A Unified Hybrid Iterative Method for Solving Variational Inequalities Involving Generalized Pseudo-contractive Mappings*

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Abstract

We study in this paper the existence and approximation of solutions of variational inequalities involving generalized pseudo-contractive mappings in Banach spaces. The convergence analysis of a proposed hybrid iterative method for approximating common zeros or fixed points of a possibly infinitely countable or uncountable family of such operators will be conducted within the conceptual framework of the "viscosity approximation technique" in reflexive Banach spaces. This technique should make existing or new results in solving variational inequalities more applicable.

1 Introduction

Variational inequalities were initially studied by Stampacchia (cf. [18]), which cover various problems from partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance, as some special cases (see, e.g. [13, 44]). Below is a famous result.

Theorem 1.1 (Projection Gradient Method [44]) *Let C be a nonempty closed convex subset of a Hilbert space H, let $\mathcal{F} : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator, and let P_C be the metric projection from H onto C. Then the following hold:*

(i) For any μ in $(0, 2\eta/\kappa^2)$, the mapping $P_C(I - \mu F) : C \to C$ is a contraction.

(ii) For any x_1 in C, the sequence $\{x_n\}$ generated by the Picard iteration process:

$$x_{n+1} = P_C(I - \mu\mathcal{F})x_n, \quad \forall n \in \mathbb{N}, \tag{1.1}$$

converges strongly to the unique solution of the variational inequality
$VI(H, C, \mathcal{F}, \eta)$: *find u in C such that $\langle \mathcal{F}u, u - v\rangle \le 0$ for all v in C.*

The fixed-point formulation (1.1) involves the projection mapping P_C, which might not be easy to compute, due to the complexity of the convex set C. In order to reduce the complexity probably caused by the projection mapping P_C, Yamada (see [40], and also [11]) recently

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introduced a hybrid steepest descent method for solving the problem

$$VI(H, F(T), \mathcal{F}, \eta): \qquad \text{find } u \text{ in } F(T) \text{ such that } \langle \mathcal{F}u, u - v\rangle \leq 0 \text{ for all } v \text{ in } F(T).$$

Here is the idea. Suppose T (e.g., $T = P_C$) is a nonexpansive mapping from a Hilbert space H into itself with a nonempty fixed point set $F(T)$, and $\mathcal{F}$ is κ-Lipschitzian and η-strongly monotone over the range $T(H)$ of T. Take a fixed number μ in $(0, 2\eta/\kappa^2)$ and a sequence $\{\lambda_n\}$ in $(0, 1)$ satisfying the conditions

$(L1)$ $\lambda_n \to 0$,

$(L2)$ $\sum_{n=1}^{\infty} \lambda_n = +\infty$, and

$(L3)$ $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$.

Starting with an arbitrary initial guess x_0 in H, one generates a sequence $\{x_n\}$ by the following algorithm:

$$x_{n+1} := Tx_n - \lambda_{n+1}\mathcal{F}(Tx_n), \quad \forall n \geq 0. \tag{1.2}$$

Yamada [40, Theorem 3.3, p.486] proved that the sequence $\{x_n\}$ defined by (1.2) converges strongly to a unique solution of $VI[H, F(T), \mathcal{F}, \eta]$.

In the case when $\{T_1, T_2, \ldots, T_N\}$ is a finite family of nonexpansive mappings satisfying

$$\bigcap_{i=1}^{N} F(T_i) = F(T_1T_2\cdots T_N) = F(T_NT_1\cdots T_{N-1}) = \cdots = F(T_2T_3\cdots T_NT_1). \tag{1.3}$$

Yamada [40] studied the following algorithm:

$$x_{n+1} = T_{[n+1]}x_n - \lambda_{n+1}\mu\mathcal{F}T_{[n+1]}x_n, \quad n \geq 0. \tag{1.4}$$

Here, $T_{[r]} = T_{r \bmod N}$ for r in $\mathbb{N}$, and the sequence $\{\lambda_n\}$ satisfies the conditions $(L1)$, $(L2)$, and

$(L4)$ $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < +\infty$.

In 2003, Xu and Kim [39] further considered and studied the hybrid steepest-descent algorithms (1.2) and (1.4). Their major contribution is that the strong convergence of algorithms (1.2) and (1.4) holds with conditions $(L1)$, $(L2)$, $(L3)$ and $(L4)$ assumed, still in the framework of Hilbert spaces. They also established the strong convergence with conditions $(L3)$ be replaced by

$(L5)$ $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})/\lambda_{n+1} = 0$,

and $(L4)$ be replaced by

$(L6)$ $\lim_{n\to\infty}(\lambda_n - \lambda_{n+r})/\lambda_{n+r} = 0$ for all r.

Following Xu and Kim [39], several authors discussed this kind of problems assuming (1.3) in the context of Hilbert spaces and q-uniformly smooth Banach spaces (see, e.g. [7, 45]). Zeng, Schaible and Yao [45] obtained a more general result in this direction, in which $\mathcal{F}$ is demicontinuous and ϕ-strongly accretive with domain $\bigcup_{i=1}^{N} T_i(H)$.

In this paper, we will study variational inequality problems concerning a closed convex subset C of a smooth Banach space X. Let D be a nonempty closed convex subset of C, let $J : X \to X^*$

be the normalized duality mapping, and let $\mathcal{F} : C \to X$ for which $A = I - \mathcal{F}$ is a generalized Φ-pseudo contractive nonlinear operator associated with a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ satisfying $\Phi(0) = 0$. We consider the following generalized variational inequality

$$GVI(1.5)[C, D, \mathcal{F}, \Phi] : \qquad \text{find } z \text{ in } D \text{ such that } \langle \mathcal{F}z, J(z - v)\rangle \leq 0 \text{ for all } v \text{ in } D. \tag{1.5}$$

Concrete definitions for the notations will be given in Section 2.

Motivated by [6, 15, 45], we will investigate in Section 3 the existence of the solutions of $GVI(1.5)$ $[C, D, \mathcal{F}, \Phi]$, under certain assumptions on the nonlinear operator $\mathcal{F}$ and that D is a set of common zeros or fixed points of a possibly infinitely countable or uncountable family of demicontinuous nonlinear operators on a reflexive Banach space. In order to broaden the scope of applicability of the existence results, we propose in Section 4 a unified hybrid iterative algorithm obtained by coupling a ϕ-strongly pseudo-contractive operator A and members of a possibly infinite family of demicontinuous non-Lipschitzian mappings. We investigate its asymptotic behavior for approximating solutions of $GVI(1.5)[C, D, \mathcal{F}, \Phi]$ under mild control conditions on iteration parameters. We also propose in Section 5 a parallel algorithm to remove the assumption (1.3). Our iterative method generalizes and improves most of the existing methods for viscosity approximation (see, e.g. [6, 8, 9, 32, 33, 37, 39, 41]) and hybrid steepest-descent methods for variational inequalities involving demicontinuous non-Lipschitzian nonlinear mappings (see, e.g. [39, 40, 45]). We shall demonstrate how to use our results in solving the image recovery problems for an example of possible applications.

