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Strong and weak convergence theorems for an infinite family of nonexpansive mappings and applications

LC Ceng[1,2], NC Wong[3] and JC Yao[4*]

*Correspondence:
yaojc@kmu.edu.tw
[4]Center for General Education, Kaohsiung Medical University, Kaohsiung, Taiwan 807
Full list of author information is available at the end of the article

Abstract

In this paper, let E be a reflexive and strictly convex Banach space which either is uniformly smooth or has a weakly continuous duality map. We consider the hybrid viscosity approximation method for finding a common fixed point of an infinite family of nonexpansive mappings in E. We prove the strong convergence of this method to a common fixed point of the infinite family of nonexpansive mappings, which solves a variational inequality on their common fixed point set. We also give a weak convergence theorem for the hybrid viscosity approximation method involving an infinite family of nonexpansive mappings in a Hilbert space.

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Keywords: hybrid viscosity approximation method; nonexpansive mapping; strictly convex Banach space; uniformly smooth Banach space; reflexive Banach space with weakly continuous duality map

1 Introduction

Let C be a nonempty closed convex subset of a (real) Banach space E, and let $T : C \to C$ be a nonlinear mapping. Denote by $F(T)$ the set of fixed points of T, *i.e.*, $F(T) = \{x \in C : Tx = x\}$. Recall that T is *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

A self-mapping $f : C \to C$ is said to be a *contraction* on C if there exists a constant α in $(0,1)$ such that

$$\|f(x) - f(y)\| \le \alpha \|x - y\|, \quad \forall x, y \in C.$$

As in [1], we use the notation Π_C to denote the collection of all contractions on C, *i.e.*,

$$\Pi_C = \{f : C \to C \text{ is a contraction}\}.$$

Note that each f in Π_C has a unique fixed point in C.

One classical way to study a nonexpansive mapping $T : C \to C$ is to use contractions to approximate T [2–4]. More precisely, for each t in $(0,1)$ we define a contraction $T_t : C \to C$

by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in C,$$

where u in C is an arbitrary but fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, how x_t behaves as $t \to 0^+$, even if T has a fixed point. However, in the case $E = H$ a Hilbert space and T having a fixed point, Browder [2] proved that x_t converges strongly to a fixed point of T. Reich [3] extends Browder's result and proves that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction $u \mapsto Q(u)$ from C onto $F(T)$. Xu [4] proved that Browder's results hold in reflexive Banach spaces with weakly continuous duality mappings. See Section 2 for definitions and notations.

Recall that the original Mann's iterative process was introduced in [5] in 1953. Let $T : C \to C$ be a map of a closed and convex subset C of a Hilbert space. The original Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily,} \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1, \end{cases} \tag{1.1}$$

where the sequence $\{\alpha_n\}$ lies in the interval $(0,1)$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = +\infty$, then the sequence $\{x_n\}$ generated by original Mann's iterative process (1.1) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with a Frechet differentiable norm [6]). In an infinite-dimensional Hilbert space, the original Mann's iterative process guarantees only the weak convergence. Therefore, many authors try to modify the original Mann's iterative process to ensure the strong convergence for nonexpansive mappings (see [3, 7–13] and the references therein).

Kim and Xu [14] proposed the following simpler modification of the original Mann's iterative process: Let C be a nonempty closed convex subset of a Banach space E and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. For an arbitrary x_0 in C, define $\{x_n\}$ in the following way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T x_n, \\ x_{n+1} = \beta_n u + (1-\beta_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where u in C is an arbitrary but fixed element in C, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. The modified Mann's Iteration scheme (1.2) is a convex combination of a particular point u in C and the original Mann's iterative process (1.1). There is no additional projection involved in iteration scheme (1.2). They proved a strong convergence theorem for the iteration scheme (1.2) under some control conditions on the parameters α_n's and β_n's.

Recently, Yao, Chen and Yao [12] combined the viscosity approximation method [1] and the modified Mann's iteration scheme [14] to develop the following hybrid viscosity approximation method. Let C be a nonempty closed convex subset of a Banach space E, let $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$, and let $f \in \Pi_C$. For any arbitrary but fixed point x_0 in C, define $\{x_n\}$ in the following way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. They proved under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ converges strongly to a fixed point of T. Their result extends and improves the main results in Kim and Xu [14].

Under the assumption that no parameter sequence converges to zero, Ceng and Yao [15] proved the strong convergence of the sequence $\{x_n\}$ generated by (1.3) to a fixed point of T, which solves a variational inequality on $F(T)$.

Theorem 1.1 (See [15, Theorem 3.1]) *Let C be a nonempty closed convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and let $f \in \Pi_C$ with a contractive constant α in $(0,1)$. Given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$ such that the following control conditions are satisfied:*

(C1) $0 \leq \beta_n \leq 1-\alpha$, $\forall n \geq n_0$ *for some integer* $n_0 \geq 1$, *and* $\sum_{n=0}^{\infty} \beta_n = +\infty$;

(C2) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(C3) $\lim_{n\to\infty}(\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}) = 0$.

For an arbitrary x_0 in C, let $\{x_n\}$ be defined by (1.3). Then,

$$x_n \textit{ converges strongly to some } Q(f) \textit{ in } F(T) \quad \Leftrightarrow \quad \beta_n\bigl(f(x_n) - x_n\bigr) \to 0.$$

In this case, $Q(f) \in F(T)$ solves the variational inequality

$$\bigl\langle (I-f)Q(f), J\bigl(Q(f)-p\bigr)\bigr\rangle \leq 0, \quad f \in \Pi_C, p \in F(T).$$

On the other hand, a similar problem concerning a family of nonexpansive mappings has also been considered by many authors. The well-known convex feasibility problem reduces to finding a common fixed point of a family of nonexpansive mappings; see, *e.g.*, [16, 17]. The problem of finding an optimal point that minimizes a given cost function over the common fixed point set of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance; see, *e.g.*, [18–20]. In particular, a simple algorithm solving the problem of minimizing a quadratic function over the common fixed point set of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation; see, *e.g.*, [20, 21].

