ITERATIVE COMMON SOLUTIONS FOR MONOTONE INCLUSION PROBLEMS, FIXED POINT PROBLEMS AND EQUILIBRIUM PROBLEMS

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ABSTRACT. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A be an α-inverse strongly-monotone mapping of C into H. Let T be a generalized hybrid mapping of C into H. Let B and W be maximal monotone operators on H such that the domains of B and W are included in C. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Suppose that $F(T) \cap (A+B)^{-1}0 \cap W^{-1}0 \neq \emptyset$. In this paper, we prove a strong convergence theorem for finding a point z_0 of $F(T) \cap (A+B)^{-1}0 \cap W^{-1}0$, where z_0 is a unique fixed point of $P_{F(T)\cap(A+B)^{-1}0\cap W^{-1}0}(I - V + \gamma g)$. This point $z_0 \in F(T) \cap (A+B)^{-1}0 \cap W^{-1}0$ is also a unique solution of the variational inequality

$$\langle (V - \gamma g)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(T) \cap (A+B)^{-1}0 \cap W^{-1}0.$$

Using this result, we obtain new and well-known strong convergence theorems in a Hilbert space. In particular, we solve a problem posed by Kurokawa and Takahashi [16].

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. A mapping $T : C \to H$ is called *generalized hybrid* [13] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. We call such a mapping an (α, β)-*generalized hybrid* mapping. Then Kocourek, Takahashi and Yao [13] proved a fixed point theorem for such mappings in a Hilbert space. Furthermore, they proved a nonlinear mean convergence theorem of Baillon's type [4] in a Hilbert space. Notice that the mapping above covers several well-known mappings. For example, an (α, β)-generalized hybrid mapping T is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

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It is also nonspreading [14, 15] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Furthermore, it is hybrid [31] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

We can also show that if $x = Tx$, then for any $y \in C$,

$$\alpha\|x - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|x - y\|^2 + (1 - \beta)\|x - y\|^2$$

and hence $\|x - Ty\| \le \|x - y\|$. This means that an $(\alpha,\ \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. The following strong convergence theorem of Halpern's type [10] was proved by Wittmannn [35]; see also [29].

Theorem 1. *Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then $\{x_n\}$ converges strongy to a fixed point of T.

Kurokawa and Takahashi [16] also proved the following strong convergence theorem for nonspreading mappings in a Hilbert space; see also Hojo and Takahashi [11] for generalized hybrid mappings.

Theorem 2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonspreading mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \ldots,$ where $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to Pu, where P is the metric projection of H onto $F(T)$.

Remark. We do not know whether Theorem 1 for nonspreading mappings holds or not; see [16] and [11].

Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \ge 0, \quad \forall y \in C. \tag{1.1}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \ge 0,\ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x,x)=0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x,y)+f(y,x) \leq 0$ for all $x,y \in C$;
(A3) for all $x,y,z \in C$,

$$\limsup_{t \downarrow 0} f(tz+(1-t)x,y) \leq f(x,y);$$

(A4) for all $x \in C$, $f(x,\cdot)$ is convex and lower semicontinuous.

Defining a set-valued mapping $A_f \subset H \times H$ by

$$A_f x = \begin{cases} \{z \in H : f(x,y) \geq \langle y-x, z\rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, \quad \forall x \notin C, \end{cases}$$

we have from [27] that A_f is a maximal monotone operator such that the domain is included in C.

In this paper, motivated by these results, we prove a strong convergence theorem for finding a point z_0 of $F(T) \cap (A+B)^{-1}0 \cap W^{-1}0$, where T is a generalized hybrid mapping of C into H, B and W are maximal monotone operators on H such that the domains of B and W are included in C, g is a k-contraction of H into itself with $0<k<1$, V is a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\overline{\gamma}>0$ and $L>0$. This point z_0 is a unique fixed point of $P_{F(T)\cap(A+B)^{-1}0\cap W^{-1}0}(I-V+\gamma g)$ and then this $z_0 \in F(T) \cap (A+B)^{-1}0 \cap W^{-1}0$ is also a unique solution of the variational inequality

$$\langle (V-\gamma g)z_0, q-z_0\rangle \geq 0, \quad \forall q \in F(T) \cap (A+B)^{-1}0 \cap W^{-1}0.$$

Using this result, we obtain new and well-known strong convergence theorems in a Hilbert space. In particular, we solve a problem posed by Kurokawa and Takahashi [16].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [30] that for any $x,y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle \tag{2.1}$$

and

$$\|\lambda x+(1-\lambda)y\|^2 = \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

Furthermore we have that for $x,y,u,v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2+\|y-u\|^2-\|x-u\|^2-\|y-v\|^2. \tag{2.3}$$

All Hilbert spaces satisfy Opial's condition, that is,

$$\liminf_{n\to\infty} \|x_n-u\| < \liminf_{n\to\infty} \|x_n-v\| \tag{2.4}$$

if $x_n \rightharpoonup u$ and $u \neq v$; see [24]. Let C be a nonempty closed convex subset of a Hilbert space H and let $T\colon C \to H$ be a mapping. We denote by $F(T)$ be the set of fixed points for T. A mapping $T : C \to H$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx-y\| \leq \|x-y\|$ for all $x \in C$ and $y \in F(T)$. If $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see [12]. For a nonempty closed convex subset C of H, the nearest point projection of H onto C is denoted by P_C, that is, $\|x-P_Cx\| \leq \|x-y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive; $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$ for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [28]. The following result is in [34].

Lemma 3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to H$ be a generalized hybrid mapping. Suppose that there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Then, $z \in F(T)$.*

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} : H \to D(B)$, which is called the resolvent of B for r. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for $r > 0$. We know from [29] that

$$A_r x \in BJ_r x, \quad \forall x \in H,\ r > 0. \tag{2.5}$$

Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that $B^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent J_r is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{2.6}$$

We also know the following lemma from [27].