2 Preliminaries

Throughout this paper $(X, \|\cdot\|)$ is a real Banach space with unit sphere $S = \{z \in X : \|z\| = 1\}$ and Banach dual space $(X^*, \|\cdot\|_*)$. A Banach space X is said to be *smooth* provided $\lim_{t\to 0^+}(\|x + ty\| - \|x\|)/t$ exists for each x and y in S. In this case, the norm of X is said to be *Gâteaux differentiable*. It is said to be *uniformly Gâteaux differentiable* if for each y in S, the above limit is attained uniformly for x in S. It is well known that every uniformly smooth space (e.g., L_p space, $1 < p < \infty$) has a uniformly Gâteaux differentiable norm (see e.g., [10]).

Let $J : X \to 2^{X^*}$ be the normalized duality mapping. In other words,

$$J(x) = \{f \in X^* : f(x) = \|x\|^2 = \|f\|^2\}, \quad \forall x \in X.$$

In case X is smooth, J is a single-valued norm to weak* continuous mapping. When X is strictly convex, $J(x) \cap J(y) = \emptyset$ for distinct x, y in X (see, e.g., [10]).

2.1 Various contractive mappings

Let $T : C \to X$ be a nonlinear mapping with domain $C \subseteq X$. The *fixed point set* of T is defined by

$$F(T) = \{x \in C : Tx = x\}.$$

The mapping T is said to be

1. *demicontinuous* if $Tx_n \to Tx$ weakly whenever $x_n \to x$ in norm in C;

2. *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in C;

3. *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall x, y \in C, \forall n \in \mathbb{N};$$

4. *pseudo-contractive* if for all x, y in C, there exists $j(x-y)$ in $J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2;$$

5. *ϕ-strongly pseudo-contractive* if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all x, y in C we have $j(x-y)$ in $J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|;$$

6. *generalized Φ-pseudo-contractive* (cf. [3, 38]) if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for all x, y in C, we have $j(x-y)$ in $J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|).$$

We remark that $\mathbb{A} := I - T$ is accretive (resp., ϕ-strongly accretive, uniformly accretive) if T is pseudo-contractive (resp., ϕ-strongly pseudo-contractive, generalized Φ-pseudo-contractive), where I is the identity operator. As accretive operators play important roles in the study of nonlinear evolution equations in Banach spaces, the pseudo-contractive mappings have been widely studied.

The *modulus of continuity* of a continuous mapping T on C is the function $w_T : [0,\infty) \to [0,\infty)$ defined by

$$w_T(t) := \sup\{\|Tx - Ty\| : x, y \in C; \|x-y\| \leq t\}.$$

Clearly, $\|Tx - Ty\| \leq w_T(\|x-y\|)$ for all x, y in C. If T is uniformly continuous, then w_T is nonnegative, nondecreasing, continuous on $(0,\infty)$, and $w_T(0) = 0$.

A nonempty closed convex subset C of a Banach space X is called a *retract* of X if there exists a continuous mapping P from X onto C such that $Px = x$ for all x in C. We call such P a *retraction* of X onto C. A retraction P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each x in X and $t \geq 0$. If a sunny retraction P is also nonexpansive, then C is said to be a *sunny nonexpansive retract* of X.

2.2 Asymptotic properties of a family of nonlinear mappings

Let C be a nonempty subset of a Banach space X and fix a sequence $\{c_n\}$ in $[0,\infty)$ with $c_n \to 0$. Throughout this paper, G denotes an unbounded subset of $\mathbb{R}^+ := [0,\infty)$ (often $G = \mathbb{N}$ or $\mathbb{R}^+$).

We adapt the following definitions from [1, 26, 27, 28]. A family $\mathcal{T} := \{T_s : s \in G\}$ of mappings from C into itself is said to be

1. *uniformly continuous on C* if each member of $\mathcal{T}$ is uniformly continuous on C;

2. *nearly uniformly L-Lipschitzian associated with net $\{c_t\}$* if there exist a constant $L > 0$ and a net $\{c_t : t \in G\}$ in $[0,\infty)$ with $\lim_{t\to\infty} c_t = 0$ such that

$$\|T_t x - T_t y\| \leq L\|x-y\| + c_t, \quad \forall x, y \in C, \forall t \in G;$$

3. *nearly asymptotically nonexpansive associated with the net $\{(c_t, \eta(T_t))\}$* if there exist two nets $\{c_t : t \in G\}$ in $[0,\infty)$ with $\lim_{t\to\infty} c_t = 0$ and $\{\eta(T_t) : t \in G\}$ in $[1,\infty)$ with $\lim_{t\to\infty} \eta(T_t) = 1$ such that

$$\|T_t x - T_t y\| \leq \eta(T_t)\|x-y\| + c_t, \quad \forall x, y \in C, \forall t \in G;$$

4. *uniformly asymptotically regular on C* if

$$\lim_{t\in G,\ t\to\infty} \sup_{x\in\widetilde{C}} \|T_t x - T_s T_t x\| = 0, \qquad \text{for all } s \in G \text{ and bounded } \widetilde{C} \subseteq C.$$

A sequence $\mathcal{S} := \{T_n\}$ of mappings from C into itself is said to be

5. *asymptotically m-regular at a point* x_0 in C if $\lim_{n\to\infty} \|T_n x_0 - T_m T_n x_0\| = 0$;

6. *asymptotically regular at a point* x_0 if it is asymptotically 1-regular at x_0.

We also say that $\mathcal{T} := \{T_s : s \in G\}$ satisfies *property* ($\mathscr{A}$) if for each bounded set $\{x_s : s \in G\}$ in C, we have

$$(\mathscr{A}) \qquad \lim_{s\to\infty} (x_s - T_s x_s) = 0 \quad \Rightarrow \quad \lim_{s\to\infty} (x_s - T_t x_s) = 0, \quad \forall t \in G.$$

It is easy to see that if $\mathcal{T}$ has property $\mathbb{A}$, then every approximate fixed point (resp. fixed point) of any member in $\mathcal{T}$ is a common approximate fixed point (resp. fixed point) of all members in $\mathcal{T}$.

Remark 2.1 (I) Let $\mathcal{T}$ be a singleton, i.e., $\mathcal{T} = \{T\}$, or $T_s = T$ for all s in G. Then $\{T\}$ always has property ($\mathscr{A}$).

