$ as $k \to \infty$ such that

$$\|z_{t_k} - v\| > k \text{ for all } k \in \mathbb{N}.$$

Since ψ is nondecreasing and $\lim_{t\to\infty} \psi(t) = \infty$, it follows from (3.8) that

$$\psi(k) < \psi(\|z_{t_k} - v\|) \le \|fv - v\|,$$

a contradiction.

Corollary 3.4. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X and $A \subset X \times X$ an accretive operator such that $A^{-1}0 \ne \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Suppose that every closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. Then we have the following:*

(a) $A^{-1}0$ is a sunny nonexpansive retract of C,

(b) for each $x \in C$, $\{J_t x\}$ converges strongly to $Q_{A^{-1}0} x$ as $t \to \infty$, where $Q_{A^{-1}0}$ is the sunny nonexpansive retraction from C onto $A^{-1}0$.

Proof. (a) In this case the mapping $f : C \to C$ defined by $fz = x_0$ for all $z \in C$ is a weakly contraction. It is easy to see from Theorem 3.1 that $\{z_t = J_t x_0\}$ converges strongly to $u \in A^{-1}0$ as $t \to \infty$.

Now, let $x \in C$. Then $\lim_{t\to\infty} J_t x \in A^{-1}0$ and there exists a mapping Q from C onto $A^{-1}0$ defined by $\lim_{t\to\infty} J_t x = Qx$ since x is an arbitrary element of C. From (3.7), we have

$$\langle z_t - x, J(z_t - v) \rangle \le 0 \text{ for all } v \in A^{-1}0,$$

it follows that

$$\langle Qx - x, J(Qx - v) \rangle \le 0 \text{ for all } v \in A^{-1}0.$$

Therefore, Q is the sunny nonexpansive retraction by Proposition 2.5.

(b) It follows from part (a). $\qquad\square$

We now replace the fixed point property assumption, mentioned in Theorem 3.1 by imposing certain conditions on the space X or on the set C.

Theorem 3.5. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $f : C \to C$ a weakly contraction mapping with the function ψ and $A \subset X \times X$ an accretive operator with $A^{-1}0 \ne \emptyset$ such that $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Suppose that X is strictly convex or C has normal structure. Then we have the following:*

(a) for each $t > 0$, $J_t f$ has a unique fixed point $z_t \in C$;

(b) $\{z_t\}$ converges strongly to $u \in A^{-1}0$ as $t \to \infty$;

(c) u is the unique solution of the variational inequality:

find $\tilde{x} \in A^{-1}0$ such that $\langle (I - f)\tilde{x}, J(\tilde{x} - v) \rangle \le 0$ for all $v \in A^{-1}0$.

Proof. Note that Remark 3.3 implies that $\{z_t\}$ is bounded. Observe that

$$\begin{aligned}
\|fz_t - v\| &\le& \|fz_t - fv\| + \|fv - v\| \\
&\le& \|z_t - v\| - \psi(\|z_t - v\|) + \|fv - v\| \\
&\le& \|z_t - v\| + \|fv - v\|,
\end{aligned}$$

which implies by boundedness of $\{z_t\}$ that $\{fz_t\}$ is bounded.

To be able to use the argument of the proof of Theorem 3.1, we just need to show that the set M defined by (2.1) has a fixed point of J_r. Suppose that X is strictly convex. Since $A^{-1}0 \neq \emptyset$, let $v \in F(J_r)$. Then the set M_0 defined by

$$M_0 = \{u \in M : \|u - v\| = \inf_{x \in M} \|x - v\|\}$$

is a singleton since X is strictly convex. Let $M_0 = \{u_0\}$ for some $u_0 \in M$. Observe that

$$\|J_r u_0 - v\| = \|J_r u_0 - J_r v\| \leq \|u_0 - v\| = \inf_{x \in M} \|x - v\|.$$

Therefore, $J_r u_0 = u_0$.

If C has normal structure, then the Kirk's fixed point theorem implies that J_r has a fixed point in M. We now follow the proof of Theorem 3.1. $\qquad\square$

Next, we derive a result for existence of solutions of $VI_{F(T)}(C, I-f)$ in a reflexive Banach space when D is a set of fixed points of nonexpansive self-mapping T on C and $f : C \to C$ is a weakly contraction.

Theorem 3.6. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $f : C \to C$ a weakly contraction mapping with the function ψ and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose that every closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. Then we have the following:*

(a) For each $t \in (0, 1)$, the mapping defined by $T_t^f : C \to C$ by

$$T_t^f x = tfx + (1-t)Tx, \quad x \in C$$

has a unique fixed point $z_t \in C$ such that $z_t \to u \in F(T)$ as $t \to 0^+$,

(b) u is the unique solution of the variational inequality:

find $\tilde{x} \in F(T)$ such that $\langle (I-f)\tilde{x}, J(\tilde{x} - v)\rangle \leq 0$ for all $v \in F(T)$.

Proof. (a) It is obvious that there exists a unique fixed point z_t of T_t^f such that

(3.9) $$z_t = tfz_t + (1-t)Tz_t.$$

First, we show that $\{z_t\}$ is bounded. Let $v \in F(T)$. Observe that

$$\begin{aligned}
\|z_t - v\| &\leq t\|fz_t - v\| + (1-t)\|Tz_t - v\| \\
&\leq t(\|fz_t - fv\| + \|fv - v\|) + (1-t)\|z_t - v\| \\
&\leq t(\|z_t - v\| - \psi(\|z_t - v\|) + \|fv - v\|) + (1-t)\|z_t - v\|,
\end{aligned}$$

which implies that

$$\psi(\|z_t - v\|) \leq \|fv - v\|.$$

Hence $\{z_t\}$ satisfies (3.8). Using the argument of Remark 3.3, we obtain that $\{z_t\}$ is bounded.