Let $T_1, T_2, \ldots$ be nonexpansive mappings of a nonempty closed and convex subset C of a Banach space E into itself. Let $\lambda_1, \lambda_2, \ldots$ be real numbers in $[0,1]$. Qin, Cho, Kang and

Kang [22] considered the nonexpansive mapping W_n defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \dots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \dots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I, \quad \forall n \geq 1. \end{cases} \tag{1.4}$$

Motivated by [7, 8, 11, 12, 14, 23], they proposed the following iterative algorithm:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) W_n x_n, \\ x_{n+1} = \beta_n u + (1-\beta_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{1.5}$$

where u in C is a given point. They proved

Theorem 1.2 (See [22, Theorem 2.1 and its proof]) *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space E with a weakly continuous duality map J_φ with gauge φ. Let T_i be a nonexpansive mapping from C into itself for $i = 1, 2, \dots$. Assume that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Given $u \in C$ and given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ in $(0, 1)$ satisfying*

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = +\infty$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(iii) $0 < \lambda_n \leq b < 1$, $\forall n \geq 1$ *for some b in* $(0, 1)$.

Then the sequence $\{x_n\}$ defined by (1.5) converges strongly to some point $Q(u)$ in F. Here, $Q : C \to F$ thus defined is the unique sunny nonexpansive retraction of Reich type from C onto F, that is, $Q(u) \in F$ solves the variational inequality

$$\langle Q(u) - u, J_\varphi(Q(u) - p) \rangle \leq 0, \quad u \in C, p \in F.$$

In this paper, let E be a reflexive and strictly convex Banach space which either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ. Combining two iterative methods (1.3) and (1.5), we give the following hybrid viscosity approximation scheme. Let C be a nonempty closed convex subset of E, let $T_i : C \to C$ be a nonexpansive mapping for each $i = 1, 2, \dots$, such that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $f \in \Pi_C$. Define $\{x_n\}$ in the following way:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) W_n x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where W_n is defined by (1.4), $\{\lambda_n\}$ is a sequence in $(0,1)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$. It is proved under some appropriate control conditions on the sequences $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ converges strongly to a common fixed point $Q(f)$ of the infinite family of nonexpansive mappings $T_1, T_2, \dots$, which solves a variational inequality on $F = \bigcap_{n=1}^{\infty} F(T_n)$. Such a result includes Theorem 1.2 as a special case. Furthermore, we also give a weak convergence theorem for the hybrid viscosity approximation method (1.6) involving an infinite family of nonexpansive mappings $T_1, T_2, \dots$ in a Hilbert space H. The results presented in this paper can be viewed as supplements, improvements and extensions of some known results in the literature, *e.g.*, [1, 7, 8, 11–15, 22–24].

2 Preliminaries

Let E be a (real) Banach space with the Banach dual space E^* in pairing $\langle \cdot, \cdot \rangle$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x, and $x_n \to x$ to indicate that $\{x_n\}$ converges strongly to x. The unit sphere of E is denoted by $U = \{x \in E : \|x\| = 1\}$.

The norm of E is said to be *Gateaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for every x, y in U. Recall that if E is reflexive, then E is smooth if and only if E^* is *strictly convex*, *i.e.*, for every distinct x^*, y^* in E^* of norm one, there holds $\|x^* + y^*\|/2 < 1$. The norm of E is said to be *uniformly Frechet differentiable* (and E is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for (x, y) in $U \times U$. Every uniformly smooth Banach space E is reflexive and smooth.

The *normalized duality mapping* J from E into the family of nonempty (by Hahn-Banach theorem) weak* compact subsets of E^* is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

If E is smooth then J is single-valued and norm-to-weak* continuous. It is also well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E.

In order to establish new strong and weak convergence theorems for hybrid viscosity approximation method (1.6), we need the following lemmas. The first lemma is a very well-known (subdifferential) inequality; see, *e.g.*, [25].

Lemma 2.1 ([25]) *Let E be a real Banach space and J the normalized duality map on E. Then, for any given x, y in E, the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 2.2 ([26, Lemma 2]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E, and let $\{\beta_n\}$ be a sequence in $[0,1]$ such that $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3 ([27]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$s_{n+1} \leq (1 - \mu_n)s_n + \mu_n \nu_n, \quad \forall n \geq 1,$$

where $\{\mu_n\}$, $\{\nu_n\}$ are sequences of real numbers such that

(i) $\{\mu_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \mu_n = +\infty$, or equivalently,

$$\prod_{n=1}^{\infty}(1 - \mu_n) := \lim_{n\to\infty} \prod_{k=1}^{n}(1 - \mu_k) = 0;$$

(ii) $\limsup_{n\to\infty} \nu_n \leq 0$, or $\sum_{n=1}^{\infty} \mu_n \nu_n$ is convergent.

Then, $\lim_{n\to\infty} s_n = 0$.

Recall that, if $D \subseteq C$ are nonempty subsets of a Banach space E such that C is nonempty, closed and convex, then a mapping $Q : C \to D$ is *sunny* [28] provided $Q(x + t(x - Q(x))) = Q(x)$ for all x in C and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A *sunny nonexpansive retraction* is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role; see, *e.g.*, [1, 22]. They are characterized as follows [28]: if E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leq 0, \quad \forall x \in C, y \in D.$$

Lemma 2.4 ([1, Theorem 4.1]) *Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \Pi_C$. Then $\{x_t\}$ defined by*

$$x_t = tf(x_t) + (1 - t)Tx_t, \quad \forall t \in (0,1),$$

converges strongly to a point in $F(T)$. Define $Q : \Pi_C \to F(T)$ by

$$Q(f) := \lim_{t\to 0^+} x_t.$$

Then, $Q(f)$ solves the variational inequality

$$\big\langle (I - f)Q(f), J\big(Q(f) - p\big)\big\rangle \leq 0, \quad \forall p \in F(T).$$

In particular, if $f = u \in C$ is a constant, then the map $u \mapsto Q(u)$ is reduced to the sunny nonexpansive retraction of Reich type from C onto $F(T)$, i.e.,

$$\big\langle Q(u) - u, J\big(Q(u) - p\big)\big\rangle \leq 0, \quad \forall p \in F(T).$$

Recall that a *gauge* is a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to gauge φ is the duality map $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \big\{x^* \in E^* : \langle x, x^*\rangle = \|x\|\varphi\big(\|x\|\big), \|x^*\| = \varphi\big(\|x\|\big)\big\}, \quad \forall x \in E.$$

Following Browder [29], we say that a Banach space E has a *weakly continuous duality map* if there exists gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous. It is known that l^p has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$ for all $1 < p < +\infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)\,d\tau, \quad \forall t \geq 0.$$

Then

$$J_\varphi(x) = \partial\Phi\big(\|x\|\big), \quad \forall x \in E,$$

where ∂ denotes the subdifferential in the sense of convex analysis; see [25, 30] for more details.