Lemma 4. *Let H be a real Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

From Lemma 4, we have that

$$\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu| / \lambda)\, \|x - J_\lambda x\|$$

for all $\lambda, \mu > 0$ and $x \in H$; see also [28, 9]. To prove our main result, we need the following lemmas.

Lemma 5 ([2]; see also [36])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 6 ([19])**.** *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(1) \le \tau(2) \le \dots$ and $\tau(n) \to \infty$;
(ii) $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ and $\Gamma_n \le \Gamma_{\tau(n)+1}$, $\forall n \in \mathbb{N}$.

3. Strong Convergence Theorems

Let H be a real Hilbert space. A mapping $g : H \to H$ is a contraction if there exists $k \in (0,1)$ such that $\|g(x) - g(y)\| \le k\|x - y\|$ for all $x, y \in H$. We call such a mapping g a k-contraction. A nonlinear operator $V : H \to H$ is called strongly monotone if there exists $\overline{\gamma} > 0$ such that $\langle x-y, Vx-Vy\rangle \ge \overline{\gamma}\|x-y\|^2$ for all $x, y \in H$. Such V is also called $\overline{\gamma}$-strongly monotone. A nonlinear operator $V : H \to H$ is called Lipschitzian continuous if there exists $L > 0$ such that $\|Vx-Vy\| \le L\|x-y\|$ for all $x, y \in H$. Such V is also called L-Lipschitzian continuous. We know the following three lemmas in a Hilbert space; see Lin and Takahashi [17].

Lemma 7 ([17]). *Let H be a Hilbert space and let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\overline{\gamma} > 0$ and $L > 0$. Let $t > 0$ satisfy $2\overline{\gamma} > tL^2$ and $1 > 2t\overline{\gamma}$. Then $0 < 1 - t(2\overline{\gamma} - tL^2) < 1$ and $I - tV : H \to H$ is a contraction, where I is the identity operator on H.*

Lemma 8 ([17]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let P_C be the metric projection of H onto C and let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\overline{\gamma} > 0$ and $L > 0$. Let $t > 0$ satisfy $2\overline{\gamma} > tL^2$ and $1 > 2t\overline{\gamma}$ and let $z \in C$. Then the following are equivalent:*

(1) $z = P_C(I - tV)z$;
(2) $\langle Vz, y - z\rangle \ge 0, \quad \forall y \in C$;
(3) $z = P_C(I - V)z$.

Such $z \in C$ exists always and is unique.

Lemma 9 ([17]). *Let H be a Hilbert space and let $g : H \to H$ be a k-contraction with $0 < k < 1$. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\overline{\gamma} > 0$ and $L > 0$. Let a real number γ satisfy $0 < \gamma < \frac{\overline{\gamma}}{k}$. Then $V - \gamma g : H \to H$ is a $(\overline{\gamma} - \gamma k)$-strongly monotone and $(L + \gamma k)$-Lipschitzian continuous mapping. Furthermore, let C be a nonempty closed convex subset of H. Then $P_C(I - V + \gamma g)$ has a unique fixed point z_0 in C. This point $z_0 \in C$ is also a unique solution of the variational inequality*

$$\langle (V - \gamma g)z_0, q - z_0\rangle \ge 0, \quad \forall q \in C.$$

Now we prove the following strong convergence theorem of Halpern's type [10] for finding a common solution of a monotone inclusion problem for the sum of two monotone mappings, of a fixed point problem for generalized hybrid mappings and of an equilibrium problem for bifunctions in a Hilbert space.

Theorem 10. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A be an α-inverse strongly-monotone mapping of C into H. Let B and W be maximal monotone operators on H such that the domains of B and W are included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ and $T_r = (I + rW)^{-1}$ be resolvents of B and W for $\lambda > 0$ and $r > 0$, respectively. Let S be a generalized hybrid mapping of C into H. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with*

$\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Suppose $F(S) \cap (A+B)^{-1}0 \cap W^{-1}0 \neq \emptyset$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\{\alpha_n \gamma g(x_n) + (I - \alpha_n V) S J_{\lambda_n}(I - \lambda_n A) T_{r_n} x_n\}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$$

$$\liminf_{n\to\infty} r_n > 0 \quad and \quad 0 < a \leq \lambda_n \leq b < 2\alpha.$$

Then $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap (A+B)^{-1}0 \cap W^{-1}0$, where z_0 is a unique fixed point in $F(S) \cap (A+B)^{-1}0 \cap W^{-1}0$ of $P_{F(S)\cap(A+B)^{-1}0\cap W^{-1}0}(I - V + \gamma g)$.

Proof. Let $z \in F(S) \cap (A+B)^{-1}0 \cap W^{-1}0$. We have that $z = Sz$, $z = J_{\lambda_n}(I - \lambda_n A)z$ and $z = T_{r_n} z$. Putting $w_n = J_{\lambda_n}(I - \lambda_n A) T_{r_n} x_n$ and $u_n = T_{r_n} x_n$, we obtain that

$$\begin{aligned}
\|Sw_n - z\|^2 &\leq \|w_n - z\|^2 \\
&= \|J_{\lambda_n}(u_n - \lambda_n A u_n) - J_{\lambda_n}(z - \lambda_n A z)\|^2 \\
&\leq \|u_n - \lambda_n A u_n - (z - \lambda_n A z)\|^2 \\
&= \|u_n - z - \lambda_n (A u_n - A z)\|^2 \\
&= \|u_n - z\|^2 - 2\lambda_n \langle u_n - z, A u_n - A z\rangle + \lambda_n^2 \|A u_n - A z\|^2 \\
&\leq \|u_n - z\|^2 - 2\lambda_n \alpha \|A u_n - A z\|^2 + \lambda_n^2 \|A u_n - A z\|^2 \\
&\leq \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha) \|A u_n - A z\|^2 \\
&\leq \|x_n - z\|^2 .
\end{aligned} \tag{3.1}$$