It is well known from Theorem 4.6.4 of Takahashi [22] that if T is nonexpansive, then $A = I - T$ is accretive and $\overline{D(A)} = C \subset \bigcap_{\lambda > 0} R(I + \lambda A)$. Hence A satisfies (3.1). Also from (3.9) we obtain that $z_t + \frac{1-t}{t}(z_t - Tz_t) = fz_t$, which implies that $J_{\mu(t)}fz_t = z_t$, where $\mu(t) = \frac{1-t}{t}$. Therefore, $z_t \to z \in A^{-1}0 = F(T)$ by Theorem 3.5.

(b) It follows from Theorem 3.5. $\qquad\square$

Theorem 3.7. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose that every closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. Then $F(T)$ is the sunny nonexpansive retract of C.*

Corollary 3.8. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $f : C \to C$ a weakly contraction mapping with the function ψ and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. For each $t \in (0,1)$, define $T_t^f : C \to C$ by*

$$T_t^f x = tfx + (1-t)Tx, \quad x \in C.$$

Suppose that X is strictly convex or C has normal structure. Then we have the following:

(a) T_t^f has a unique fixed point $z_t \in C$,

(b) $\{z_t\}$ converges strongly to $u \in F(T)$ as $t \to 0^+$,

(c) u is the unique solution of the variational inequality:

$$\text{find } \tilde{x} \in F(T) \text{ such that } \langle (I - f)\tilde{x}, J(\tilde{x} - v) \rangle \leq 0 \text{ for all } v \in F(T).$$

Remark 3.9. Corollary 3.8 improves a number of results concerning viscosity approximation methods, particularly the results of Moudafi [16] and Xu [24] in the following way:

(1) in variational inequality $VI_{F(T)}(C, I - f)$, the mapping f is not necessarily a contraction,

(2) the underlying space X is not necessarily Hilbert or uniformly smooth.

4. Variational inequalities involving nonexpansive mappings

In this section we give sufficient conditions on X, C and T which provides affirmative answer of Question 1.3.

Theorem 4.1. *Let X be a reflexive strictly convex smooth Banach space. Let C be a nonempty closed convex bounded subset of X with normal structure and let $f, T : C \to C$ be two nonexpansive mappings. If T is demicompact at zero, then we have the following:*

(a) there exists a sequence $\{z_n\}$ in C such that

(4.1) $$z_n = t_n f z_n + (1 - t_n)T z_n \text{ for all } n \in \mathbb{N},$$

where $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$,

(b) the sequence $\{z_n\}$ in C generated by (4.1) converges strongly to an element of $F(T) \cap \Omega_{F(T)}(I - f)$.

Proof. (a) For each $t \in (0,1)$, the mapping $T_t^f : C \to C$ defined by

$$T_t^f x = tfx + (1-t)Tx, \ x \in C$$

is nonexpansive. Indeed, for $x, y \in C$, we have

$$\|T_t^f x - T_t^f y\| \leq t\|fx - fy\| + (1-t)\|Tx - Ty\| \leq \|x - y\|.$$

By Kirk's theorem, $F(T_t^f) \neq \emptyset$ and hence Lemma 2.8 implies that there exists the unique minimum norm solution z_t of equation $x = T_t^f x$, i.e.,

$$(4.2) \qquad\qquad z_t = t f z_t + (1-t) T z_t.$$

Observe that

$$\|z_t - T z_t\| \leq \|f z_t - T z_t\| \leq t \; diam(C) \to 0 \text{ as } t \to 0^+,$$

where $diam(C)$ is the diameter of C. Since T is demicompact at zero, we may assume that $z_{t_n} \to z \in C$ as $n \to \infty$, where $\{t_n\}$ is a sequence in $(0,1)$ such that $\lim_{n \to \infty} t_n = 0$. Set $z_n := z_{t_n}$. Then from (4.2), there exists a sequence in C satisfying (4.1).

(b) Since $z_n - T z_n \to 0$ and $z_n \to z$, it follows that $z \in F(T)$. So, it remains to show that z is a solution of the variational inequality:

find $x \in F(T)$ such that $\langle (I-f)x, J(x-v) \rangle \leq 0$ for all $v \in F(T)$.

Let $v \in F(T)$. Observe that

$$
\begin{aligned}
\langle z_t - T z_t, J(z_t - v) \rangle &= \langle z_t - v + Tv - T z_t, j(z_t - v) \rangle \\
&= \|z_t - v\|^2 - \langle T z_t - Tz, J(z_t - v) \rangle \\
&\geq \|z_t - v\|^2 - \|T z_t - Tv\| \; \|z_t - v\| \\
(4.3) \qquad &\geq 0.
\end{aligned}
$$

Hence from (4.2), we have

$$
\begin{aligned}
\langle z_t - f z_t, J(z_t - v) \rangle &= (1-t)\langle T z_t - f z_t, J(z_t - v) \rangle \\
&\leq (1-t)\langle T z_t - z_t + z_t - f z_t, J(z_t - v) \rangle,
\end{aligned}
$$

which implies form (4.3) that

$$\langle z_t - f z_t, J(z_t - v) \rangle \leq 0.$$

and hence, we get

$$(4.4) \qquad\qquad \langle z_n - f z_n, J(z_n - v) \rangle \leq 0 \text{ for all } n \in \mathbb{N}.$$

Since J is norm-to-weak* continuous, it follows from $z_n \to z$ that $J(z_n - v) \rightharpoonup^* J(z-v)$. Observe that

$$
\begin{aligned}
| \langle z_n &- f z_n, J(z_n - v) \rangle - \langle z - fz, J(z-v) \rangle | \\
&= | \langle z_n - f z_n - (z - fz), J(z_n - v) \rangle + \langle z - fz, J(z_n - v) - J(z-v) \rangle | \\
&\leq \|z_n - f z_n - (z - fz)\| \; \|z_n - v\| + | \langle z - Az, J(z_n - v) - J(z-v) \rangle | \to 0,
\end{aligned}
$$

it follows from (4.4) that

$$\langle z - fz, J(z-v) \rangle = \lim_{n \to \infty} \langle z_n - f z_n, J(z_n - v) \rangle \leq 0.$$

$\square$

It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure.