(II) Assume $\mathcal{T} = \{T_s : s \in G\}$ is a uniformly continuous semigroup and uniformly asymptotically regular on C. Then $\mathcal{T}$ has property ($\mathscr{A}$). Indeed, for a bounded set $\{y_s : s \in G\}$ in C with $\lim_{s\to\infty} (x_s - T_s x_s) = 0$, we have

$$\begin{aligned} \|y_s - T_t y_s\| &\leq \|y_s - T_s y_s\| + \|T_s y_s - T_t T_s y_s\| + \|T_t T_s y_s - T_t y_s\| \\ &\leq \|y_s - T_s y_s\| + \sup_{y\in\{y_\gamma:\gamma\in G\}} \|T_s y - T_t T_s y\| + w_{T_t}(\|y_s - T_s y_s\|) \to 0 \end{aligned}$$

as $s \to \infty$ for all t in G.

The example below shows that there exists a nonexpansive mapping which is not asymptotically regular. However, a nontrivial convex combination $T_\lambda = (1-\lambda)I + \lambda T$ of nonexpansive mappings turns out to be asymptotically regular in a general Banach space (see [1]).

Example 2.2 Let C be a bounded symmetric subset of a Banach space X containing 0. Let $Tx = -x$, $\forall x \in C$. Let $\mathcal{T} = \{T^n : n \in \mathbb{N}\}$ be the semigroup generated by the nonexpansive mapping $T : C \to C$ with $F(T) = \{0\}$. Then $\sup_{x\in\widetilde{C}} \|T^n x - T^{n+1} x\| = 2\sup_{x\in\widetilde{C}} \|x\|$ for all $\widetilde{C} \subset C$. Clearly, $\mathcal{T}$ is not uniformly asymptotically regular on C, but it is 2-regular on C.

2.3 Some known results

In subsequent sections, we shall make use of the following results.

Lemma 2.3 (see [1, 22]) *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space X, and let $x \in X$. Then there exists a unique element x_0 in C such that $\|x - x_0\| = \inf_{y\in C} \|x - y\|$.*

Let f be a continuous linear functional on ℓ_∞. We use $f_n(x_{n+m})$ to denote

$$f(x_{m+1}, x_{m+2}, x_{m+3}, \cdots, x_{m+n}, \cdots),$$

for $m = 0, 1, 2, \ldots$. A continuous linear functional j on l_∞ is called a *Banach limit* if $\|j\|_* = j(1) = 1$ and $j_n(x_n) = j_n(x_{n+1})$ for each $x = (x_1, x_2, \cdots)$ in l_∞.

Fix any Banach limit and denote it by LIM. Note that $\|\operatorname{LIM}\|_* = 1$,

$$\liminf_{n\to\infty}\ t_n \leq \operatorname{LIM}_n t_n \leq \limsup_{n\to\infty}\ t_n,$$

and

$$\text{LIM}_n \ t_n = \text{LIM}_n \ t_{n+1}, \quad \forall (t_n) \in l_\infty.$$

Let $\mathcal{T} = \{T_s : s \in G\}$ be a collection of self-mappings on $C \subseteq X$. For a bounded sequence $\{y_n\}$ in C, let

$$M_{\{y_n\}} = \{y \in C : \text{LIM}_n \|y_n - y\|^2 = \inf_{x \in C} \text{LIM}_n \|y_n - x\|^2\}.$$

Lemma 2.4 (Ha and Jung [14, Lemma 1]) *Let X be a Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, and $\{x_n\}$ a bounded sequence in X. Let* LIM *be a Banach limit and $y \in C$. Then*

$$y \in M_{\{x_n\}} \quad \Leftrightarrow \quad \text{LIM}_n \langle x - y, J(x_n - y)\rangle \leq 0, \quad \forall x \in C.$$

Lemma 2.5 (Goebel and Reich [12, Lemma 13.1]) *Let C be a nonempty convex subset of a smooth Banach space X, let D be a non-empty subset of C, and let P be a retraction from C onto D. Then the following are equivalent:*

(a) P is sunny and nonexpansive.

(b) $\langle x - Px, J(z - Px)\rangle \leq 0$, for all x in C and z in D.

(c) $\langle x - y, J(Px - Py)\rangle \geq \|Px - Py\|^2$, for all x, y in C.

Proposition 2.6 (Bruck [5]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\mathcal{S} = \{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings from C into itself such that $F(\mathcal{S}) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$. Then $Tx = \sum_{n=1}^{\infty} \alpha_n T_n x$ defines a nonexpansive mapping on C with $F(\mathcal{S}) = F(T)$.*

Lemma 2.7 (Alber and Guerre-Delabriere [2]) *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of nonnegative real numbers such that $\lim_{n\to\infty} \beta_n/\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$. Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous and nondecreasing function such that $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. Let $\{\lambda_n\}$ be a sequence of nonnegative real numbers satisfying the recursive inequality:*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \phi(\lambda_n) + \beta_n, \quad \forall n \in \mathbb{N}.$$

Then $\{\lambda_n\}$ converges to zero.

3 Existence Results

Proposition 3.1 *Let $A : C \to C$ be a continuous generalized Φ-pseudo-contractive mapping of a nonempty closed convex subset C of a smooth Banach space X. Let $\mathcal{T} = \{T_s : s \in G\}$ be a family of continuous pseudo-contractive mappings from C into itself.*

(a) For any s in G and scalar b_s in $(0, 1)$, there exists a unique point y_s in C such that

$$y_s \quad = \quad b_s A y_s + (1 - b_s) T_s y_s. \tag{3.1}$$

(b) If v is a common fixed point of $\mathcal{T}$, then

$$\langle y_s - A y_s, J(y_s - v)\rangle \leq 0.$$

Proof. (a) Set $\Phi_s(\cdot) := b_s\Phi(\cdot)$ for each s in G. The mapping

$$T_s^A(y) := b_sAy + (1-b_s)T_sy, \quad \forall y \in C,$$

is continuous and generalized Φ_s-pseudo-contractive. Indeed, for x, y in C,

$$\begin{aligned}\langle T_s^Ax - T_s^Ay, J(x-y)\rangle &= b_s\langle Ax - Ay, J(x-y)\rangle + (1-b_s)\langle T_sx - T_sy, J(x-y)\rangle \\ &\leq b_s(\|x-y\|^2 - \Phi(\|x-y\|)) + (1-b_s)\|x-y\|^2 \\ &= \|x-y\|^2 - \Phi_s(\|x-y\|).\end{aligned}$$

Note also that $\Phi_s(\cdot)$ is a strictly increasing function with $\Phi_s(0) = 0$. By Xiang [38, Theoerm 2.1], T_s^A has a unique fixed point y_s in C satisfying (3.1).