The first part of the following lemma is an immediate consequence of the subdifferential inequality, and the proof of the second part can be found in [31].

Lemma 2.5 *Assume that E has a weakly continuous duality map J_φ with gauge φ.*

(i) For all $x, y \in E$, there holds the inequality

$$\Phi\big(\|x+y\|\big) \leq \Phi\big(\|x\|\big) + \big\langle y, J_\varphi(x+y)\big\rangle.$$

(ii) Assume a sequence $\{x_n\}$ in E is weakly convergent to a point x. Then there holds the identity

$$\limsup_{n\to\infty} \Phi\big(\|x_n - y\|\big) = \limsup_{n\to\infty} \Phi\big(\|x_n - x\|\big) + \Phi\big(\|y - x\|\big), \quad \forall y \in E.$$

Xu [4] showed that, if E is a reflexive Banach space and has a weakly continuous duality map J_φ with gauge φ, then there is a sunny nonexpansive retraction from C onto $F(T)$. Further this result is extended to the following general case.

Lemma 2.6 ([32, Theorem 3.1 and its proof]) *Let E be a reflexive Banach space and have a weakly continuous duality map J_φ with gauge φ, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and let $f \in \Pi_C$. Then $\{x_t\}$ defined by*

$$x_t = tf(x_t) + (1-t)Tx_t, \quad \forall t \in (0,1),$$

converges strongly to a point in $F(T)$ as $t \to 0^+$. Define $Q : \Pi_C \to F(T)$ by

$$Q(f) := \lim_{t\to 0^+} x_t.$$

Then, $Q(f)$ solves the variational inequality

$$\big\langle (I-f)Q(f), J_\varphi\big(Q(f) - p\big)\big\rangle \leq 0, \quad \forall p \in F(T).$$

In particular, if $f = u \in C$ is a constant, then the map $u \mapsto Q(u)$ is reduced to the sunny nonexpansive retraction of Reich type from C onto $F(T)$, i.e.,

$$\langle Q(u) - u, J_\varphi\big(Q(u) - p\big)\rangle \leq 0, \quad \forall p \in F(T).$$

Recall that E satisfies Opial's property [33] provided, for each sequence $\{x_n\}$ in E, the condition $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in E, y \neq x.$$

It is known in [33] that each l^p $(1 \leq p < +\infty)$ enjoys this property, while L^p does not unless $p = 2$. It is known in [34] that every separable Banach space can be equivalently renormed so that it satisfies Opial's property. We denote by $\omega_w(x_n)$ the *weak ω-limit set* of $\{x_n\}$, *i.e.*,

$$\omega_w(x_n) = \big\{\bar{x} \in E : x_{n_i} \rightharpoonup \bar{x} \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\big\}. \tag{2.2}$$

Finally, recall that in a Hilbert space H, there holds the following equality

$$\big\|\lambda x + (1-\lambda)y\big\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in H, \forall \lambda \in [0,1]. \tag{2.3}$$

See, *e.g.*, Takahashi [35].

We will also use the following elementary lemmas in the sequel.

Lemma 2.7 ([36]) *Let $\{a_n\}$ and $\{b_n\}$ be the sequences of nonnegative real numbers such that $\sum_{n=0}^\infty b_n < \infty$ and $a_{n+1} \leq a_n + b_n$ for all $n \geq 0$. Then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.8 (Demiclosedness Principle [25, 30]) *Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $x_n \rightharpoonup x$ in C and $(I - T)x_n \to y$ in H, it follows that $(I - T)x = y$. Here, I is the identity operator of H.*

3 Main results

Lemma 3.1 ([24]) *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings from C into itself such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \leq b < 1$ for all $n \geq 1$. Then, for every x in C and $k \geq 1$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 3.1, one can define the mapping W from C into itself as follows.

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C. \tag{3.1}$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Throughout this paper, we always assume that $0 < \lambda_n \leq b < 1$ for some real constant b and for all $n \geq 1$.

Lemma 3.2 ([24]) *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$*

and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \le b < 1$ for any $n \ge 1$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.

Here comes the main result of this paper.

Theorem 3.3 *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space E. Assume, in addition, E either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ. Let $T_i : C \to C$ be a nonexpansive mapping for each $i = 1, 2, \ldots$ such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and $f \in \Pi_C$ with contractive constant α in $(0,1)$. Given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ in $[0,1]$, the following conditions are satisfied:*

- *(C1) $0 \le \beta_n \le 1 - \alpha$, $\forall n \ge n_0$ for some $n_0 \ge 1$, and $\sum_{n=0}^{\infty} \beta_n = +\infty$;*
- *(C2) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;*
- *(C3) $\lim_{n\to\infty}\left(\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}\right) = 0$;*
- *(C4) $0 < \lambda_n \le b < 1$, $\forall n \ge 1$ for some constant b in $(0,1)$.*

For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) W_n x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) y_n, \quad \forall n \ge 0. \end{cases} \tag{3.2}$$

Then,

$$x_n \text{ converges strongly to some point } Q(f) \text{ in } F$$
$$\iff \quad \beta_n\bigl(f(x_n) - x_n\bigr) \to 0.$$

In this case,

(i) if E is uniformly smooth, then $Q(f) \in F$ solves the variational inequality

$$\bigl\langle (I-f)Q(f), J\bigl(Q(f)-p\bigr)\bigr\rangle \le 0, \quad f \in \Pi_C, p \in F;$$

(ii) if E has a weakly continuous duality map J_φ with gauge φ, then $Q(f) \in F$ solves the variational inequality

$$\bigl\langle (I-f)Q(f), J_\varphi\bigl(Q(f)-p\bigr)\bigr\rangle \le 0, \quad f \in \Pi_C, p \in F.$$

Proof First, let us show that $\{x_n\}$ is bounded. Indeed, taking an element p in $F = \bigcap_{i=1}^{\infty} F(T_i)$ arbitrarily, we obtain that $p = W_n p$ for all $n \ge 0$. It follows from the nonexpansivity of W_n that

$$\|y_n - p\| \le \alpha_n \|x_n - p\| + (1-\alpha_n)\|W_n x_n - p\| \le \|x_n - p\|.$$