Theorem 4.2. *Let X be a smooth uniformly convex Banach space. Let C be a nonempty closed convex bounded subset of X and let $f, T : C \to C$ be two nonexpansive mappings. If T is demicompact at zero, then $F(T) \cap \Omega_{F(T)}(I - f) \neq \emptyset$.*

Corollary 4.3. *Let X be a reflexive strictly convex smooth Banach space. Let C be a nonempty compact convex subset of X and let $f, T : C \to C$ be two nonexpansive mappings. Then $F(T) \cap \Omega_{F(T)}(I - f) \neq \emptyset$.*

Next, we consider the variational inequality $VI_{F(T)}(C, I - f)$ for pseudocontractive mappings $f : C \to C$.

Theorem 4.4. *Let X be a reflexive strictly convex smooth Banach space. Let C be a nonempty closed convex bounded subset of X with normal structure, $f : C \to C$ a continuous pseudocontractive mapping and $T : C \to C$ a nonexpansive mapping. If T is demicompact at zero, then there exists a sequence $\{z_n\}$ in C such that*

$$(4.5) \qquad z_n = t_n f z_n + (1 - t_n) T z_n \text{ for all } n \in \mathbb{N},$$

where $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$, and the sequence $\{z_n\}$ in C generated by (4.5) converges strongly to an element of $F(T) \cap \Omega_{F(T)}(I - f)$.

Proof. For each $t \in (0, 1)$, the mapping $T_t^f : C \to C$ defined by

$$T_t^f x = t f x + (1 - t) T x, \ \ x \in C$$

is pseudocontractive. Indeed, for $x, y \in C$, we have

$$\begin{aligned}
\langle T_t^f x - T_t^f y, J(x - y) \rangle &= t \langle f x - f y, J(x - y) \rangle + (1 - t) \langle T x - T y, J(x - y) \rangle \\
&\leq \|x - y\|^2.
\end{aligned}$$

It follows from Theorem 6 of Martin [14] that $g_t^f = (2I - T_t^f)^{-1}$ is a nonexpansive mapping from C into itself with $F(g_t^f) = F(T_t^f)$. By Kirk's theorem, $F(g_t^f) \neq \emptyset$ and hence Lemma 2.8 implies that there exists the unique minimum norm solution z_t of $x = g_t^f x$, i.e.,

$$z_t = t f z_t + (1 - t) T z_t.$$

Similar to the argument of proof of Theorem 4.1, we have a sequence $\{z_n\}$ in C such that

$$z_n = t_n f z_n + (1 - t_n) T z_n \text{ for all } n \in \mathbb{N},$$

where $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$, and the sequence $\{z_n\}$ in C converges strongly to an element of $F(T) \cap \Omega_{F(T)}(I - f)$. □

It is well known that if C is a closed convex subset of a Hilbert space H, then a sunny nonexpansive retraction is coincident with the metric projection mapping P_C from H onto C. This is not true in general, since outside Hilbert space, the metric projection mappings, although sunny, are no longer nonexpansive. From Theorem 3.8 we can derive the following result for existence of sunny nonexpansive retractions (from C onto $F(T)$) in Banach spaces.

Theorem 4.5. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that X is strictly convex or C has normal structure. Then $F(T)$ is a sunny nonexpansive retract of C.*

We now turn our attention to deal with the problem of existence of solutions of $VI_D(C, I - f)$ by sunny nonexpansive retractions.

Let C be a convex subset of a smooth Banach space X and D a nonempty subset of C. Let Q_D be the sunny nonexpansive retraction from C onto D. Let $x \in C$ and $x_0 \in D$. Then from Lemma 2.5, we have

(4.6) $x_0 = Q_D x$ if and only if $\langle x_0 - x, J(x_0 - v) \rangle \leq 0$ for all $v \in D$.

Following relation (4.6), we can show that the variational inequality $VI_D(C, I - f)$ is equivalent to the fixed point problem.

Proposition 4.6. *Let C be a convex subset of a smooth Banach space X and D a nonempty subset of C. Let $f : C \to C$ be a mapping and let Q_D be the sunny nonexpansive retraction from C onto D. Then we have the following:*

(a) $\Omega_D(I - f) = F(Q_D f)$.

(b) If $f : C \to C$ is a Lipschitzian mapping with Lipschitz constant L, then $Q_D f$ is also a Lipschitzian mapping with the same Lipschitz constant.

Proof. (a) For any $z \in C$, we obtain from (4.6) that

$$z = Q_D f z \text{ if and only if } \langle z - fz, J(z - v) \rangle \leq 0 \text{ for all } v \in D.$$

(b) For $x, y \in C$, we have

$$\|Q_D fx - Q_D fy\| \leq \|fx - fy\| \leq L\|x - y\|.$$

$\square$

The following result shows that the Kirk's fixed point theorem plays an important role in existence of common element of D and set of solutions of variational inequality $VI_D(C, I - f)$ when $f : C \to C$ is a nonexpansive mapping.