(b) Suppose that v is a common fixed point of the family $\mathcal{T}$. Since each T_s is a pseudo-contractive,

$$\begin{aligned}\langle y_s - T_sy_s, J(y_s - v)\rangle &= \langle y_s - v + T_sv - T_sy_s, J(y_s - v)\rangle \\ &= \|y_s - v\|^2 - \langle T_sy_s - T_sv, J(y_s - v)\rangle \geq 0.\end{aligned}$$

From (3.1), we have

$$\begin{aligned}\langle y_s - Ay_s, J(y_s - v)\rangle &= (1-b_s)\langle T_sy_s - Ay_s, J(y_s - v)\rangle \\ &= (1-b_s)\langle T_sy_s - y_s + y_s - Ay_s, J(y_s - v)\rangle.\end{aligned}$$

It follows that

$$\langle y_s - Ay_s, J(y_s - v)\rangle = \frac{1-b_s}{b_s}\langle T_sy_s - y_s, J(y_s - v)\rangle \leq 0,$$

as asserted. □

Let $\mathcal{T} = \{T_s : s \in G\}$ be a family of mappings from a nonempty convex subset C of a Banach space into C and $A : C \to C$. Denote by

$$E_{\mathcal{T}}(C) = \{x \in C : T_sx = \lambda x + (1-\lambda)Ax \text{ for some } \lambda > 1 \text{ and } s \text{ in } G\}.$$

Theorem 3.2 *Let C be a nonempty closed convex subset of be a reflexive Banach space X with a uniformly Gâteaux differentiable norm. Let $A : C \to C$ be a continuous generalized Φ-pseudo-contractive mapping with a bounded range $A(C)$. Set $\mathcal{F} = I - A$. Let $\mathcal{T} = \{T_s : s \in G\}$ be a family of continuous pseudo-contractive mappings with property $(\mathscr{A})$ such that $E_{\mathcal{T}}(C)$ is bounded. Suppose every nonempty closed convex bounded subset of C has the fixed point property for nonexpansive self-mappings. Then we have the following:*

(i) $GVI(3.2)[C, F(\mathcal{T}), \mathcal{F}, \Phi]$:

$$\text{find } z \text{ in } F(\mathcal{T}) \text{ such that } \langle \mathcal{F}z, J(z-v)\rangle \leq 0 \text{ for all } v \text{ in } F(\mathcal{T}) \tag{3.2}$$

has a unique solution y^* in $F(\mathcal{T})$.

(ii) Let $\{b_s\}_{s\in G}$ be a scalar net in $(0,1)$ such that $\lim_{s\to\infty} b_s = 0$. The net $\{y_s\}$ described by (3.1) converges strongly to y^* as $s \to \infty$.

Proof. (i) From [21, Theorem 6] we know that the mapping $2I - T_t$ has a nonexpansive inverse, denoted by g_t, which maps C into itself with $F(T_t) = F(g_t)$ for t in G. Proposition 3.1(a) shows that there exists a unique point y_s in C satisfying (3.1). From (3.1), we have

$$y_s - T_sy_s = b_s(1-b_s)^{-1}(Ay_s - y_s).$$

One can easily see, by the boundedness of $E_{\mathcal{T}}(C)$ and $A(C)$, that $y_s - T_s y_s \to 0$ as $s \to \infty$. By property $(\mathscr{A})$, we have $y_s - T_t y_s \to 0$ as $s \to \infty$ for all t in G. This implies that $y_s - g_t y_s \to 0$ as $s \to \infty$ for all t in G. We can choose a sequence $\{s_n\}$ in G such that $\lim_{n\to\infty} s_n = \infty$. Set $y_n := y_{s_n}$. Fix an arbitrary Banach limit LIM, and define a function $\varphi : C \to \mathbb{R}^+$ by

$$\varphi(x) := \mathrm{LIM}_n \|y_n - x\|^2, \quad \forall x \in C.$$

Set

$$M := \{y \in C : \varphi(y) = \inf_{x\in C} \varphi(x)\}. \tag{3.3}$$

Note that X is reflexive, $\varphi(x) \to \infty$ as $\|x\| \to \infty$, and φ is a continuous convex function. By Barbu and Precupanu [4, Theorem 1.2, p. 79], the set M is nonempty. By Takahashi [34], we see that M is also closed, convex and bounded. Moreover, M is invariant under g_γ, i.e., $g_\gamma(M) \subset M$ for all γ in G. In fact, we have for each y in M,

$$\varphi(g_\gamma y) = \mathrm{LIM}_n \|y_n - g_\gamma y\|^2 \leq \mathrm{LIM}_n \|g_\gamma y_n - g_\gamma y\|^2 \leq \mathrm{LIM}_n \|y_n - y\|^2 = \varphi(y).$$

By assumption, each g_γ has a fixed point y_γ in M. As the family $\{g_\gamma : \gamma \in G\}$ has property $(\mathcal{A})$, it has a common fixed point in M, that is, $M \cap F(\mathcal{T}) \neq \emptyset$. Let $y^* \in F(\mathcal{T}) \cap M$. By Lemma 2.4, we have $\mathrm{LIM}_n\langle z - y^*, J(y_n - y^*)\rangle \leq 0$ for all z in C. In particular,

$$\mathrm{LIM}_n\langle Ay^* - y^*, J(y_n - y^*)\rangle \leq 0. \tag{3.4}$$

By Proposition 3.1(b), we have

$$\langle y_n - Ay_n, J(y_n - v)\rangle \leq 0, \quad \forall n \in \mathbb{N}, \forall v \in F(\mathcal{T}). \tag{3.5}$$

From (3.5), we have

$$\begin{aligned} \|y_n - y^*\|^2 &= \langle y_n - Ay_n + Ay_n - Ay^* + Ay^* - y^*, J(y_n - y^*)\rangle \\ &\leq \|y_n - y^*\|^2 - \Phi(\|y_n - y^*\|) + \langle Ay^* - y^*, J(y_n - y^*)\rangle. \end{aligned} \tag{3.6}$$

From (3.4) and (3.6), we obtain $\mathrm{LIM}_n \Phi(\|y_n - y^*\|) \leq 0$. Thus, there exists a subsequence of $\{y_n\}$, still denoted by $\{y_n\}$, such that $y_n \to y^*$.