Put $\tau = \overline{\gamma} - \frac{L^2\mu}{2}$. Using $\lim_{n\to\infty} \alpha_n = 0$, we have that for any $x, y \in H$,

$$\begin{aligned}
\|(I - \alpha_n V)x - (I - \alpha_n V)y\|^2 &= \|x - y - \alpha_n (Vx - Vy)\|^2 \\
&= \|x - y\|^2 - 2\alpha_n \langle x - y, Vx - Vy\rangle + \alpha_n^2 \|Vx - Vy\|^2 \\
&\leq \|x - y\|^2 - 2\alpha_n \overline{\gamma} \|x - y\|^2 + \alpha_n^2 L^2 \|x - y\|^2 \\
&= (1 - 2\alpha_n \overline{\gamma} + \alpha_n^2 L^2)\|x - y\|^2 \\
&= (1 - 2\alpha_n \tau - \alpha_n L^2 \mu + \alpha_n^2 L^2)\|x - y\|^2 \\
&\leq \big(1 - 2\alpha_n \tau - \alpha_n (L^2\mu - \alpha_n L^2) + \alpha_n^2 \tau^2\big)\|x - y\|^2 \\
&\leq (1 - 2\alpha_n \tau + \alpha_n^2 \tau^2)\|x - y\|^2 \\
&= (1 - \alpha_n \tau)^2 \|x - y\|^2.
\end{aligned} \tag{3.2}$$

Since $1 - \alpha_n \tau > 0$, we obtain that for any $x, y \in H$,

$$\|(I - \alpha_n V)x - (I - \alpha_n V)y\| \leq (1 - \alpha_n \tau)\|x - y\|. \tag{3.3}$$

Putting $y_n = \alpha_n \gamma g(x_n) + (I - \alpha_n V) S J_{\lambda_n}(I - \lambda_n A) T_{r_n} x_n$, from $z = \alpha_n V z + z - \alpha_n V z$, (3.1) and (3.3) we have that

$$\begin{aligned}\|y_n - z\| &= \|\alpha_n(\gamma g(x_n) - Vz) + (I - \alpha_n V) S w_n - (I - \alpha_n V) z\| \\ &\le \alpha_n \gamma\, k \,\|x_n - z\| + \alpha_n \|\gamma g(z) - Vz\| + (1 - \alpha_n \tau)\,\|S w_n - z\| \\ &\le \{1 - \alpha_n(\tau - \gamma\, k)\}\,\|x_n - z\| + \alpha_n \|\gamma g(z) - Vz\|.\end{aligned}$$

Using this, we get

$$\begin{aligned}\|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1 - \beta_n)(y_n - z)\| \\ &\le \beta_n \|x_n - z\| + (1 - \beta_n)\,\|y_n - z\| \\ &\le \beta_n \|x_n - z\| \\ &\quad + (1 - \beta_n)(\{1 - \alpha_n(\tau - \gamma\, k)\}\,\|x_n - z\| + \alpha_n \|\gamma g(z) - Vz\|) \\ &= \{1 - (1 - \beta_n)\alpha_n(\tau - \gamma\, k)\}\|x_n - z\| \\ &\quad + (1 - \beta_n)\alpha_n(\tau - \gamma\, k)\frac{\|\gamma g(z) - Vz\|}{\tau - \gamma\, k}.\end{aligned}$$

Putting $K = \max\{\|x_1 - z\|, \frac{\|\gamma g(z) - Vz\|}{\tau - \gamma\, k}\}$, we have that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are bounded. Using Lemma 9, we can take a unique $z_0 \in F(S) \cap (A + B)^{-1}0 \cap W^{-1}0$ such that

$$z_0 = P_{F(S) \cap (A+B)^{-1}0 \cap W^{-1}0}(I - V + \gamma g) z_0.$$

From the definition of $\{x_n\}$, we have that

$$x_{n+1} - x_n = \beta_n x_n + (1 - \beta_n)\{\alpha_n \gamma g(x_n) + (I - \alpha_n V) S w_n\} - x_n$$

and hence

$$\begin{aligned}x_{n+1} - x_n - (1 - \beta_n)\alpha_n \gamma g(x_n) &= \beta_n x_n + (1 - \beta_n)(I - \alpha_n V) S w_n - x_n \\ &= (1 - \beta_n)\{(I - \alpha_n V) S w_n - x_n\} \\ &= (1 - \beta_n)(S w_n - x_n - \alpha_n V S w_n).\end{aligned}$$

Thus we have that

$$\begin{aligned}&\langle x_{n+1} - x_n - (1 - \beta_n)\alpha_n \gamma g(x_n), x_n - z_0\rangle \\ &\quad = \langle (1 - \beta_n)(S w_n - x_n - \alpha_n V S w_n), x_n - z_0\rangle \\ &\quad = -(1 - \beta_n)\langle x_n - S w_n, x_n - z_0\rangle - (1 - \beta_n)\alpha_n \langle V S w_n, x_n - z_0\rangle.\end{aligned} \tag{3.4}$$

From (2.3) and (3.1), we have that

$$\begin{aligned}2\langle x_n - S w_n, x_n - z_0\rangle &= \|x_n - z_0\|^2 + \|S w_n - x_n\|^2 - \|S w_n - z_0\|^2 \\ &\ge \|x_n - z_0\|^2 + \|S w_n - x_n\|^2 - \|x_n - z_0\|^2 \\ &= \|S w_n - x_n\|^2.\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we also have that

$$\begin{aligned}-2\langle x_n - x_{n+1}, x_n - z_0\rangle &= 2(1 - \beta_n)\alpha_n \langle \gamma g(x_n), x_n - z_0\rangle \\ &\quad - 2(1 - \beta_n)\langle x_n - S w_n, x_n - z_0\rangle - 2(1 - \beta_n)\alpha_n \langle V S w_n, x_n - z_0\rangle \\ &\le 2(1 - \beta_n)\alpha_n \langle \gamma g(x_n), x_n - z_0\rangle \\ &\quad - (1 - \beta_n)\|S w_n - x_n\|^2 - 2(1 - \beta_n)\alpha_n \langle V S w_n, x_n - z_0\rangle.\end{aligned} \tag{3.6}$$