Observe that

$$\begin{aligned} \|x_{n+1} - p\| &= \bigl\|\beta_n\bigl(f(x_n) - p\bigr) + (1-\beta_n)(y_n - p)\bigr\| \\ &\le \beta_n\bigl(\bigl\|f(x_n) - f(p)\bigr\| + \bigl\|f(p) - p\bigr\|\bigr) + (1-\beta_n)\|y_n - p\| \\ &\le \beta_n\bigl(\alpha\|x_n - p\| + \bigl\|f(p) - p\bigr\|\bigr) + (1-\beta_n)\|x_n - p\| \end{aligned}$$

$$\begin{aligned} &= \bigl(1-(1-\alpha)\beta_n\bigr)\|x_n-p\| + \beta_n\bigl\|f(p)-p\bigr\| \\ &\le \max\left\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-\alpha}\right\}. \end{aligned}$$

By simple induction, we have

$$\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-\alpha}\right\}.$$

Hence $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{W_nx_n\}$ and $\{f(x_n)\}$.

Suppose that $x_n \to Q(f) \in F$ as $n \to \infty$. Then $Q(f) = W_nQ(f)$ for all $n \ge 0$. From (3.2) it follows that

$$\begin{aligned} \bigl\|y_n-Q(f)\bigr\| &\le \alpha_n\bigl\|x_n-Q(f)\bigr\| + (1-\alpha_n)\bigl\|W_nx_n-Q(f)\bigr\| \\ &\le \alpha_n\bigl\|x_n-Q(f)\bigr\| + (1-\alpha_n)\bigl\|x_n-Q(f)\bigr\| \\ &= \bigl\|x_n-Q(f)\bigr\| \to 0 \quad (n\to\infty), \end{aligned}$$

that is, $y_n \to Q(f)$. Again from (3.2) we obtain that

$$\begin{aligned} \bigl\|\beta_n\bigl(f(x_n)-x_n\bigr)\bigr\| &= \bigl\|x_{n+1}-x_n-(1-\beta_n)(y_n-x_n)\bigr\| \\ &\le \|x_{n+1}-x_n\| + (1-\beta_n)\|y_n-x_n\| \to 0. \end{aligned}$$

Conversely, suppose that $\beta_n(f(x_n)-x_n) \to 0$ $(n\to\infty)$. Put

$$\gamma_n = (1-\beta_n)\alpha_n, \quad \forall n \ge 0.$$

Then, it follows from (C1) and (C2) that

$$\alpha_n \ge \gamma_n = (1-\beta_n)\alpha_n \ge \bigl(1-(1-\alpha)\bigr)\alpha_n = \alpha\alpha_n, \quad \forall n \ge n_0,$$

and hence

$$0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1. \tag{3.3}$$

Define z_n by

$$x_{n+1} = \gamma_nx_n + (1-\gamma_n)z_n. \tag{3.4}$$

Observe that

$$\begin{aligned} &z_{n+1}-z_n \\ &\quad= \frac{x_{n+2}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \frac{x_{n+1}-\gamma_nx_n}{1-\gamma_n} \\ &\quad= \frac{\beta_{n+1}f(x_{n+1})+(1-\beta_{n+1})y_{n+1}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_nf(x_n)+(1-\beta_n)y_n-\gamma_nx_n}{1-\gamma_n} \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\beta_n f(x_n)}{1-\gamma_n}\right) - \frac{(1-\beta_n)[\alpha_n x_n + (1-\alpha_n)W_n x_n] - \gamma_n x_n}{1-\gamma_n} \\
&\quad + \frac{(1-\beta_{n+1})[\alpha_{n+1}x_{n+1} + (1-\alpha_{n+1})W_{n+1}x_{n+1}] - \gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\
&= \left(\frac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\beta_n f(x_n)}{1-\gamma_n}\right) \\
&\quad + \frac{(1-\beta_{n+1})(1-\alpha_{n+1})W_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \frac{(1-\beta_n)(1-\alpha_n)W_n x_n}{1-\gamma_n} \\
&= \left(\frac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\beta_n f(x_n)}{1-\gamma_n}\right) + (W_{n+1}x_{n+1} - W_n x_n) - \frac{\beta_{n+1}}{1-\gamma_{n+1}}W_{n+1}x_{n+1} + \frac{\beta_n}{1-\gamma_n}W_n x_n \\
&= \left(\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right)f(x_{n+1}) + \frac{\beta_n}{1-\gamma_n}\bigl(f(x_{n+1}) - f(x_n)\bigr) + (W_{n+1}x_{n+1} - W_n x_n) \\
&\quad - \left(\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right)W_{n+1}x_{n+1} - (W_{n+1}x_{n+1} - W_n x_n)\frac{\beta_n}{1-\gamma_n} \\
&= \left(\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right)\bigl(f(x_{n+1}) - W_{n+1}x_{n+1}\bigr) + \frac{\beta_n}{1-\gamma_n}\bigl(f(x_{n+1}) - f(x_n)\bigr) \\
&\quad + \frac{1-\gamma_n-\beta_n}{1-\gamma_n}(W_{n+1}x_{n+1} - W_n x_n).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\quad \le \left|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right| \bigl\|f(x_{n+1}) - W_{n+1}x_{n+1}\bigr\| + \frac{\beta_n}{1-\gamma_n}\bigl\|f(x_{n+1}) - f(x_n)\bigr\| \\
&\qquad + \frac{1-\gamma_n-\beta_n}{1-\gamma_n}\|W_{n+1}x_{n+1} - W_n x_n\| \\
&\quad \le \left|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right| \bigl(\bigl\|f(x_{n+1})\bigr\| + \|W_{n+1}x_{n+1}\|\bigr) + \frac{\alpha\beta_n}{1-\gamma_n}\|x_{n+1} - x_n\| \\
&\qquad + \frac{1-\gamma_n-\beta_n}{1-\gamma_n}\bigl(\|W_{n+1}x_{n+1} - W_{n+1}x_n\| + \|W_{n+1}x_n - W_n x_n\|\bigr) \\
&\quad \le \left|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right| \bigl(\bigl\|f(x_{n+1})\bigr\| + \|W_{n+1}x_{n+1}\|\bigr) + \frac{\alpha\beta_n}{1-\gamma_n}\|x_{n+1} - x_n\| \\
&\qquad + \frac{1-\gamma_n-\beta_n}{1-\gamma_n}\bigl(\|x_{n+1} - x_n\| + \|W_{n+1}x_n - W_n x_n\|\bigr) \\
&\quad \le \|x_{n+1} - x_n\| + \left|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right| \bigl(\|f(x_{n+1})\| + \|W_{n+1}x_{n+1}\|\bigr) \\
&\qquad + \|W_{n+1}x_n - W_n x_n\|. \qquad (3.5)
\end{aligned}
$$