We now show that y^* is a solution of $GVI(3.2)[C, F(\mathcal{T}), \mathcal{F}, \Phi]$. For fixed v in $F(\mathcal{T})$, we have $\{y_n - v\}$ is bounded. It follows from (3.5) that

$$\begin{aligned} \langle y^* - Ay^*, J(y_n - v)\rangle &= \langle y^* - y_n + y_n - Ay_n + Ay_n - Ay^*, J(y_n - v)\rangle \\ &\leq \langle y^* - y_n + Ay_n - Ay^*, J(y_n - v)\rangle \\ &\leq \|y^* - y_n + Ay_n - Ay^*\| \, \|y_n - v\| \\ &\leq \|y^* - y_n + Ay_n - Ay^*\| \sup_{m\in\mathbb{N}} \|y_m - v\|, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.7}$$

Since the duality mapping J is single-valued and norm to weak* continuous, by passing to a subsequence we have

$$\langle y^* - Ay^*, J(y_{n_i} - v)\rangle \to \langle y^* - Ay^*, J(y^* - v)\rangle.$$

Using the continuity of A we get from (3.7) that

$$\langle y^* - Ay^*, J(y^* - v)\rangle \leq 0, \quad \forall v \in F(\mathcal{T}).$$

It follows that y^* is a solution of the variational inequality $GVI(3.2)[C, F(\mathcal{T}), \mathcal{F}, \Phi]$. One can easily see that y^* is the unique solution of $GVI(3.2)[C, F(\mathcal{T}), \mathcal{F}, \Phi]$.

(ii) Assume that $\{t_n\}$ is another sequence in G with $\lim_{n\to\infty} t_n = \infty$ such that $y_{t_n} \to z^*$. Then as shown in (i), z^* is also a solution of $GVI(3.2)[C, F(\mathcal{T}), \mathcal{F}, \Phi]$. By uniqueness, $z^* = y^*$. Therefore, $\{y_s\}$ converges strongly to y^*. $\square$

The next theorem extends the results of Kikkawa and Takahashi [17], Takahashi [35], Wong, Sahu and Yao [37] and many others from nonexpansive mappings to a semigroup of nonexpansive mappings.

Theorem 3.3 *Let C be a nonempty closed convex subset of a reflexive Banach space with a uniform Găteaux differentiable norm. Assume that C has normal structure. Let $A : C \to C$ be a continuous generalized Φ-pseudo-contractive mapping with a bounded range $A(C)$, and let $\mathcal{T} = \{T_s : s \in G\}$ be a semigroup of nonexpansive mappings from C into itself with property ($\mathscr{A}$) such that $E_{\mathcal{T}}(C)$ is bounded. Then the conclusions of Theorem 3.2 hold.*

Proof. Using the argument of the proof of Theorem 3.2, we obtain that the set M defined by (3.3) is nonempty, closed, convex, bounded, and invariant under $\{T_s : s \in G\}$. Theorem 1 of Lim [19] implies that the commuting family $\{T_s : s \in G\}$ has a common fixed point in M. Similar to the proof of Theorem 3.2, one can show that $\{y_s : s \in G\}$ converges strongly to a common fixed point y^* of $\mathcal{T}$ as $s \to \infty$. $\square$

Theorem 3.4 *Let C be a nonempty closed convex subset of a strictly convex reflexive Banach space with a uniform Găteaux differentiable norm. Let $A : C \to C$ be a continuous generalized Φ-pseudo-contractive mapping with a bounded range $A(C)$, and let $\mathcal{T} = \{T_s : s \in G\}$ be a family of continuous pseudo-contractive mappings from C into itself with property ($\mathscr{A}$) such that $E_{\mathcal{T}}(C)$ is bounded and $F(\mathcal{T}) \neq \emptyset$. Then the conclusions of Theorem 3.2 hold.*

Proof. To utilize the arguments in the proof of Theorem 3.2, we need to show that the set M defined by (3.3) has a common fixed point of the family $\{g_\gamma : \gamma \in G\}$. As $F(\mathcal{T}) \neq \emptyset$, we can find $v \in \bigcap_{\gamma\in\Gamma} F(g_\gamma)$. Then the set

$$M_0 = \{u \in M : \|u - v\| = \inf_{x\in M} \|x - v\|\}$$

is a singleton since X is strictly convex (see Lemma 2.3). Let $M_0 = \{u_0\}$ for some u_0 in M. Observe that

$$\|g_\gamma u_0 - v\| = \|g_\gamma u_0 - g_\gamma v\| \leq \|u_0 - v\| = \inf_{x\in M} \|x - v\|.$$

Thus, $g_\gamma u_0 = u_0$ for all γ in G and hence $\bigcap_{\gamma\in G} F(g_\gamma) \cap M \neq \emptyset$. $\square$

We remark that Theorem 3.4 is a far more general result than those in the existing literature of this nature. In particular, it extends Jung and Sahu [16, Theorem 1], Morales [23, Theorem 2], Morales and Jung [24, Theorem 2], Takahashi [35] from the class of nonexpansive or Lipschitzian pseudo-contractive self-mappings to the more general family of pseudo-contractive self-mappings of a Banach space.

Let C be a closed convex subset of a Banach space X. A mapping $T : C \to X$ is said to be *c-pseudo-contractive* (cf. [15]), if there exists a monotonic function $h : [0, \infty) \to [0, \infty)$ with $\lim_{t\to 0^+} h(t) = 0$, and $L > 0$ such that for all u, v, x, y in C, there exists some j in $J(x - y)$ (depending on u, v, x, y) such that

$$\langle Tu - Tv, j\rangle \leq h(\|u - x\| + \|v - y\|) + L\|x - y\|^2.$$

The mapping T is said to be *locally c-pseudo-contractive* if for each z in C, a number $r > 0$ exists such that $T : C \cap B(z; r) \to C$ is c-pseudo-contractive. We remark that any locally Lipschitzian mapping is c-pseudo-contractive.

Proposition 3.5 *Let C be a nonempty closed convex subset of a Banach space X and T a demicontinuous, pseudo-contractive, and locally c-pseudo-contractive mapping from C into itself. Let $A_T : C \to X$ be a mapping defined by $A_T := I + r(I - T)$ for any $r > 0$. Then we have the following:*

(a) The range of A_T contains C, i.e., $C \subseteq A_T(C)$.

(b) A_T^{-1} is nonexpansive from $A_T(C)$ into C, and the fixed point sets $F(A_T^{-1}) = F(T)$.

(c) If X is strictly convex, then $F(T)$ is closed and convex.

Proof. (a) Let z be a point in C. Then it suffices to show that there exists x in C such that $z = A_T(x)$. Define $g : C \to C$ by $g(x) = (1 + r)^{-1}(rTx + z)$. Then g is a demicontinuous, $r/(1+r)$-strongly pseudo-contractive, and locally c-pseudo-contractive mapping. By Hester and Morales [15, Theorem 4], there exists x in C with $g(x) = x$, i.e., $z = A_T(x)$.

(b) By the pseudocontractivity of T, we have

$$\|x - y\| \leq \|[I + r(I - T)]x - [I + r(I - T)]y\| = \|A_T(x) - A_T(y)\|, \quad \forall x, y \in C.$$

It follows that A_T is one-one. Therefore, A_T^{-1} is nonexpansive from $A_T(C)$ into C. Clearly, we have $F(A_T^{-1}) = F(T)$.