Furthermore using (2.3) and (3.6), we have that

$$\begin{aligned}\|x_{n+1}-z_0\|^2-&\|x_n-x_{n+1}\|^2-\|x_n-z_0\|^2\\ &\le 2(1-\beta_n)\alpha_n\langle\gamma g(x_n),x_n-z_0\rangle\\ &\quad-(1-\beta_n)\|Sw_n-x_n\|^2-2(1-\beta_n)\alpha_n\langle VSw_n,x_n-z_0\rangle.\end{aligned}$$

Setting $\Gamma_n=\|x_n-z_0\|^2$, we have that

$$\begin{aligned}\Gamma_{n+1}-\Gamma_n-&\|x_n-x_{n+1}\|^2\\ &\le 2(1-\beta_n)\alpha_n\langle\gamma g(x_n),x_n-z_0\rangle\\ &\quad-(1-\beta_n)\|Sw_n-x_n\|^2-2(1-\beta_n)\alpha_n\langle VSw_n,x_n-z_0\rangle.\end{aligned}\tag{3.7}$$

Noting that

$$\begin{aligned}\|x_{n+1}-x_n\|&=\|(1-\beta_n)\alpha_n(\gamma g(x_n)-VSw_n)+(1-\beta_n)(Sw_n-x_n)\|\\ &\le(1-\beta_n)\big(\|Sw_n-x_n\|+\alpha_n\|\gamma g(x_n)-VSw_n\|\big),\end{aligned}\tag{3.8}$$

we have that

$$\begin{aligned}\|x_{n+1}-x_n\|^2&\le(1-\beta_n)^2\big(\|Sw_n-x_n\|+\alpha_n\|\gamma g(x_n)-VSw_n\|\big)^2\\ &=(1-\beta_n)^2\|Sw_n-x_n\|^2+(1-\beta_n)^2 2\alpha_n\|Sw_n-x_n\|\|\gamma g(x_n)-VSw_n\|\\ &\quad+(1-\beta_n)^2\alpha_n^2\|\gamma g(x_n)-VSw_n\|^2.\end{aligned}\tag{3.9}$$

Thus we have from (3.7) and (3.9) that

$$\begin{aligned}\Gamma_{n+1}-\Gamma_n&\le\|x_n-x_{n+1}\|^2+2(1-\beta_n)\alpha_n\langle\gamma g(x_n),x_n-z_0\rangle\\ &\quad-(1-\beta_n)\|Sw_n-x_n\|^2-2(1-\beta_n)\alpha_n\langle VSw_n,x_n-z_0\rangle\\ &\le(1-\beta_n)^2\|Sw_n-x_n\|^2+(1-\beta_n)^2 2\alpha_n\|Sw_n-x_n\|\|\gamma g(x_n)-VSw_n\|\\ &\quad+(1-\beta_n)^2\alpha_n^2\|\gamma g(x_n)-VSw_n\|^2+2(1-\beta_n)\alpha_n\langle\gamma g(x_n),x_n-z_0\rangle\\ &\quad-(1-\beta_n)\|Sw_n-x_n\|^2-2(1-\beta_n)\alpha_n\langle VSw_n,x_n-z_0\rangle\end{aligned}$$

and hence

$$\begin{aligned}\Gamma_{n+1}-\Gamma_n+&\beta_n(1-\beta_n)\|Sw_n-x_n\|^2\le(1-\beta_n)^2 2\alpha_n\|Sw_n-x_n\|\|\gamma g(x_n)-VSw_n\|\\ &+(1-\beta_n)^2\alpha_n^2\|\gamma g(x_n)-VSw_n\|^2+2(1-\beta_n)\alpha_n\langle\gamma g(x_n),x_n-z_0\rangle\\ &-2(1-\beta_n)\alpha_n\langle VSw_n,x_n-z_0\rangle.\end{aligned}\tag{3.10}$$

We will divide the proof into two cases.

Case 1: Suppose that $\Gamma_{n+1}\le\Gamma_n$ for all $n\in\mathbb{N}$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1}-\Gamma_n)=0$. Using $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$ and $\lim_{n\to\infty}\alpha_n=0$, we have from (3.10) that

$$\lim_{n\to\infty}\|Sw_n-x_n\|=0.\tag{3.11}$$

Using (3.8), we also have that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{3.12}$$

Since $x_{n+1}-x_n=(1-\beta_n)(y_n-x_n)$, we have from (3.12) that

$$\lim_{n\to\infty}\|y_n-x_n\|=0.\tag{3.13}$$

We also have from (2.6) that

$$\begin{aligned}2\|u_n - z_0\|^2 &= 2\|T_{r_n}x_n - T_{r_n}z_0\|^2\\ &\le 2\langle x_n - z_0, u_n - z_0\rangle\\ &= \|x_n - z_0\|^2 + \|u_n - z_0\|^2 - \|u_n - x_n\|^2\end{aligned}$$

and hence

$$\|u_n - z_0\|^2 \le \|x_n - z_0\|^2 - \|u_n - x_n\|^2. \tag{3.14}$$

From (3.1) we have that

$$\|Sw_n - z_0\|^2 \le \|u_n - z_0\|^2 \le \|x_n - z_0\|^2 - \|u_n - x_n\|^2$$

and hence

$$\|u_n - x_n\|^2 \le \|x_n - z_0\|^2 - \|Sw_n - z_0\|^2 \le M\|Sw_n - x_n\|^2,$$

where $M = \sup\{\|x_n - z_0\| + \|Sw_n - z_0\| : n \in \mathbb{N}\}$. Thus from (3.11) we have that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.15}$$