Theorem 4.7. *Let X be a reflexive smooth Banach space, C a nonempty closed convex bounded subset of X with normal structure and D a nonempty subset of C which is a sunny nonexpansive retract of C. Let $f : C \to C$ be a nonexpansive mapping and let Q_D be the sunny nonexpansive retraction from C onto D. Then variational inequality $VI(C, I - f)$ has a solution in D.*

Proof. Since $Q_D f$ is a nonexpansive mapping from C into $D \subseteq C$ with $\Omega_D(I - f) = F(Q_D f)$. It follows from Kirk's fixed point theorem that $F(Q_D f) \neq \emptyset$. Let $z \in F(Q_D f)$. Then $z = Q_D f z \in D$. $\square$

Invoking Theorem 4.5, we obtain the following existence theorem:

Theorem 4.8. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable and C a nonempty closed convex bounded subset of X with normal structure. Let $f, T : C \to C$ be two nonexpansive mappings. Then variational inequality $VI(C, I - f)$ has a solution in $F(T)$.*

Proof. Note that $F(T) \neq \emptyset$ by Theorem 2.4. By Theorem 4.5, there exists a sunny nonexpansive retraction C onto $F(T)$. Let $Q_{F(T)}$ be the sunny nonexpansive retraction C onto $F(T)$. Then the result follows from Theorem 4.7. $\square$

Theorem 4.9. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable and C a nonempty closed convex bounded subset of X with normal structure. Let $f : C \to C$ be a nonexpansive mapping and $T : C \to C$ a continuous pseudocontractive mapping. Then variational inequality $VI_{F(T)}(C, I - f)$ has a solution in $F(T)$.*

Proof. Since $g = (2I - T)^{-1}$ is nonexpansive mapping from C into itself with $F(g) = F(T)$. Then $F(g) \neq \emptyset$ by Theorem 2.4. Let $Q_{F(g)}$ be the sunny nonexpansive retraction from C onto $F(g)$. Then the result follows from Theorem 4.7. $\qquad\square$

5. STRONG CONVERGENCE OF ITERATIVE ALGORITHMS

In this section, we apply the results investigated in Sections 3 and 4 for computing solutions of variational inequalities $VI_C(I - f)$ in Banach spaces when f is a weakly contraction or nonexpansive mapping. In order to prove our main convergence theorems, we start with a key preliminary result.

Lemma 5.1. *Let C be a nonempty closed convex subset of a smooth Banach space X, $f : C \to C$ a weakly contraction mapping with the function ψ and $\{T_n\}$ a sequence of nonexpansive self-mappings on C such that $\bigcap_{n\in\mathbb{N}} F(T_n) \neq \emptyset$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by algorithm:*

$$(5.1) \qquad x_{n+1} = \alpha_n f x_n + (1 - \alpha_n) T_n x_n \text{ for all } \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in [0,1] such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then we have the following:

(a) If $\inf\{\psi(\|x_n - p\|)/\|x_n - p\| : x_n \neq p, n \in \mathbb{N}\} = \delta > 0$ for $p \in \bigcap_{n\in\mathbb{N}} F(T_n)$, then $\{x_n\}$ is bounded.

(b) If $\{x_n\}$ is bounded and there exists a point $z \in \bigcap_{n\in\mathbb{N}} F(T_n)$ such that

$$(5.2) \qquad \limsup_{n\to\infty} \langle fz - z, J(x_n - z) \rangle \leq 0,$$

then $\{x_n\}$ converges strongly to z.

Proof. (a) Let $p \in \bigcap_{n\in\mathbb{N}} F(T_n)$. From (5.1), we have

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(fx_n - fp + fp - p) + (1 - \alpha_n)(T_n x_n - p)\| \\
&\leq \alpha_n\|fx_n - fp\| + \alpha_n\|fp - p\| + (1 - \alpha_n)\|T_n x_n - p\| \\
&\leq \alpha_n(\|x_n - p\| - \psi(\|x_n - p\|)) + \alpha_n\|fp - p\| + (1 - \alpha_n)\|x_n - p\| \\
(5.3) \qquad &= \|x_n - p\| - \alpha_n \psi(\|x_n - p\|) + \alpha_n\|fp - p\|.
\end{aligned}
$$

Since $0 < \delta = \inf\{\psi(\|x_n - p\|)/\|x_n - p\| : x_n \neq p, n \in \mathbb{N}\}$, it follows from (5.3) that

$$
\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - \alpha_n \delta)\|x_n - p\| + \alpha_n\|fp - p\| \\
&\leq \max\{\|x_n - p\|, \|fp - p\|/\delta\} \\
&\;\;\vdots \\
&\leq \max\{\|x_1 - p\|, \|fp - p\|/\delta\}.
\end{aligned}
$$

Consequently, $\{x_n\}$ is bounded.

(b) From (5.2), there exists a sequence $\{\gamma_n\}$ in $(0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$(5.4) \qquad \langle fz - z, J(x_{n+1} - z) \rangle \leq \gamma_n \text{ for all } \in \mathbb{N}.$$