(c) By the continuity of A_T^{-1}, one sees that $F(T)$ is closed. The convexity of $F(T)$ follows from Agarwal, Regan and Sahu [1, Theorem 5.2.27]. □

Recall that an accretive operator $\mathbb{A}$ is said to be *m-accretive* if $R(I + r\mathbb{A}) = X$ for all $r > 0$. It is well known that every continuous accretive operator on X is m-accretive (see, Martin [20]). As a direct consequence of Proposition 3.5(a), we derive an interesting new result, which is a significant improvement of a corresponding result of Martin [20].

Corollary 3.6 *Let X be a Banach space and let $\mathbb{A} : X \to X$ be a demicontinuous accretive operator such that $I - \mathbb{A}$ is locally c-pseudo-contractive. Then $\mathbb{A}$ is m-accretive.*

Theorem 3.7 *Let C be a nonempty closed convex subset of a strictly convex reflexive Banach space with a uniformly Găteaux differentiable norm. Let $\mathcal{T} = \{T_s : s \in G\}$ be a family of demicontinuous, pseudo-contractive, and locally c-pseudo-contractive mappings from C into itself with property ($\mathscr{A}$), and $F(\mathcal{T}) \neq \emptyset$. Let $\{b_s\}$ be a scalar net in $(0, 1)$ such that $\lim_{s\to\infty} b_s = 0$, and for each u in C, suppose*

$$E_u = \{x \in C : T_s x = tx + (1 - t)u \text{ for some } t > 1 \text{ and } s \text{ in } G\}$$

is a bounded set. Then, for each s in G, there exists a unique point y_s in C such that

$$y_s \quad = \quad b_s u + (1 - b_s)T_s y_s, \tag{3.8}$$

Moreover, $\{y_s\}$ described by (3.8) converges strongly to Qu in $F(\mathcal{T})$ as $s \to \infty$, where Q so defined is a sunny nonexpansive retraction from C onto $F(\mathcal{T})$.

Proof. For each s in G, the mapping $T_s^u(y) := b_s u + (1 - b_s)T_s y$ is a demicontinuous, $(1 - b_s)$-strongly pseudo-contractive and locally c-pseudo-contractive mapping. Hester and Morales [15, Theorem 4] implies that there exists a unique point y_s in C satisfying (3.8). The constant map $Ax := u$ is a continuous generalized Φ-pseudo-contractive mapping from C into C. Using the argument of the proof of Theorem 3.4, we obtain that $\langle y_s - u, j(y_s - v)\rangle \leq 0$ for all v in $F(\mathcal{T})$, and $\{y_s\}$ converges strongly to y^* in $F(\mathcal{T})$. Let $Qu := \lim_{s\to\infty} y_s$. Note that Qu is the unique solution the following variational inequality:

$$\langle Qu - u, J(Qu - v)\rangle \leq 0 \text{ for all } v \text{ in } F(\mathcal{T}).$$

One can easily see from Proposition 3.5 that $F(\mathcal{T})$ is closed and convex. Therefore, Lemma 2.5 shows that Q is a sunny nonexpansive retraction from C onto $F(\mathcal{T})$. □

4 Convergence of a hybrid iterative method

In this section, we assume that X is a Banach space with a uniformly Gâteaux differentiable norm, and C is a nonempty closed convex subset of X. Let $A : C \to C$ be a ϕ-strongly pseudo-contractive mapping, and let $\mathcal{F} = I - A$. Let $\mathcal{T} = \{T_s : s \in G\}$ be a family of pseudo-contractive mappings from C into C with a nonempty common fixed point set $F(\mathcal{T})$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences of real numbers in $(0, 1]$ with $\lambda_n(1+\theta_n) \leq 1$ for all n in $\mathbb{N}$, and let $\{s_n\}$ be another sequence in G such that $\lim_{n\to\infty} s_n = \infty$. Assume that the following $GVI(4.1)[C, F(\mathcal{T}), \mathcal{F}, \phi]$ has a unique solution y^* in C:

$$\text{Find } z \text{ in } F(\mathcal{T}) \text{ such that } \langle \mathcal{F}z, J(z - v)\rangle \leq 0 \text{ for all } v \text{ in } F(\mathcal{T}). \tag{4.1}$$

Motivated by Bruck [6], we now introduce a hybrid iterative method, called a *functional Bruck method*, for finding the unique solution y^* of $GVI(4.1)[C, F(\mathcal{T}), \mathcal{F}, \phi]$.

Algorithm 4.1 Given x_1 in C, a sequence $\{x_n\}$ in C is constructed as follows:

$$x_{n+1} := (1 - \lambda_n(1 + \theta_n))x_n + \lambda_n T_{s_n} x_n + \lambda_n\theta_n A x_n \quad \text{for all } n \text{ in } \mathbb{N}. \tag{4.2}$$

In order to establish the main result of this section, we need the following lemma.

Lemma 4.2 *Let C be a nonempty closed convex subset of a Banach space X with a uniformly Gâteaux differentiable norm, and let $A : C \to C$ be a continuous strongly ϕ-pseudo-contractive mapping with a bounded range $A(C)$. Let $\mathcal{T} = \{T_s : s \in G\}$ be a semigroup of demicontinuous pseudo-contractive and nearly uniformly L-Lipschitzian mappings from C into itself associated with the net $\{a_s\}$ with property ($\mathscr{A}$) and $F(\mathcal{T}) \neq \emptyset$. Let $\{\lambda_m\}$ and $\{b_t\}$ be in $(0, 1)$ such that* $\lim_{m\to\infty} \lambda_m = \lim_{t\to\infty} b_t = 0$. *For m in $\mathbb{N}$, let $\mathcal{S}_m = \{S_{m,t} : t \in G\}$ be the family of mappings $S_{m,t} : C \to C$ defined by*

$$S_{m,t}x = (1 - \lambda_m)x + \lambda_m T_t x, \quad \forall x \in C,\ \forall t \in G. \tag{4.3}$$

For m in $\mathbb{N}$ and t in G, let $z_{m,t}$ be the unique point in C described by

$$z_{m,t} = b_t A z_{m,t} + (1 - b_t) S_{m,t} z_{m,t}. \tag{4.4}$$

Suppose $\lim_{t\to\infty} z_{m,t} = z_m$ *exists for each m in $\mathbb{N}$. Then all $z_m = y^*$, where y^* in C is the unique solution of* $GVI(4.1)[C, F(\mathcal{T}), \mathcal{F}, \phi]$. *Moreover, for any bounded sequence $\{x_n\}$ in C we have*

$$\limsup_{n\to\infty} \langle Ay^* - y^*, J(x_n - y^*)\rangle \leq 0.$$

Proof. Let $m \in \mathbb{N}$. Note that

$$\begin{aligned} \|S_{m,t}x - S_{m,t}y\| &\leq (1 - \lambda_m)\|x - y\| + \lambda_m\|T_t x - T_t y\| \\ &\leq (1 - \lambda_m)\|x - y\| + \lambda_m[L\|x - y\| + a_t] \\ &\leq L'_m\|x - y\| + a_t, \quad \forall x, y \in C, \forall t \in G. \end{aligned} \tag{4.5}$$