We will show $\lim_{n\to\infty} \|Sw_n - w_n\| = 0$. Since $\|\cdot\|^2$ is a convex function, we have that

$$\|x_{n+1} - z_0\|^2 \le \beta_n \|x_n - z_0\|^2 + (1-\beta_n)\|y_n - z_0\|^2. \tag{3.16}$$

From $z_0 = \alpha_n V z_0 + z_0 - \alpha_n V z_0$ and (2.1) we also have that

$$\begin{aligned}\|y_n - z_0\|^2 &= \|\alpha_n(\gamma g(x_n) - Vz_0) + (I - \alpha_n V)Sw_n - (I-\alpha_n V)z_0\|^2\\ &\le (1-\alpha_n\tau)^2\|Sw_n - z_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz_0, y_n - z_0\rangle\\ &\le (1-\alpha_n\tau)^2\|w_n - z_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz_0, y_n - z_0\rangle\\ &\le \|w_n - z_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz_0, y_n - z_0\rangle\\ &\le \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Az\|^2\\ &\qquad + 2\alpha_n\langle \gamma g(x_n) - Vz_0, y_n - z_0\rangle.\end{aligned} \tag{3.17}$$

Using (3.16) and (3.17), we have that

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &\le \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|x_n - z_0\|^2\\ &\quad + (1-\beta_n)(\lambda_n(\lambda_n - 2\alpha)\|Au_n - Az_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz_0, y_n - z_0\rangle)\\ &= \|x_n - z_0\|^2 + (1-\beta_n)(\lambda_n(\lambda_n - 2\alpha)\|Au_n - Az_0\|^2\\ &\quad + 2\alpha_n\langle \gamma g(x_n) - Vz_0, y_n - z_0\rangle).\end{aligned} \tag{3.18}$$

Thus we have that

$$\begin{aligned}&(1-\beta_n)\lambda_n(2\alpha - \lambda_n)\|Au_n - Az\|^2\\ &\quad \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1-\beta_n)2\alpha_n\langle \gamma g(x_n) - Vz_0, y_n - z_0\rangle.\end{aligned} \tag{3.19}$$

Then we have that

$$\lim_{n\to\infty} \|Au_n - Az_0\| = 0. \tag{3.20}$$

Since J_{λ_n} is firmly nonexpansive, we have that

$$\begin{aligned}
2\|w_n-z_0\|^2 &= 2\left\|J_{\lambda_n}(u_n-\lambda_n Au_n)-J_{\lambda_n}(z_0-\lambda_n Az_0)\right\|^2\\
&\le 2\left\langle u_n-\lambda_n Au_n-(z_0-\lambda_n Az_0), w_n-z_0\right\rangle\\
&= \left\|u_n-\lambda_n Au_n-(z_0-\lambda_n Az_0)\right\|^2+\left\|w_n-z_0\right\|^2\\
&\qquad -\left\|u_n-\lambda_n Au_n-(z_0-\lambda_n Az_0)-(w_n-z_0)\right\|^2\\
&\le \left\|u_n-z_0\right\|^2+\left\|w_n-z_0\right\|^2\\
&\qquad -\left\|u_n-w_n-\lambda_n(Au_n-Az_0)\right\|^2\\
&\le \left\|x_n-z_0\right\|^2+\left\|w_n-z_0\right\|^2-\left\|u_n-w_n\right\|^2\\
&\qquad +2\lambda_n\left\langle u_n-w_n, Au_n-Az_0\right\rangle-\lambda_n^2\left\|Au_n-Az_0\right\|^2.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\|w_n-z_0\|^2 &\le \left\|x_n-z_0\right\|^2-\left\|u_n-w_n\right\|^2\\
&\qquad +2\lambda_n\left\langle u_n-w_n, Au_n-Az_0\right\rangle-\lambda_n^2\left\|Au_n-Az_0\right\|^2.
\end{aligned}\tag{3.21}$$

Using (3.17), we obtain

$$\begin{aligned}
\|x_{n+1}-z_0\|^2 &\le \beta_n\left\|x_n-z_0\right\|^2+(1-\beta_n)\|y_n-z_0\|^2\\
&\le \beta_n\left\|x_n-z_0\right\|^2+(1-\beta_n)(\|w_n-z_0\|^2+2\alpha_n\langle\gamma g(x_n)-Vz_0, y_n-z_0\rangle)\\
&\le \beta_n\left\|x_n-z_0\right\|^2+(1-\beta_n)\left\|x_n-z_0\right\|^2\\
&\quad -(1-\beta_n)\left\|u_n-w_n\right\|^2+(1-\beta_n)2\lambda_n\left\langle u_n-w_n, Au_n-Az_0\right\rangle\\
&\quad -(1-\beta_n)\lambda_n^2\left\|Au_n-Az_0\right\|^2+(1-\beta_n)2\alpha_n\langle\gamma g(x_n)-Vz_0, y_n-z_0\rangle\\
&= \|x_n-z_0\|^2-(1-\beta_n)\left\|u_n-w_n\right\|^2\\
&\quad +(1-\beta_n)2\lambda_n\left\langle u_n-w_n, Au_n-Az_0\right\rangle-(1-\beta_n)\lambda_n^2\left\|Au_n-Az_0\right\|^2\\
&\quad +(1-\beta_n)2\alpha_n\langle\gamma g(x_n)-Vz_0, y_n-z_0\rangle.
\end{aligned}$$

So we have that

$$\begin{aligned}
(1-\beta_n)\left\|x_n-w_n\right\|^2 &\le \left\|x_n-z_0\right\|^2\\
&\quad -\left\|x_{n+1}-z_0\right\|^2+2\lambda_n\left\langle u_n-w_n, Au_n-Az_0\right\rangle\\
&\quad -\lambda_n^2\left\|Au_n-Az_0\right\|^2+2\alpha_n\langle\gamma g(x_n)-Vz_0, y_n-z_0\rangle).
\end{aligned}$$