From (5.1), we obtain

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \langle \alpha_n(fx_n - fz + fz - z) + (1 - \alpha_n)(T_n x_n - z), J(x_{n+1} - z)\rangle \\
&\leq \|(1 - \alpha_n)(T_n x_n - z) + \alpha_n(fx_n - fz)\| \; \|x_{n+1} - z\| \\
&\quad + \alpha_n \langle fz - z, J(x_{n+1} - z)\rangle \\
&\leq \frac{1}{2}[\|(1 - \alpha_n)(T_n x_n - z) + \alpha_n(fx_n - fz)\|^2 + \|x_{n+1} - z\|^2] + \alpha_n \gamma_n \\
&\leq \frac{1}{2}[(1 - \alpha_n)\|x_n - z\| + \alpha_n(\|x_n - z\| - \psi(\|x_n - z\|))]^2 \\
&\quad + \frac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n \gamma_n,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|x_n - z\|^2 - 2\alpha_n \psi(\|x_n - z\|)\|x_n - z\| + \alpha_n^2(\psi(\|x_n - z\|))^2 \\
&\quad + 2\alpha_n \gamma_n \\
&\leq \|x_n - z\|^2 - 2\alpha_n \psi(\|x_n - z\|)\|x_n - z\| + \alpha_n^2(\psi(K_3))^2 + 2\alpha_n \gamma_n
\end{aligned}
$$

for some $K_3 > 0$ since $\{\|x_n - z\|\}$ is bounded. Thus, for $\lambda_n = \|x_n - z\|^2$, we obtain the following recursive inequality:

$$
\lambda_{n+1} \leq \lambda_n - \alpha_n \phi(\lambda_n) + \beta_n,
$$

where $\beta_n = \alpha_n(K_3\,\alpha_n + 2\gamma_n)$ and $\phi(t) = 2\sqrt{t}\,\psi(\sqrt{t})$. Therefore, $\{x_n\}$ converges strongly to z by Lemma 2.9. $\qquad \square$

Remark 5.2. If $f : C \to C$ is a contraction mapping with Lipschitz constant $k \in (0, 1)$, then the assumption $\inf\{\psi(\|x_n - p\|)/\|x_n - p\| : x_n \neq p, n \in \mathbb{N}\} = \delta > 0$ for $p \in \bigcap_{n \in \mathbb{N}} F(T_n)$ is satisfied with $\delta = k$. Thus, the sequence $\{x_n\}$ defined by (5.1) is always bounded for contraction mappings $f : C \to C$.

Theorem 5.3. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $f : C \to C$ a weakly contraction mapping with the function ψ and $A \subset X \times X$ an accretive operator with resolvent J_t for $t > 0$ such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Suppose that X is strictly convex or C has normal structure. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by algorithm:*

(5.5) $\qquad x_{n+1} = \alpha_n fx_n + (1 - \alpha_n)J_{t_n}x_n$ *for all $n \in \mathbb{N}$,*

where $\{\alpha_n\}$ is a sequence in [0,1] and $\{t_n\}$ is a sequence in $(0, \infty)$ such that

$$
\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \lim_{n \to \infty} t_n = \infty.
$$

Then $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

find $\tilde{x} \in A^{-1}0$ such that $\langle (I - f)\tilde{x}, J(\tilde{x} - v)\rangle \leq 0$ for all $v \in A^{-1}0$

provided that $\inf\{\psi(\|x_n - p\|)/\|x_n - p\| : x_n \neq p, n \in \mathbb{N}\} = \delta > 0$ for $p \in A^{-1}0$.

Proof. Note that $\{x_n\}$ is bounded by Lemma 5.1(a) and hence both $\{fx_n\}$ and $\{J_{t_n}x_n\}$ are bounded. Hence form (5.5), we have

$$
\|x_{n+1} - J_{t_n}x_n\| = \alpha_n \|fx_n - J_{t_n}x_n\| \to 0 \text{ as } n \to \infty.
$$

Observe that

$$\|A_{t_n} x_n\| = \frac{1}{t_n}\|x_n - J_{t_n} x_n\| \to 0 \text{ as } n \to \infty.$$

By Theorem 3.5 for each $t > 0$, the mapping $T_t^f : C \to C$ defined by

$$T_t^f x = J_t f x, \ x \in C$$

has a unique fixed point $z_t \in C$ such that $\{z_t\}$ converges strongly to an element $z \in A^{-1}0$ as $t \to \infty$, and z is the unique solution of the variational inequality $VI_{A^{-1}0}(C, I - f)$. From (3.5) we have

$$\frac{1}{t}(fz_t - z_t) = A_t f z_t \in AJ_t f z_t = Az_t,$$

it follows from the fact $A_{t_n} x_n \in AJ_{t_n} x_n$ that

$$0 \le \langle A_{t_n} x_n - \frac{1}{t}(fz_t - z_t), J(J_{t_n} x_n - z_t)\rangle.$$

By boundedness of $\{J_{t_n} x_n - z_t\}$, we have

$$\begin{aligned}
\langle fz_t - z_t, J(J_{t_n} x_n - z_t)\rangle &\le& t\langle A_{t_n} x_n, J(J_{t_n} x_n - z_t)\rangle \\
&\le& t\|A_{t_n} x_n\| \ \|J_{t_n} x_n - z_t\| \\
&\le& t\|A_{t_n} x_n\| \ K_4
\end{aligned}$$

for some $K_4 > 0$. Since $\|A_{t_n} x_n\| \to 0$ as $n \to \infty$, we see that

$$(5.6) \qquad \limsup_{n\to\infty}\langle fz_t - z_t, J(J_{t_n} x_n - z_t)\rangle \le 0.$$

Further, since $z_t \to z$ as $t \to \infty$, the set $\{z_t - x_n\}$ is bounded and the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X, it follows that

$$\begin{aligned}
| \langle fz &-& z, J(J_{t_n} x_n - z)\rangle - \langle fz_t - z_t, J(J_{t_n} x_n - z_t)\rangle | \\
&=& | \langle fz - z, J(J_{t_n} x_n - z) - J(J_{t_n} x_n - z_t)\rangle \\
&& +\langle fz - z - (fz_t - z_t), J(J_{t_n} x_n - z_t)\rangle | \\
&\le& | \langle fz - z, J(J_{t_n} x_n - z) - J(J_{t_n} x_n - z_t)\rangle | \\
&& +\|fz - z - (fz_t - z_t)\| \ \|x_n - z_t\| \to 0 \text{ as } t \to \infty.
\end{aligned}$$