Here, $L'_m = 1 - \lambda_m + L\lambda_m$. By choosing a y from $F(\mathcal{T})$ and noting $\{z_{m,t}\}$ converges strongly, we have from (4.5) that $\{S_{m,t}z_{m,t} : t \in G\}$ is eventually bounded. From (4.4) and the boundedness of $A(C)$, we have

$$\|z_{m,t} - S_{m,t}z_{m,t}\| = b_t\|Az_{m,t} - S_{m,t}z_{m,t}\| \to 0 \text{ as } t \to \infty.$$

It follows from (4.3) that

$$\|z_{m,t} - T_t z_{m,t}\| = \frac{\|z_{m,t} - S_{m,t}z_{m,t}\|}{\lambda_m} \to 0 \text{ as } t \to \infty.$$

Property ($\mathscr{A}$) implies that $T_s z_{m,t} \to z_m$ for any s in G, and also $T_{s+s'} z_{m,t} \to z_m$ for any s, s' in G. By the demicontinuity of $T_{s'}$, we have $T_{s+s'} z_{m,t} = T_{s'}(T_s z_{m,t}) \to T_{s'} z_m$ weakly. By the uniqueness of the weak limit of $\{T_s z_{m,t}\}_{t\in G}$, we have $z_m = T_{s'} z_m$. Thus, $z_m \in \bigcap_{s\in G} F(T_s)$ for each m in $\mathbb{N}$. We now show that z_m is a solution $GVI(4.1)[C, F(\mathcal{T}), \mathcal{F}, \phi]$ for each m. For v in $F(\mathcal{T})$ and m in $\mathbb{N}$, we have $\{z_{m,t} - v\}$ is bounded. In the lines of (3.7), one can obtain that

$$\langle z_m - Az_m, J(z_{m,t} - v)\rangle \leq \|z_m - z_{m,t} + Az_{m,t} - Az_m\| \sup_{s\in G} \|z_{m,s} - v\|, \quad \forall t \in G.$$

Due to the facts that $\lim_{t\to\infty} z_{m,t} = z_m$, that A is continuous and that the duality mapping J is norm-to-weak* continuous, we have

$$\langle z_m - Az_m, J(z_m - v)\rangle \leq 0.$$

By the uniqueness of the solution of $GVI(4.1)[C, F(\mathcal{T}), \mathcal{F}, \phi]$, we have $z_m = y^*$ in C for each m in $\mathbb{N}$. In particular,

$$\lim_{t\to\infty} z_{m,t} = y^*, \quad \forall m \in \mathbb{N}.$$

Since for each m in $\mathbb{N}$ and t in G, the mapping $S_{m,t}$ is pseudocontractive, it follows from (4.4) that

$$\begin{aligned}
\|z_{m,t} - y^*\|^2 &= \langle b_t(Az_{m,t} - y^*) + (1 - b_t)(S_{m,t}z_{m,t} - y^*), J(z_{m,t} - y^*)\rangle \\
&\leq b_t\langle Az_{m,t} - Ay^* + Ay^* - y^*, J(z_{m,t} - y^*)\rangle + (1 - b_t)\|z_{m,t} - y^*\|^2 \\
&\leq b_t[\|z_{m,t} - y^*\|^2 - \|z_{m,t} - y^*\|\phi(\|z_{m,t} - y^*\|)] + b_t\langle Ay^* - y^*, J(z_{m,t} - y^*)\rangle \\
&\quad +(1 - b_t)\|z_{m,t} - y^*\|^2.
\end{aligned}$$

Consequently, $\phi(\|z_{m,t} - y^*\|) \leq \|Ay^* - y^*\|$. Hence

$$\|z_{m,t} - y^*\| \leq \phi^{-1}(\|Ay^* - y^*\|), \quad \forall m \in \mathbb{N}, \forall t \in G.$$

Thus, by the boundedness of $\{x_n\}$, we may assume that

$$\|x_n - z_{m,t}\| \leq K_1, \quad \forall m, n \in \mathbb{N}, \forall t \in G.$$

By (4.3), we have

$$\lim_{m\to\infty} \|x_n - S_{m,t}x_n\| = \lim_{m\to\infty} \lambda_m\|x_n - T_t x_n\| = 0. \tag{4.6}$$

Since $(1 - b_t)(z_{m,t} - S_{m,t}z_{m,t}) = b_t(Az_{m,t} - z_{m,t})$, we have

$$\begin{aligned}
b_t\langle Az_{m,t} - z_{m,t}, J(x_n - z_{m,t})\rangle &= (1 - b_t)\langle(z_{m,t} - S_{m,t}z_{m,t}), J(x_n - z_{m,t})\rangle \\
&= (1 - b_t)\langle z_{m,t} - x_n + x_n - S_{m,t}x_n \\
&\quad +S_{m,t}x_n - S_{m,t}z_{m,t}, J(x_n - z_{m,t})\rangle \\
&\leq (1 - b_t)\langle x_n - S_{m,t}x_n, J(x_n - z_{m,t})\rangle \\
&\leq (1 - b_t)\|x_n - S_{m,t}x_n\|K_1.
\end{aligned}$$

Using (4.6), we obtain

$$\limsup_{m\to\infty}\langle Az_{m,t} - z_{m,t}, J(x_n - z_{m,t})\rangle \leq 0, \quad \forall n \in \mathbb{N}, \forall t \in G.$$

Clearly, for each $\varepsilon > 0$, there exists m_0 in $\mathbb{N}$ such that

$$\langle Az_{m,t} - z_{m,t}, J(x_n - z_{m,t})\rangle \leq \frac{\varepsilon}{2}, \quad \forall m \geq m_0, \ \forall n \in \mathbb{N}, \forall t \in G. \tag{4.7}$$

Noting again that $\lim_{t\to\infty} z_{m_0,t} = y^*$, that A is continuous and that the duality mapping J is norm-to-weak* continuous, we have

$$\begin{aligned}
& \mid \langle Ay^* - y^*, J(x_n - y^*)\rangle - \langle Az_{m_0,t} - z_{m_0,t}, J(x_n - z_{m_0,t})\rangle \mid \\
= & \mid \langle Ay^* - y^*, J(x_n - y^*) - J(x_n - z_{m_0,t})\rangle + \langle Ay^* - y^* - (Az_{m_0,t} - z_{m_0,t}), J(x_n - z_{m_0,t})\rangle \mid \\
\leq & \mid \langle Ay^* - y^*, J(x_n - y^*) - J(x_n - z_{m_0,t})\rangle \mid + \|Ay^* - y^* - (Az_{m_0,t} - z_{m_0,t})\| K_1 \\
\to & \quad 0 \text{ as } t \to \infty.
\end{aligned}$$