Then we have

$$\lim_{n\to\infty}\|u_n-w_n\|=0. \tag{3.22}$$

From (3.22) and (3.15) we have that

$$\lim_{n\to\infty}\|x_n-w_n\|=0. \tag{3.23}$$

Since $\|Sw_n-w_n\|\le\|Sw_n-x_n\|+\|x_n-w_n\|$, we have that

$$\lim_{n\to\infty}\|Sw_n-w_n\|=0. \tag{3.24}$$

Take $\lambda_0 \in \mathbb{R}$ with $0 < a \leq \lambda_0 \leq b < 2\alpha$ arbitrarily. Put $s_n = (I - \lambda_n A)u_n$. Using $u_n = T_{r_n} x_n$ and $w_n = J_{\lambda_n}(I - \lambda_n A)u_n$, we have from Lemma 4 that

$$\begin{aligned}
\|J_{\lambda_0}(I - \lambda_0 A)u_n - w_n\| &= \|J_{\lambda_0}(I - \lambda_0 A)u_n - J_{\lambda_n}(I - \lambda_n A)u_n\| \\
&= \|J_{\lambda_0}(I - \lambda_0 A)u_n - J_{\lambda_0}(I - \lambda_n A)u_n \\
&\quad + J_{\lambda_0}(I - \lambda_n A)u_n - J_{\lambda_n}(I - \lambda_n A)u_n\| \\
&\leq \|(I - \lambda_0 A)u_n - (I - \lambda_n A)u_n\| + \|J_{\lambda_0} s_n - J_{\lambda_n} s_n\| \\
&\leq |\lambda_0 - \lambda_n|\|Au_n\| + \frac{|\lambda_0 - \lambda_n|}{\lambda_0}\|J_{\lambda_0} s_n - s_n\|.
\end{aligned} \tag{3.25}$$

We also have from (3.25) that

$$\|u_n - J_{\lambda_0}(I - \lambda_0 A)u_n\| \leq \|u_n - w_n\| + \|w_n - J_{\lambda_0}(I - \lambda_0 A)u_n\|. \tag{3.26}$$

We will use (3.25) and (3.26) later.

Let us show that $\limsup_{n\to\infty} \langle (V - \gamma g)z_0, x_n - z_0\rangle \geq 0$. Put

$$A = \limsup_{n\to\infty} \langle (V - \gamma g)z_0, x_n - z_0\rangle .$$

Without loss of generality, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $A = \lim_{i\to\infty} \langle (V - \gamma g)z_0, x_{n_i} - z_0\rangle$ and $\{x_{n_i}\}$ converges weakly some point $w \in H$. From $\|x_n - w_n\| \to 0$ and $\|x_n - u_n\| \to 0$, we also have that $\{w_{n_i}\}$ and $\{u_{n_i}\}$ converge weakly to $w \in C$. On the other hand, from $\{\lambda_{n_i}\} \subset [a, b]$ there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $\lambda_{n_{i_j}} \to \lambda_0$ for some $\lambda_0 \in [a, b]$. Without loss of generality, we assume that $w_{n_i} \to w$, $u_{n_i} \to w$ and $\lambda_{n_i} \to \lambda_0$. From (3.24) we know $\lim_{n\to\infty} \|Sw_n - w_n\| = 0$. Thus we have from Lemma 3 that $w = Sw$. Since W is a monotone operator and $\frac{x_{n_i} - u_{n_i}}{r_{n_i}} \in Wu_{n_i}$, we have that for any $(u, v) \in W$,

$$\langle u - u_{n_i}, v - \frac{x_{n_i} - u_{n_i}}{r_{n_i}}\rangle \geq 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, $u_{n_i} \rightharpoonup w$ and $x_{n_i} - u_{n_i} \to 0$, we have

$$\langle u - w, v\rangle \geq 0.$$

Since W is a maximal monotone operator, we have $0 \in Ww$ and hence $w \in W^{-1}0$. Since $\lambda_{n_i} \to \lambda_0$, we have from (3.25) that

$$\|J_{\lambda_0}(I - \lambda_0 A)u_{n_i} - w_{n_i}\| \to 0.$$

Furthermore, we have from (3.26) that

$$\|u_{n_i} - J_{\lambda_0}(I - \lambda_0 A)u_{n_i}\| \to 0.$$

Since $J_{\lambda_0}(I - \lambda_0 A)$ is nonexpansive, we have that $w = J_{\lambda_0}(I - \lambda_0 A)w$. This means that $0 \in Aw + Bw$. Thus we have

$$w \in F(T) \cap (A + B)^{-1}0 \cap W^{-1}0.$$

Then we have

$$A = \lim_{i\to\infty} \langle (V - \gamma g)z_0, x_{n_i} - z_0\rangle = \langle (V - \gamma g)z_0, w - z_0\rangle \geq 0. \tag{3.27}$$

Since $y_n - z_0 = \alpha_n(\gamma g(x_n) - Vz_0) + (I - \alpha_n V)Sw_n - (I - \alpha_n V)z_0$, we have

$$\|y_n - z_0\|^2 \leq (1 - \alpha_n\tau)^2 \|Sw_n - z_0\|^2 + 2\alpha_n \langle \gamma g(x_n) - Vz_0, y_n - z_0\rangle .$$

Thus we have

$$\|y_n - z_0\|^2 \leq (1 - \alpha_n\tau)^2 \|x_n - z_0\|^2 + 2\alpha_n \langle \gamma g(x_n) - Vz_0, y_n - z_0\rangle .$$