Let $\varepsilon > 0$. Then there exists $t_0 > 0$ such that

$$\langle fz - z, J(J_{t_n} x_n - z)\rangle < \langle fz_t - z_t, J(J_{t_n} x_n - z_t)\rangle + \varepsilon \text{ for all } n \in \mathbb{N} \text{ and } t \ge t_0.$$

Using (5.6), we get

$$\begin{aligned}
\limsup_{n\to\infty}\langle fz - z, J(J_{t_n} x_n - z)\rangle &\le& \limsup_{n\to\infty}\langle fz_t - z_t, J(J_{t_n} x_n - z_t)\rangle + \varepsilon \\
&\le& \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we obtain that

$$\limsup_{n\to\infty}\langle fz - z, J(J_{t_n} x_n - z)\rangle \le 0.$$

Since $x_{n+1} - J_{t_n} x_n \to 0$ as $n \to \infty$ and the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X, it follows that

$$\lim_{n\to\infty} | \langle fz - z, J(x_{n+1} - z)\rangle - \langle fz - z, J(J_{t_n} x_n - z)\rangle | = 0.$$

Therefore,

$$\limsup_{n\to\infty}\langle fz - z, J(x_{n+1} - z)\rangle \le 0.$$

So, there exists a sequence $\{\gamma_n\}$ in $(0,\infty)$ satisfying (5.4) such that $\lim_{n\to\infty}\gamma_n = 0$. Therefore, since $z \in F(J_{t_n})$ for all $n \in \mathbb{N}$, $\{x_n\}$ converges strongly to z by Lemma 5.1. □

We now give a more general proximal point algorithm for resolvent operators in a Banach space which provides an affirmative answer of Question 1.5.

Theorem 5.4. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $f : C \to C$ a weakly contraction mapping with the function ψ and $A \subset X \times X$ an accretive operator with resolvent J_t for $t > 0$ such that $A^{-1}0 \ne \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Suppose that X is strictly convex or C has normal structure. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by algorithm:*

$$x_{n+1} = \alpha_n f x_n + (1 - \alpha_n)J_{t_n}x_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in [0,1] and $\{t_n\}$ is a sequence in $(0,\infty)$ such that

$$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty \text{ and } \lim_{n\to\infty}t_n = \infty.$$

If $\{x_n\}$ is bounded, then it converges strongly to the unique solution of the variational inequality:

find $\tilde{x} \in A^{-1}0$ such that $\langle (I - f)\tilde{x}, J(\tilde{x} - v)\rangle \le 0$ for all $v \in A^{-1}0$.

The following result improves and unifies several known results that were obtained using stronger assumptions (see e.g. Theorem 2.5 of Benavides, Acedo and Xu [2], Theorem 2 of Kim and Xu [9], Theorem 3.1 of Mainge [13], and Theorem 4.2 of Nakajo [17]).

Corollary 5.5. *Let X be a reflexive Banach space with uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $f : C \to C$ a contraction mapping with Lipschitz constant $k \in [0,1)$ and $A \subset X \times X$ an accretive operator with resolvent J_t for $t > 0$ such that $A^{-1}0 \ne \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Suppose that X is strictly convex or C has normal structure. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by algorithm:*

$$x_{n+1} = \alpha_n f x_n + (1 - \alpha_n)J_{t_n}x_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in [0,1] and $\{t_n\}$ is a sequence in $(0,\infty)$ such that

$$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty \text{ and } \lim_{n\to\infty}t_n = \infty.$$

Then $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

find $\tilde{x} \in A^{-1}0$ such that $\langle (I - f)\tilde{x}, J(\tilde{x} - v)\rangle \le 0$ for all $v \in A^{-1}0$.

Theorem 5.6. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X, $f : C \to C$ a weakly contraction mapping with the function ψ and $T : C \to C$ a nonexpansive mapping with $F(T) \ne \emptyset$. Suppose that X is strictly convex or C has normal structure. Let $\{\alpha_n\}$ be a sequence in [0,1] such that*

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ and \ \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.$$

For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by algorithm:

$$(5.7) \qquad\qquad x_{n+1} = \alpha_n f x_n + (1 - \alpha_n) T x_n \ for \ all \ \in \mathbb{N}.$$

If $\{x_n\}$ is bounded, then it converges strongly to the unique solution of the variational inequality:

$$find \ \tilde{x} \in F(T) \ such \ that \ \langle (I - f)\tilde{x}, J(\tilde{x} - v) \rangle \le 0 \ for \ all \ v \in F(T).$$

Proof. Boundedness of $\{x_n\}$ implies that both $\{fx_n\}$ and $\{Tx_n\}$ are bounded. Hence form (5.7), we have

$$\|x_{n+1} - Tx_n\| = \alpha_n \|fx_n - Tx_n\| \to 0 \ as \ n \to \infty.$$

Observe that

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f x_n + (1 - \alpha_n) T x_n - (\alpha_{n-1} f x_{n-1} + (1 - \alpha_{n-1}) T x_{n-1})\| \\
&\le \|(1 - \alpha_n)(T x_n - T x_{n-1}) + (\alpha_n - \alpha_{n-1})(f x_{n-1} - T x_{n-1})\| \\
&\quad + \alpha_n \|f x_n - f x_{n-1}\| \\
&\le (1 - \alpha_n)\|x_n - x_{n-1}\| + \alpha_n(\|x_n - x_{n-1}\| - \psi(\|x_n - x_{n-1}\|)) \\
&\quad + |\alpha_n - \alpha_{n-1}| \ K_5 \\
&= \|x_n - x_{n-1}\| - \alpha_n \psi(\|x_n - x_{n-1}\|) + |\alpha_n - \alpha_{n-1}| \ K_5
\end{aligned}
$$

for some constant $K_5 > 0$. Since $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$, it follows from Lemma 2.9 that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Hence

$$\|x_n - Tx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \to 0 \ as \ n \to \infty.$$

For each $t \in (0, 1)$, define $T_t^f : C \to C$ by

$$T_t^f x = t f x + (1 - t) T x, \quad x \in C.$$

Note that Theorem 3.8 implies that T_t^f has a unique fixed point $z_t \in C$ such that $\{z_t\}$ converges strongly to an element $z \in F(T)$ as $t \to 0^+$, and z is the unique solution of the variational inequality $VI_{F(T)}(C, I - f)$.