Since T_i and $U_{n,i}$ are nonexpansive, from (1.4) we have

$$
\begin{aligned}
\|W_{n+1}x_n - W_n x_n\| &= \|\lambda_1 T_1 U_{n+1,2}x_n - \lambda_1 T_1 U_{n,2}x_n\| \\
&\le \lambda_1\|U_{n+1,2}x_n - U_{n,2}x_n\| \\
&= \lambda_1\|\lambda_2 T_2 U_{n+1,3}x_n - \lambda_2 T_2 U_{n,3}x_n\| \\
&\le \lambda_1\lambda_2\|U_{n+1,3}x_n - U_{n,3}x_n\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \cdots \\
&\leq \lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
&= \lambda_1\lambda_2\cdots\lambda_{n+1}\|T_{n+1}x_n - x_n\|.
\end{aligned} \tag{3.6}$$

Since $\{x_n\}$ is a bounded sequence and all T_n are nonexpansive with a common fixed point p, there is $M_1 \geq 0$ such that

$$\|T_{n+1}x_n - x_n\| \leq \|T_{n+1}x_n - T_{n+1}p\| + \|p - x_n\| \leq M_1, \quad \forall n \geq 0.$$

Substituting (3.6) into (3.5), we have

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq \left|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right|\big(\|f(x_{n+1})\| + \|W_{n+1}x_{n+1}\|\big) + M_1\prod_{i=1}^{n+1}\lambda_i.$$

From conditions (C3), (C4) and the boundedness of $\{f(x_n)\}$ and $\{W_nx_n\}$, it follows that

$$\limsup_{n\to\infty}\big(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

Hence by Lemma 2.2 we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

It follows from (3.3) and (3.4) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\gamma_n)\|z_n - x_n\| = 0.$$

From (3.2), we have

$$x_{n+1} - x_n = \beta_n\big(f(x_n) - x_n\big) + (1-\beta_n)(y_n - x_n).$$

This implies that

$$\begin{aligned}
\alpha\|y_n - x_n\| &\leq (1-\beta_n)\|y_n - x_n\| \\
&= \big\|x_{n+1} - x_n - \beta_n\big(f(x_n) - x_n\big)\big\| \\
&\leq \|x_{n+1} - x_n\| + \big\|\beta_n\big(f(x_n) - x_n\big)\big\|.
\end{aligned}$$

Since $x_{n+1} - x_n \to 0$ and $\beta_n(f(x_n) - x_n) \to 0$, we get

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.7}$$

Observe that

$$y_n - x_n = (1-\alpha_n)(W_nx_n - x_n). \tag{3.8}$$

It follows from (C2), (3.7) and (3.8) that

$$\lim_{n\to\infty} \|x_n - W_n x_n\| = 0.$$

Also, note that

$$\|Wx_n - x_n\| \le \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\|.$$

From [37, Remark 2.2] (see also [38, Remark 3.1]), we have

$$\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0.$$

It follows

$$\lim_{n\to\infty} \|Wx_n - x_n\| = 0. \tag{3.9}$$

In terms of (3.1) and Lemma 3.2, $W : C \to C$ is a nonexpansive mapping such that $F(W) = F$. In the following, we discuss two cases.

(i) Firstly, suppose that E is uniformly smooth. Let x_t be the unique fixed point of the contraction mapping T_t given by

$$T_t x = tf(x) + (1-t)Wx, \quad t \in (0,1).$$

By Lemma 2.4, we can define

$$Q(f) := \lim_{t\to 0^+} x_t,$$

and $Q(f) \in F(W) = F$ solves the variational inequality

$$\big\langle (I-f)Q(f), J\big(Q(f)-p\big)\big\rangle \le 0, \quad \forall p \in F.$$

Let us show that

$$\limsup_{n\to\infty} \big\langle f(z) - z, J(x_n - z)\big\rangle \le 0, \tag{3.10}$$

where $z = Q(f)$. Note that

$$x_t - x_n = t\big(f(x_t) - x_n\big) + (1-t)(Wx_t - x_n).$$

Applying Lemma 2.1 we derive

$$\begin{aligned}
&\|x_t - x_n\|^2 \\
&\quad \le (1-t)^2 \|Wx_t - x_n\|^2 + 2t\big\langle f(x_t) - x_n, J(x_t - x_n)\big\rangle \\
&\quad \le (1-t)^2 \big(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|\big)^2 + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\|x_t - x_n\|^2 \\
&\quad \le (1-t)^2 \|x_t - x_n\|^2 + a_n(t) + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\|x_t - x_n\|^2,
\end{aligned}$$

where

$$a_n(t) = \|Wx_n - x_n\|\bigl(2\|x_t - x_n\| + \|Wx_n - x_n\|\bigr) \to 0 \quad \bigl(\text{due to (3.9)}\bigr).$$

The last inequality implies

$$\bigl\langle x_t - f(x_t), J(x_t - x_n)\bigr\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}a_n(t).$$

It follows that

$$\limsup_{n\to\infty}\bigl\langle x_t - f(x_t), J(x_t - x_n)\bigr\rangle \le M\frac{t}{2}, \tag{3.11}$$

where $M > 0$ is a constant such that $M \ge \|x_t - x_n\|^2$ for all $n \ge 0$ and small enough t in $(0,1)$. Taking the limsup as $t \to 0^+$ in (3.11) and noticing the fact that the two limits are interchangeable due to the fact that the duality map J is uniformly norm-to-norm continuous on any bounded subset of E, we obtain (3.10).