Then we have that

$$\begin{aligned}
\|x_{n+1}-z_0\|^2 &\le \beta_n \|x_n-z_0\|^2 + (1-\beta_n)\|y_n-z_0\|^2 \\
&\le \beta_n \|x_n-z_0\|^2 \\
&\quad + (1-\beta_n)\Big((1-\alpha_n\tau)^2\|x_n-z_0\|^2 + 2\alpha_n\langle \gamma g(x_n)-Vz_0, y_n-z_0\rangle\Big) \\
&= \big(\beta_n+(1-\beta_n)(1-\alpha_n\tau)^2\big)\|x_n-z_0\|^2 \\
&\quad + 2(1-\beta_n)\alpha_n\langle \gamma g(x_n)-Vz_0, y_n-z_0\rangle \\
&\le \big(1-(1-\beta_n)(2\alpha_n\tau-(\alpha_n\tau)^2)\big)\|x_n-z_0\|^2 \\
&\quad + 2(1-\beta_n)\alpha_n\gamma\, k\|x_n-z_0\|^2 + 2(1-\beta_n)\alpha_n\langle \gamma g(z_0)-Vz_0, y_n-z_0\rangle \\
&= (1-2(1-\beta_n)\alpha_n(\tau-\gamma\, k))\|x_n-z_0\|^2 \\
&\quad + (1-\beta_n)(\alpha_n\tau)^2\|x_n-z_0\|^2 + 2(1-\beta_n)\alpha_n\langle \gamma g(z_0)-Vz_0, y_n-z_0\rangle \\
&= (1-2(1-\beta_n)\alpha_n(\tau-\gamma\, k))\|x_n-z_0\|^2 \\
&\quad + 2(1-\beta_n)\alpha_n(\tau-\gamma\, k)\left(\frac{\alpha_n\tau^2\|x_n-z_0\|^2}{2(\tau-\gamma\, k)} + \frac{\langle \gamma g(z_0)-Vz_0, y_n-z_0\rangle}{\tau-\gamma\, k}\right).
\end{aligned}$$

By (3.27) and Lemma 5, we obtain that $x_n \to z_0$, where

$$z_0 = P_{F(S)\cap(A+B)^{-1}0\cap W^{-1}0}(I-V+\gamma g)z_0.$$

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 6 that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. Thus we have from (3.10) that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\beta_{\tau(n)}(1-&\beta_{\tau(n)})\|Sw_{\tau(n)}-x_{\tau(n)}\|^2 \\
&\le (1-\beta_{\tau(n)})^2 2\alpha_{\tau(n)}\|Sw_{\tau(n)}-x_{\tau(n)}\|\|\gamma g(x_{\tau(n)})-VSw_{\tau(n)}\| \\
&\quad + (1-\beta_{\tau(n)})^2\alpha_{\tau(n)}^2\|\gamma g(x_{\tau(n)})-VSw_{\tau(n)}\|^2 \\
&\quad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle \gamma g(x_{\tau(n)}), x_{\tau(n)}-z_0\rangle \\
&\quad - 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle VSw_{\tau(n)}, x_{\tau(n)}-z_0\rangle.
\end{aligned} \tag{3.28}$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, we have from (3.28) and Lemma 6 that

$$\lim_{n\to\infty}\|Sw_\tau(n)-x_\tau(n)\| = 0. \tag{3.29}$$

As in the proof of Case 1 we also have that

$$\lim_{n\to\infty}\|x_{\tau(n)+1}-x_{\tau(n)}\| = 0 \tag{3.30}$$

and

$$\lim_{n\to\infty}\|y_{\tau(n)}-x_{\tau(n)}\| = 0. \tag{3.31}$$

Furthermore, we have that $\lim_{n\to\infty}\|u_{\tau(n)}-x_{\tau(n)}\| = 0$, $\lim_{n\to\infty}\|Au_{\tau(n)}-Az_0\| = 0$, $\lim_{n\to\infty}\left\|u_{\tau(n)}-w_{\tau(n)}\right\| = 0$ and $\lim_{n\to\infty}\left\|x_{\tau(n)}-w_{\tau(n)}\right\| = 0$. From these we have that $\lim_{n\to\infty}\|Sw_{\tau(n)}-w_{\tau(n)}\| = 0$. As in the proof of Case 1, we can show that

$$\limsup_{n\to\infty}\left\langle (V-\gamma g)z_0, x_{\tau(n)}-z_0\right\rangle \ge 0.$$

We also have that

$$\left\|y_{\tau(n)} - z_0\right\|^2 \leq (1-\alpha_{\tau(n)}\tau)^2 \left\|x_{\tau(n)} - z_0\right\|^2 + 2\alpha_{\tau(n)} \left\langle \gamma g(x_{\tau(n)}) - Vz_0, y_{\tau(n)} - z_0 \right\rangle$$

and then

$$\begin{aligned}\|x_{\tau(n)+1} - z_0\|^2 &\leq \left(1 - 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}(\tau - \gamma\ k)\right) \left\|x_{\tau(n)} - z_0\right\|^2 \\ &+ (1-\beta_{\tau(n)})(\alpha_{\tau(n)}\tau)^2\|x_{\tau(n)} - z_0\|^2 + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle \gamma g(z_0) - Vz_0, y_{\tau(n)} - z_0\rangle.\end{aligned}$$

From $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we have that

$$\begin{aligned}&2(1-\beta_{\tau(n)})\alpha_{\tau(n)}(\tau - \gamma\ k)) \left\|x_{\tau(n)} - z_0\right\|^2 \\ &\leq (1-\beta_{\tau(n)})(\alpha_{\tau(n)}\tau)^2\|x_{\tau(n)} - z_0\|^2 + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle \gamma g(z_0) - Vz_0, y_{\tau(n)} - z_0\rangle.\end{aligned}$$

Since $(1-\beta_{\tau(n)})\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}&2(\tau - \gamma\ k) \left\|x_{\tau(n)} - z_0\right\|^2 \\ &\qquad \leq \alpha_{\tau(n)}\tau^2\|x_{\tau(n)} - z_0\|^2 + 2\langle \gamma g(z_0) - Vz_0, y_{\tau(n)} - z_0\rangle.\end{aligned}$$