Since $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ and $\{z_t\}$ converges strongly to $z \in F(T)$, it follows from Lemma 2.12 that $\limsup_{n\to\infty} \langle fz - z, J(x_n - z) \rangle \le 0$. Since $z \in F(T)$ and (5.2) is satisfied, then we conclude from Lemma 5.1 that $\{x_n\}$ converges strongly to z. $\qquad\square$

As a direct consequence of Theorem 5.6, we have the following:

Corollary 5.7. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X, $f : C \to C$ a contraction mapping with Lipschitz constant $k \in [0, 1)$, and $T : C \to C$ a nonexpansive mapping with $F(T) \ne \emptyset$. Suppose that X is strictly convex or C has normal structure. Let $\{\alpha_n\}$ be a sequence in [0,1] such that*

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ and \ \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.$$

For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by the functional Halpern iteration process:

$$x_{n+1} = \alpha_n f x_n + (1 - \alpha_n) T x_n \text{ for all } n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

$$\text{find } \tilde{x} \in F(T) \text{ such that } \langle (I - f)\tilde{x}, J(\tilde{x} - v) \rangle \leq 0 \text{ for all } v \in F(T).$$

Remark 5.8. Corollary 5.7 improves a number of results concerning viscosity approximation methods, particularly the results of Moudafi [16] and Xu [24] in the following way:

(1) in variational inequality $VI_{F(T)}(C, I - f)$, the mapping f is not necessarily contraction,

(2) the underlying space X is not necessarily Hilbert or uniformly smooth.

Corollary 5.9. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that X is strictly convex or C has normal structure. Let $\{\alpha_n\}$ be a sequence in [0,1] such that*

$$\lim_{n \to \infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ and \ \lim_{n \to \infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.$$

For given $x, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by the Halpern iteration process:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n \text{ for all } n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $Q_{F(T)} x$, where $Q_{F(T)}$ is the sunny nonexpansive retraction from C onto $F(T)$.

Note that the nonexpansive mapping T involving in algorithm (5.7) plays an important role in strong convergence of the sequence $\{x_n\}$ defined by (5.7). In our next result, we construct some iterative algorithms using sunny nonexpansive retraction which are independent of such nonexpansive mappings.

Theorem 5.10. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X and D a nonempty subset of C. Suppose that D is a sunny nonexpansive retract of C. Let $f : C \to C$ be a weakly contraction mapping with the function ψ and let Q_D be the sunny nonexpansive retraction from C onto D. Then the sequence $\{x_n\}$ degenerated by the algorithm:*

$$x_{n+1} = Q_D f x_n \text{ for all } n \in \mathbb{N},$$

converges strongly to z, the unique solution of $VI_D(C, I - f)$, with the following error estimate:

$$\|x_n - z\| \leq \Phi^{-1}(\Phi(\|x_1 - z\|)) - (n - 1).$$

Proof. Note that $Q_D f$ is a weakly contraction mapping from C into itself with the function ψ. In fact,

$$\|Q_D f x - Q_D f y\| \leq \|fx - fy\| \leq \|x - y\| - \psi(\|x - y\|) \text{ for all } x, y \in C.$$

Since $Q_D f$ is a self-mapping on C, $\{x_n\}$ is well defined. Applying Theorem 2.3, we obtain that $\{x_n\}$ converges strongly to the unique fixed point z of $Q_D f$. The reminder estimate follows from Theorem 2.3. $\qquad\square$

Theorem 5.11. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex bounded subset of X and D a nonempty subset of C with normal structure. Suppose that D is a sunny nonexpansive retract of C. Let $f : C \to C$ be a nonexpansive mapping and let Q_D be the sunny nonexpansive retraction from C onto D. Let x, x_1 be elements in C and let $\{\alpha_n\}$ be a sequence in [0,1] such that*

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ and \ \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.$$

Then the sequence $\{x_n\}$ degenerated by the Helpern iteration process:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) Q_D f x_n \ for \ all \ n \in \mathbb{N},$$

converges strongly to an element of $\Omega_D(I - f)$.

Proof. Since $Q_D f$ is a nonexpansive mapping from C into itself, hence $F(Q_D f) \neq \emptyset$ and $\{x_n\}$ is well defined. Applying Corollary 5.9, we obtain that $\{x_n\}$ converges strongly to a fixed point z of $Q_D f$. $\qquad\square$

Next, we replace D by $F(T)$, the fixed point of a nonexpansive mapping $T : C \to C$.

Theorem 5.12. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable norm and C a nonempty closed convex bounded subset of X with normal structure. Let $f, T : C \to C$ be two nonexpansive mappings. Let x, x_1 be elements in C and let $\{\alpha_n\}$ be a sequence in [0,1] such that*

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ and \ \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.$$

Then the sequence $\{x_n\}$ degenerated by the Helpern iteration process:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) Q_{F(T)} f x_n \ for \ all \ n \in \mathbb{N},$$

converges strongly to $z = P_{F(Q_{F(T)}f)} x$ which a solution of the variational inequality:

$$find \ \tilde{x} \in F(T) \ such \ that \ \ \langle (I - f)\tilde{x}, J(\tilde{x} - v) \rangle \leq 0 \ for \ all \ v \in F(T),$$

where $P_{F(Q_{F(T)}f)}$ is the sunny nonexpansive retraction form C onto $F(Q_{F(T)}f)$.