Now, let us show that $x_n \to z$ as $n \to \infty$. Indeed, observe

$$\begin{aligned} x_{n+1} - z &= \beta_n\bigl(f(x_n) - z\bigr) + (1-\beta_n)(y_n - z) \\ &= \beta_n\bigl(f(x_n) - z\bigr) + (1-\beta_n)(1-\alpha_n)(W_n x_n - z) + (1-\beta_n)\alpha_n(x_n - z). \end{aligned}$$

Then, utilizing Lemma 2.1 we get

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &\quad\le \bigl\|(1-\beta_n)\alpha_n(x_n - z) + (1-\beta_n)(1-\alpha_n)(W_n x_n - z)\bigr\|^2 + 2\beta_n\bigl\langle f(x_n) - z, J(x_{n+1} - z)\bigr\rangle \\ &\quad\le \bigl[(1-\beta_n)\alpha_n\|x_n - z\| + (1-\beta_n)(1-\alpha_n)\|x_n - z\|\bigr]^2 + 2\beta_n\bigl\langle f(x_n) - f(z), J(x_{n+1} - z)\bigr\rangle \\ &\qquad + 2\beta_n\bigl\langle f(z) - z, J(x_{n+1} - z)\bigr\rangle \\ &\quad\le (1-\beta_n)^2\|x_n - z\|^2 + 2\alpha\beta_n\|x_n - z\|\|x_{n+1} - z\| + 2\beta_n\bigl\langle f(z) - z, J(x_{n+1} - z)\bigr\rangle \\ &\quad\le (1-\beta_n)^2\|x_n - z\|^2 + \alpha\beta_n\bigl(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\bigr) + 2\beta_n\bigl\langle f(z) - z, J(x_{n+1} - z)\bigr\rangle. \end{aligned}$$

It follows that, for all $n \ge n_0$, we have

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &\quad\le \frac{1 - (2-\alpha)\beta_n + \beta_n^2}{1 - \alpha\beta_n}\|x_n - z\|^2 + \frac{2\beta_n}{1 - \alpha\beta_n}\bigl\langle f(z) - z, J(x_{n+1} - z)\bigr\rangle \\ &\quad\le \left(1 - \frac{(1-\alpha)\beta_n}{1 - \alpha\beta_n}\right)\|x_n - z\|^2 + \frac{2\beta_n}{1 - \alpha\beta_n}\bigl\langle f(z) - z, J(x_{n+1} - z)\bigr\rangle, \end{aligned}$$

due to (C1). For every $n \ge n_0$, put

$$\mu_n = \frac{(1-\alpha)\beta_n}{1 - \alpha\beta_n}$$

and

$$\nu_n = \frac{2}{1-\alpha}\big\langle f(z) - z, J(x_{n+1} - z)\big\rangle.$$

Since $0 < 1 - \alpha\beta_n \le 1$, we have $\mu_n \ge (1-\alpha)\beta_n$. Now, we have

$$\|x_{n+1} - z\|^2 \le (1 - \mu_n)\|x_n - z\|^2 + \mu_n\nu_n, \quad \forall n \ge n_0. \tag{3.12}$$

It is readily seen from (C1) and (3.10) that

$$\sum_{n=0}^{\infty} \mu_n = +\infty \quad \text{and} \quad \limsup_{n\to\infty} \nu_n \le 0.$$

Therefore, applying Lemma 2.3 to (3.12), we conclude that $x_n \to z$ as $n \to \infty$.

(ii) Secondly, suppose that E has a weakly continuous duality map J_φ with gauge φ. Let x_t be the unique fixed point of the contraction mapping T_t given by

$$T_t x = tf(x) + (1-t)Wx, \quad t \in (0,1).$$

By Lemma 2.6, we can define $Q(f) := \lim_{t\to 0^+} x_t$, and $Q(f) \in F(W) = F$ solves the variational inequality

$$\big\langle (I-f)Q(f), J_\varphi\big(Q(f) - p\big)\big\rangle \le 0, \quad \forall p \in F. \tag{3.13}$$

Let us show that

$$\limsup_{n\to\infty} \big\langle f(z) - z, J_\varphi(x_n - z)\big\rangle \le 0, \tag{3.14}$$

where $z = Q(f)$. We take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \big\langle f(z) - z, J_\varphi(x_n - z)\big\rangle = \lim_{k\to\infty} \big\langle f(z) - z, J_\varphi(x_{n_k} - z)\big\rangle. \tag{3.15}$$

Since E is reflexive and $\{x_n\}$ is bounded, we may further assume that $x_{n_k} \rightharpoonup \bar{x}$ for some $\bar{x}$ in C. Since J_φ is weakly continuous, utilizing Lemma 2.5, we have

$$\limsup_{k\to\infty} \Phi\big(\|x_{n_k} - x\|\big) = \limsup_{k\to\infty} \Phi\big(\|x_{n_k} - \bar{x}\|\big) + \Phi\big(\|x - \bar{x}\|\big), \quad \forall x \in E.$$

Put

$$\Gamma(x) = \limsup_{k\to\infty} \Phi\big(\|x_{n_k} - x\|\big), \quad \forall x \in E.$$

It follows that

$$\Gamma(x) = \Gamma(\bar{x}) + \Phi\big(\|x - \bar{x}\|\big), \quad \forall x \in E.$$

From (3.9), we have

$$\begin{aligned}\Gamma(W\bar{x}) &= \limsup_{k\to\infty}\Phi\bigl(\|x_{n_k}-W\bar{x}\|\bigr)=\limsup_{k\to\infty}\Phi\bigl(\|Wx_{n_k}-W\bar{x}\|\bigr)\\ &\le \limsup_{k\to\infty}\Phi\bigl(\|x_{n_k}-\bar{x}\|\bigr)=\Gamma(\bar{x}).\end{aligned}\tag{3.16}$$

Furthermore, observe that

$$\Gamma(W\bar{x})=\Gamma(\bar{x})+\Phi\bigl(\|W\bar{x}-\bar{x}\|\bigr).\tag{3.17}$$

Combining (3.16) with (3.17), we obtain

$$\Phi\bigl(\|W\bar{x}-\bar{x}\|\bigr)\le 0.$$

Hence $W\bar{x}=\bar{x}$ and $\bar{x}\in F(W)=F$ (by Lemma 3.2). Thus, from (3.13) and (3.15), it is easy to see that

$$\limsup_{n\to\infty}\bigl\langle f(z)-z,J_\varphi(x_n-z)\bigr\rangle=\bigl\langle f(z)-z,J_\varphi(\bar{x}-z)\bigr\rangle\le 0.$$

Therefore, we deduce that (3.14) holds.