Thus we have that

$$\limsup_{n\to\infty} 2(\tau - \gamma\ k) \left\|x_{\tau(n)} - z_0\right\|^2 \leq 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$ as $n \to \infty$. Since $x_{\tau(n)} - x_{\tau(n)+1} \to 0$, we have $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 6 again, we obtain that

$$\|x_n - z_0\| \leq \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

4. Applications

In this section, using Theorem 10, we can obtain well-known and new strong convergence theorems for in a Hilbert space. Let H be a Hilbert space and let f be a proper lower semicontinuous convex function of H into $(-\infty, \infty]$. Then, the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \leq f(y),\ \forall y \in H\}$$

for all $x \in H$. From Rockafellar [25], we know that ∂f is a maximal monotone operator. Let C be a nonempty closed convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then, i_C is a proper lower semicontinuous convex function on H and then the subdifferential ∂i_C of i_C is a maximal monotone operator. So, we can define the resolvent J_λ of ∂i_C for $\lambda > 0$, i.e.,

$$J_\lambda x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$. We know that $J_\lambda x = P_C x$ for all $x \in H$ and $\lambda > 0$; see [30].

Theorem 11. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let S be a generalized hybrid mapping of C into C. Suppose $F(S) \neq \emptyset$. Let $u, x_1 \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\{\alpha_n u + (1-\alpha_n)Sx_n\}$$

for all $n \in \mathbb{N}$, where $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$$
$$\text{and} \quad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 \in F(S)$, where $z_0 = P_{F(S)}u$.

Proof. Put $A = 0$, $B = W = \partial i_C$ and $\lambda_n = r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 10. Then we have $J_{\lambda_n} = T_{r_n} = P_C$ for all $n \in \mathbb{N}$. Furthermore, put $g(x) = u$ and $V(x) = x$ for all $x \in H$. Then, we can take $\overline{\gamma} = L = 1$. Thus we can take $\mu = 1$. On the other hand, since $\|g(x) - g(y)\| = 0 \le \frac{1}{3}\|x - y\|$ for all $x, y \in H$, we can take $k = \frac{1}{3}$. So, we can take $\gamma = 1$. Then for $u, x_1 \in C$, we get that

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n u + (I - \alpha_n)Sx_n\}$$

for all $n \in \mathbb{N}$. So, we have $\{x_n\} \subset C$. We also have

$$z_0 = P_{F(S)\cap C}(I - V + \gamma g)z_0 = P_{F(S)}(z_0 - z_0 + 1 \cdot u) = P_{F(S)}u.$$

Thus we obtain the desired result by Theorem 10. □

Theorem 11 solves a problem posed by Kurokawa and Takahashi [16]. The following result is a strong convergence theorem of Halpern's type [10] for finding a common solution of a monotone inclusion problem for the sum of two monotone mappings, of a fixed point problem for nonexpansive mappings and of an equilibrium problem for bifunctions in a Hilbert space.

Theorem 12. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A be an α-inverse strongly-monotone mapping of C into H. Let B and W be maximal monotone operators on H such that the domains of B and W are included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ and $T_r = (I + rW)^{-1}$ be resolvents of B and W for $\lambda > 0$ and $r > 0$, respectively. Let S be a nonexpansive mapping of C into H. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Suppose $F(S) \cap (A + B)^{-1}0 \cap W^{-1}0 \neq \emptyset$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n \gamma g(x_n) + (I - \alpha_n V)SJ_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n\}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1,$$
$$\liminf_{n\to\infty} r_n > 0 \quad \text{and} \quad 0 < a \le \lambda_n \le b < 2\alpha.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap (A + B)^{-1}0 \cap W^{-1}0$, where $z_0 = P_{F(S)\cap(A+B)^{-1}0\cap W^{-1}0}(I - V + \gamma g)z_0$.

Proof. We know that a nonexpansive mapping T of C into H is a (1,0)-generalized hybrid mapping. So, we obtain the desired result by Theorem 10. □

The following lemmas were given in Combettes and Hirstoaga [8] and Takahashi, Takahashi and Toyoda [27]; see also [1].

Lemma 13 ([8]). *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Assume that $f : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

We call such T_r the resolvent of f for $r > 0$.

Lemma 14 ([27]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy $(A1)-(A4)$. Let A_f be a set-valued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, \quad \forall x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $D(A_f) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, i.e.,

$$T_r x = (I + rA_f)^{-1} x.$$

Using Lemmas 13, 14 and Theorem 10, we also obtain the following result for generalized hybrid mappings of C into H with equilibrium problem in a Hilbert space.

Theorem 15. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let S be a generalised hybrid mapping of C into H. Let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator of H into itself with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Suppose that $F(S) \cap EP(f) \neq \emptyset$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n \gamma g(x_n) + (I - \alpha_n V)Su_n\}$$

for all $n \in \mathbb{N}$*, where* $\{\beta_n\} \subset (0,1)$*,* $\{\alpha_n\} \subset (0,1)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \liminf_{n\to\infty} r_n > 0,$$

$$\text{and} \quad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 \in F(S) \cap EP(f)$*, where* $z_0 = P_{F(S)\cap EP(f)}(I - V + \gamma g)z_0$*.*

Proof. Put $A = 0$ and $B = \partial i_C$ in Theorem 10. Futhermore, for the bifunction $f : C \times C \to \mathbb{R}$, define A_f as in Lemma 14. Put $W = A_f$ in Theorem 10 and let T_{r_n} be the resolvent of A_f for $r_n > 0$. Then we obtain that the domain of A_f is included in C and $T_{r_n} x_n = u_n$ for all $n \in \mathbb{N}$. Thus we obtain the desired result by Theorem 10. □

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