6. Applications

In this section, we give two convergence theorems for finding common fixed points of nonexpansive mappings and common zeros of accretive operators in Banach spaces.

Proposition 6.1. *Let C be a nonempty closed convex subset of a strictly convex Banach space X and let $\lambda_i > 0$ $(i = 1, 2, \cdots, r)$ such that $\sum_{i=1}^{r} \lambda_i = 1$. Let $T_1, T_2, \cdots, T_r : C \to C$ be nonexpansive mappings with $\cap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $T = \sum_{i=1}^{r} \lambda_i T_i$. Then T is nonexpansive from C into itself and $F(T) = \cap_{i=1}^{r} F(T_i)$.*

Proof. Since $\cap_{i=1}^r F(T_i) \subseteq F(T)$ is trivial, we show that $F(T) \subseteq \cap_{i=1}^r F(T_i)$. Let $z \in F(T)$ and $v \in \cap_{i=1}^r F(T_i)$. Observe that

$$
\begin{aligned}
\|z - v\| &= \|\sum_{i=1}^r \lambda_i T_i z - v\| \\
&\leq \lambda_1 \|T_1 z - v\| + \lambda_2 \|T_2 z - v\| + \cdots + \lambda_r \|T_r z - v\| \\
&\leq \lambda_1 \|z - v\| + \lambda_2 \|z - v\| + \cdots + \lambda_r \|z - v\| = \|z - v\|,
\end{aligned}
$$

which implies that

$$
\|T_1 z - v\| = \|T_2 z - v\| = \cdots = \|T_r z - v\| = \|z - v\|.
$$

Since X is a strictly convex, it follows that $T_1 z = T_2 z = \cdots = T_r z = z$. $\qquad\square$

Recall that the so-called problem of image recovery is essentially to find a common element of finitely many nonexpansive retracts $C_1, C_2, \ldots, C_r$ of C with $\cap_{i=1}^r C_i \neq \emptyset$. It is easy to see that every nonexpansive retraction P_i of C onto C_i is a nonexpansive mapping of C into itself. There is no doubt that the problem of image recovery is equivalent to finding a common fixed point of finitely many nonexpansive mappings $P_1, \ldots, P_r$ of C into itself.

Applying Proposition 6.1, we obtain the following result which improves a number of results connected to the problem of image recovery.

Theorem 6.2. *Let X be a reflexive strictly convex Banach space whose norm is uniformly Gâteaux differentiable and C a nonempty closed convex subset of X. Let $\lambda_i \geq 0$ $(i = 1, 2, \cdots, r)$ such that $\sum_{i=1}^r \lambda_i = 1$ and let $T_1, T_2, \cdots, T_r : C \to C$ be nonexpansive mappings with $\cap_{i=1}^r F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in [0,1] such that*

$$
\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^\infty \alpha_n = \infty \ \text{and} \ \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.
$$

For given $x, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by algorithm:

$$
x_{n+1} = \alpha_n x + (1 - \alpha_n) \sum_{i=1}^r \lambda_i T_i x_n \ \text{for all } n \in \mathbb{N}.
$$

Then $\{x_n\}$ converges strongly to $Q_{\cap_{i=1}^r F(T_i)} x$, where $Q_{\cap_{i=1}^r F(T_i)}$ is the sunny nonexpansive retraction from C onto $\cap_{i=1}^r F(T_i)$.

Proof. Let $T = \sum_{i=1}^r \lambda_i T_i$. Proposition 6.1 implies that T is nonexpansive from C into itself and $F(T) = \cap_{i=1}^r F(T_i)$. Hence the result follows from Corollary 5.9. $\quad\square$

The following result in an improvement of Theorem 3.3 of Zegeye and Shahzad [26].

Theorem 6.3. *Let X be a reflexive strictly convex Banach space with uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X and $f : C \to C$ a contraction mapping with Lipschitz constant $k \in [0, 1)$. Let $A_i \subset X \times X$, $i = 1, 2, \cdots, r$ be a family of accretive operators with resolvent $J_t^{A_i}$ for $t > 0$ such that $\cap_{i=1}^r A_i^{-1} 0 \neq \emptyset$ and $\overline{D(A_i)} \subset C \subset \bigcap_{t>0} R(I + tA_i)$. Let $\lambda_i \geq 0$ $(i = 1, 2, \cdots, r)$*

such that $\sum_{i=1}^{r} \lambda_i = 1$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by algorithm:

$$(6.1) \qquad x_{n+1} = \alpha_n f x_n + (1 - \alpha_n) \sum_{i=1}^{r} \lambda_i J_t^{A_i} x_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in [0,1] such that

$$\lim_{n \to \infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ \text{and} \ \lim_{n \to \infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.$$

Then $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

find $\tilde{x} \in \cap_{i=1}^{r} A_i^{-1} 0$ such that $\langle (I - f)\tilde{x}, J(\tilde{x} - v) \rangle \leq 0$ for all $v \in \cap_{i=1}^{r} A_i^{-1} 0$.

Proof. Note that each $J_t^{A_i}$ is nonexpansive. Let $T = \sum_{i=1}^{r} \lambda_i J_t^{A_i}$. Proposition 6.1 implies that T is nonexpansive from C into itself and $F(T) = \cap_{i=1}^{r} A_i^{-1} 0$. Hence the result follows from Corollary 5.7. $\square$

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