Now, let us show that $x_n\to z$ as $n\to\infty$. Indeed, observe that

$$\begin{aligned}\Phi\bigl(\|y_n-z\|\bigr) &= \Phi\bigl(\bigl\|\alpha_n(x_n-z)+(1-\alpha_n)(W_nx_n-z)\bigr\|\bigr)\\ &\le \Phi\bigl(\alpha_n\|x_n-z\|+(1-\alpha_n)\|W_nx_n-z\|\bigr)\\ &\le \Phi\bigl(\|x_n-z\|\bigr).\end{aligned}$$

Therefore, by applying Lemma 2.5, we have

$$\begin{aligned}\Phi\bigl(\|x_{n+1}-z\|\bigr) &= \Phi\bigl(\bigl\|\beta_n\bigl(f(x_n)-z\bigr)+(1-\beta_n)(y_n-z)\bigr\|\bigr)\\ &= \Phi\bigl(\bigl\|\beta_n\bigl(f(x_n)-f(z)+f(z)-z\bigr)+(1-\beta_n)(y_n-z)\bigr\|\bigr)\\ &\le \Phi\bigl(\bigl\|(1-\beta_n)(y_n-z)+\beta_n\bigl(f(x_n)-f(z)\bigr)\bigr\|\bigr)+\beta_n\bigl\langle f(z)-z,J_\varphi(x_{n+1}-z)\bigr\rangle\\ &\le \Phi\bigl((1-\beta_n)\|y_n-z\|+\beta_n\bigl\|f(x_n)-f(z)\bigr\|\bigr)+\beta_n\bigl\langle f(z)-z,J_\varphi(x_{n+1}-z)\bigr\rangle\\ &\le \Phi\bigl((1-\beta_n)\|y_n-z\|+\alpha\beta_n\|x_n-z\|\bigr)+\beta_n\bigl\langle f(z)-z,J_\varphi(x_{n+1}-z)\bigr\rangle\\ &\le \bigl(1-(1-\alpha)\beta_n\bigr)\Phi\bigl(\|x_n-z\|\bigr)+\beta_n\bigl\langle f(z)-z,J_\varphi(x_{n+1}-z)\bigr\rangle.\end{aligned}$$

Applying Lemma 2.3, we get

$$\Phi\bigl(\|x_n-z\|\bigr)\to 0\quad (n\to\infty),$$

which implies that $\|x_n-z\|\to 0(n\to\infty)$, *i.e.*, $x_n\to z(n\to\infty)$. This completes the proof. □

Corollary 3.4 *The conclusion in Theorem 3.3 still holds, provided the conditions (C1)-(C4) are replaced by the following:*

(D1) $0 \le \beta_n \le 1-\alpha$, $\forall n \ge n_0$ *for some integer* $n_0 \ge 1$;
(D2) $\lim_{n\to\infty}(\beta_n - \beta_{n+1}) = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = +\infty$;
(D3) $\lim_{n\to\infty}(\alpha_n - \alpha_{n+1}) = 0$ *and* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(D4) $0 < \lambda_n \le b < 1$, $\forall n \ge 1$ *for some* b *in* $(0,1)$.

Proof Observe that

$$\begin{aligned}
&\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n} \\
&\quad = \frac{(\beta_{n+1}-\beta_n) - \beta_{n+1}\alpha_n + \beta_n\alpha_{n+1} + \beta_{n+1}\beta_n\alpha_n - \beta_n\beta_{n+1}\alpha_{n+1}}{(1-(1-\beta_{n+1})\alpha_{n+1})(1-(1-\beta_n)\alpha_n)} \\
&\quad = \frac{(\beta_{n+1}-\beta_n) - \beta_{n+1}(\alpha_n-\alpha_{n+1}) - \alpha_{n+1}(\beta_{n+1}-\beta_n) + \beta_n\beta_{n+1}(\alpha_n-\alpha_{n+1})}{(1-(1-\beta_{n+1})\alpha_{n+1})(1-(1-\beta_n)\alpha_n)} \\
&\quad = \frac{(\beta_{n+1}-\beta_n)(1-\alpha_{n+1}) - \beta_{n+1}(\alpha_n-\alpha_{n+1})(1-\beta_n)}{(1-(1-\beta_{n+1})\alpha_{n+1})(1-(1-\beta_n)\alpha_n)}.
\end{aligned}$$

Since $\lim_{n\to\infty}(\beta_n - \beta_{n+1}) = 0$ and $\lim_{n\to\infty}(\alpha_n - \alpha_{n+1}) = 0$, it follows that

$$\lim_{n\to\infty}\left(\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}\right) = 0.$$

Consequently, all conditions of Theorem 3.3 are satisfied. So, utilizing Theorem 3.3, we obtain the desired result. □

Corollary 3.5 *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space E. Assume, in addition, E either is uniformly smooth or has a weakly continuous duality map* J_φ *with gauge* φ. *Let* $T_i : C \to C$ *be a nonexpansive mapping for each* $i = 1, 2, \ldots$ *such that* $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, *and let* $f \in \Pi_C$ *with contractive constant* α *in* $(0,1)$. *Given sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *in* $[0,1]$, *the following conditions are satisfied:*
(E1) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = +\infty$;
(E2) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(E3) $0 < \lambda_n \le b < 1, \forall n \ge 1$ *for some* $b \in (0,1)$.
Then. for an arbitrary but fixed x_0 *in C, the sequence* $\{x_n\}$ *defined by (3.2) converges strongly to a common fixed point* $Q(f)$ *in F. Moreover,*
(i) if E is uniformly smooth, then $Q(f) \in F$ *solves the variational inequality*

$$\big\langle (I-f)Q(f), J\big(Q(f)-p\big)\big\rangle \le 0, \quad f \in \Pi_C, p \in F;$$

(ii) if E has a weakly continuous duality map J_φ *with gauge* φ, *then* $Q(f) \in F$ *solves the variational inequality*

$$\big\langle (I-f)Q(f), J_\varphi\big(Q(f)-p\big)\big\rangle \le 0, \quad f \in \Pi_C, p \in F.$$

Proof Repeating the arguments in the proof of Theorem 3.3, we know that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{W_n x_n\}$ and $\{f(x_n)\}$. Since $\lim_{n\to\infty} \beta_n = 0$, it is easy to see that there hold the following:
(i) $\beta_n(f(x_n) - x_n) \to 0$ $(n \to \infty)$;

(ii) $0 \le \beta_n \le 1 - \alpha$, $\forall n \ge n_0$ for some integer $n_0 \ge 1$;
(iii) $\lim_{n\to\infty}(\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}) = 0$.

Therefore, all conditions of Theorem 3.3 are satisfied. So, utilizing Theorem 3.3, we obtain the desired result. □

To end this paper, we give a weak convergence theorem for hybrid viscosity approximation method (3.2) involving an infinite family of nonexpansive mappings $T_1, T_2, \ldots$ in a Hilbert space H.

Theorem 3.6 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be a nonexpansive mapping for each $i = 1, 2, \ldots$ such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $f \in \Pi_C$. Given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ in $[0,1]$, the following conditions are satisfied:*
(F1) $\sum_{n=0}^{\infty} \beta_n < +\infty$;
(F2) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(F3) $0 < \lambda_n \le b < 1$, $\forall n \ge 1$ for some $b \in (0,1)$.

Then, for an arbitrary but fixed x_0 in C, the sequence $\{x_n\}$ defined by (3.2) converges weakly to a common fixed point of the infinite family of nonexpansive mappings $T_1, T_2, \ldots$.

Proof Take an arbitrary p in $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Repeating the arguments in the proof of Theorem 3.3, we know that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{W_n x_n\}$ and $\{f(x_n)\}$.

It follows from (2.3) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le (1-\beta_n)\|y_n - p\|^2 + \beta_n \|f(x_n) - p\|^2 \\
&\quad \le \|y_n - p\|^2 + \beta_n \|f(x_n) - p\|^2 \\
&\quad = \|\alpha_n(x_n - p) + (1-\alpha_n)(W_n x_n - p)\|^2 + \beta_n \|f(x_n) - p\|^2 \\
&\quad = \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|W_n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - W_n x_n\|^2 + \beta_n \|f(x_n) - p\|^2 \\
&\quad \le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - W_n x_n\|^2 + \beta_n \|f(x_n) - p\|^2 \\
&\quad = \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - W_n x_n\|^2 + \beta_n \|f(x_n) - p\|^2 \\
&\quad \le \|x_n - p\|^2 + \beta_n \|f(x_n) - p\|^2.
\end{aligned} \tag{3.18}$$

Since $\sum_{n=0}^{\infty} \beta_n < +\infty$ and $\{f(x_n)\}$ is bounded, we get $\sum_{n=0}^{\infty} \beta_n \|f(x_n) - p\|^2 < +\infty$. Utilizing Lemma 2.7, we conclude that $\lim_{n\to\infty} \|x_n - p\|$ exists. Furthermore, it follows from (3.18) that for all $n \ge 0$, we have

$$\alpha_n(1-\alpha_n)\|x_n - W_n x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n \|f(x_n) - p\|^2. \tag{3.19}$$

Since $\beta_n \to 0$ and $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, it follows from (3.19) that $\lim_{n\to\infty} \|x_n - W_n x_n\| = 0$. Also, observe that

$$\|Wx_n - x_n\| \le \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\|.$$

From [37, Remark 2.2] (see also [38, Remark 3.1]), we have

$$\lim_{n\to\infty} \|Wx_n - W_nx_n\| = 0.$$

This implies immediately that

$$\lim_{n\to\infty} \|Wx_n - x_n\| = 0.$$

Now, let us show that $\omega_w(x_n) \subset F$ (see (2.2)). Indeed, let $\bar{x} \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$. Since $(I - W)x_n \to 0$, by Lemma 2.8, $\bar{x} \in F(W) = F$.

Finally, let us show that $\omega_w(x_n)$ is a singleton. Indeed, let $\{x_{m_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{m_j} \rightharpoonup \hat{x}$. Then $\hat{x}$ also lies in F. If $\bar{x} \neq \hat{x}$, by Opial's property of H, we reach the following contradiction:

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \bar{x}\| &= \lim_{i\to\infty} \|x_{n_i} - \bar{x}\| \\
&< \lim_{i\to\infty} \|x_{n_i} - \hat{x}\| = \lim_{j\to\infty} \|x_{m_j} - \hat{x}\| \\
&< \lim_{j\to\infty} \|x_{m_j} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\|.
\end{aligned}$$

This implies that $\omega_w(x_n)$ is a singleton. Consequently, $\{x_n\}$ converges weakly to an element of F. □

Remark 3.7 As pointed out in [22, Remark 2.1], the mild conditions are imposed on the parameter sequence $\{\lambda_n\}$, which are different from those in [8, 11, 18, 23]. Theorem 2.1 in [22] is a supplement to Remark 5 of Zhou, Wei and Cho [23] in reflexive Banach spaces. Moreover, it extends Theorem 1 in [14] from the case of a single nonexpansive mapping to that of an infinite family of nonexpansive mappings, and relaxes the restrictions imposed on the parameters in [14, Theorem 1]. Compared with Theorem 2.1 in [22] (*i.e.*, Theorem 1.2), our Theorems 3.3 and 3.6 supplement, improve and extend them in the following aspects:

(1) The hybrid viscosity approximation method (3.2) includes their modified Mann's iterative process (1.5) as a special case.
(2) We relax the restrictions imposed on the parameters in [22, Theorem 2.1]; for instance, there can be no parameter sequence convergent to zero in our Theorem 3.3.
(3) In Theorem 3.3, the problem of finding a common fixed point of an infinite family of nonexpansive mappings is also considered in the framework of uniformly smooth Banach space.
(4) In order to show the strong convergence of the hybrid viscosity approximation method (3.2), we use the techniques very different from those in the proof of [22, Theorem 2.1]; for instance, we use Theorem 4.1 in [1] and Theorem 3.1 in [32].
(5) Theorem 3.3 shows that the hybrid viscosity approximation method (3.2) converges strongly to a common fixed point of an infinite family of nonexpansive mappings, which solves a variational inequality on their common fixed point set.

(6) In Theorem 3.6, the conditions imposed on $\{\alpha_n\}$ and $\{\beta_n\}$ are very different from those in [22, Theorem 2.1].

(7) In the proof of Theorem 3.6, we use the techniques very different from those in the proof of [22, Theorem 2.1]; for instance, we use Opial's property of Hilbert space and Tan and Xu's lemma in [36].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China. [2]Scientific Computing Key Laboratory of Shanghai Universities, Shanghai, China. [3]Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan 804. [4]Center for General Education, Kaohsiung Medical University, Kaohsiung, Taiwan 807.

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