Hence, there exists t_0 in G such that

$$\mid \langle Ay^* - y^*, J(x_n - y^*)\rangle - \langle Az_{m_0,t} - z_{m_0,t}, J(x_n - z_{m_0,t})\rangle \mid < \frac{\varepsilon}{2}, \quad \forall t \geq t_0, n \in \mathbb{N}.$$

Using (4.7), we obtain

$$\langle Ay^* - y^*, J(x_n - y^*)\rangle \leq \langle Az_{m_0,t} - z_{m_0,t}, J(x_n - z_{m_0,t})\rangle + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \in \mathbb{N}.$$

Therefore, $\limsup_{n\to\infty}\langle Ay^* - y^*, J(x_n - y^*)\rangle \leq 0$. □

Theorem 4.3 *Let C be a nonempty closed convex subset of a Banach space X with a uniformly Gâteaux differentiable norm, and $A : C \to C$ be a uniformly continuous ϕ-strongly-pseudo-contractive mapping with a bounded range $A(C)$. Let $\mathcal{T} = \{T_s : s \in G\}$ be a semigroup of demicontinuous pseudo-contractive and nearly uniformly L-Lipschitzian mappings from C into itself associated with the net $\{a_s\}$ with property ($\mathscr{A}$). Let $\{s_n\}$ be a sequence in G such that $\lim_{n\to\infty} s_n = \infty$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0,1]$ satisfying the following conditions:*

(S1) $\lambda_n(1+\theta_n) \leq 1$ for all n in $\mathbb{N}$, $\lambda_n \to 0$ and $\lim_{n\to\infty} \lambda_n/\theta_n = 0$.

(S2) $\sum_{n=1}^{\infty} \lambda_n\theta_n = +\infty$.

(S3) $\lim_{n\to\infty} a_{s_n}/\theta_n = 0$.

Suppose $F(\mathcal{T})$ is nonempty and y^* in C is the unique solution of $GVI(4.1)[C, F(\mathcal{T}), \mathcal{F}, \phi]$. Let $\{b_s\}$ be a net in $(0,1)$ such that $\lim_{s\to\infty} b_s = 0$. For m in $\mathbb{N}$ and s in G, define $S_{m,s} = (1-\lambda_m)I + \lambda_m T_s$, and let $z_{m,s}$ be the unique point in C described by

$$z_{m,s} = b_s A z_{m,s} + (1 - b_s) S_{m,s} z_{m,s}.$$

Suppose for each m in $\mathbb{N}$, the net $\{z_{m,s} : s \in G\}$ is strongly convergent in C as $s \to \infty$. Suppose also the sequence $\{x_n\}$ generated by Algorithm 4.1 is bounded. Then $\{x_n\}$ converges strongly to y^*.

Proof. Since $\{a_{s_n}\}$ is bounded, we may assume that $a_{s_n} \leq \overline{a}$ for all n in $\mathbb{N}$. Set

$$\overline{d} := \|y^* - Ay^*\|, \quad \delta_n := 1 - \lambda_n\theta_n, \quad \text{and} \quad \sigma_n := \|x_n - y^*\|.$$

We now estimate

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
= & \ \lambda_n\|T_{s_n}x_n - x_n - \theta_n(x_n - Ax_n)\| \\
\leq & \ \lambda_n[\|T_{s_n}x_n - x_n\| + \theta_n\|x_n - Ax_n\|] \\
\leq & \ \lambda_n[\|T_{s_n}x_n - T_{s_n}y^*\| + \|x_n - y^*\| + \theta_n(\|x_n - y^*\| + \|y^* - Ay^*\| + \|Ay^* - Ax_n\|)] \\
\leq & \ \lambda_n[(1+L)\|x_n - y^*\| + a_{s_n} + \theta_n(\|x_n - y^*\| + \|y^* - Ay^*\| + w_A(\|y^* - x_n\|))] \\
\leq & \ \lambda_n[(1+L)\sigma_n + a_{s_n}] + \lambda_n\theta_n[\sigma_n + \overline{d} + w_A(\sigma_n)]
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\sigma_{n+1}^2 &= \langle (1-\lambda_n\theta_n)(x_n-y^*)+\lambda_n(T_{s_n}x_n-x_n)+\lambda_n\theta_n(Ax_n-y^*), J(x_{n+1}-y^*)\rangle \\
&= \langle (1-\lambda_n\theta_n)(x_n-y^*)+\lambda_n(T_{s_n}x_n-x_n+x_{n+1}-T_{s_n}x_{n+1}-(x_{n+1}-T_{s_n}x_{n+1})) \\
&\quad +\lambda_n\theta_n(Ax_n-Ax_{n+1}+Ax_{n+1}-Ay^*+Ay^*-y^*), J(x_{n+1}-y^*)\rangle \\
&\le \langle (1-\lambda_n\theta_n)(x_n-y^*)+\lambda_n(T_{s_n}x_n-x_n+x_{n+1}-T_{s_n}x_{n+1}) \\
&\quad +\lambda_n\theta_n(Ax_n-Ax_{n+1}+Ay^*-y^*), J(x_{n+1}-y^*)\rangle \\
&\quad +\lambda_n\theta_n[\|x_{n+1}-y^*\|^2-\|x_{n+1}-y^*\|\phi(\|x_{n+1}-y^*\|)] \\
&\le [(1-\lambda_n\theta_n)\|x_n-y^*\|+\lambda_n(\|T_{s_n}x_n-T_{s_n}x_{n+1}\|+\|x_{n+1}-x_n\|) \\
&\quad +\lambda_n\theta_n w_A(\|x_n-x_{n+1}\|)]\|x_{n+1}-y^*\|+\lambda_n\theta_n\langle Ay^*-y^*, J(x_{n+1}-y^*)\rangle \\
&\quad +\lambda_n\theta_n[\|x_{n+1}-y^*\|^2-\|x_{n+1}-y^*\|\phi(\|x_{n+1}-y^*\|)] \\
&\le [(1-\lambda_n\theta_n)\sigma_n+\lambda_n((1+L)\|x_{n+1}-x_n\|+a_{s_n})+\lambda_n\theta_n w_A(\|x_n-x_{n+1}\|)]\sigma_{n+1} \\
&\quad +\lambda_n\theta_n[\langle Ay^*-y^*, J(x_{n+1}-y^*)\rangle+\sigma_{n+1}^2-\sigma_{n+1}\phi(\sigma_{n+1})] \\
&\le (1-\lambda_n\theta_n)(\sigma_n^2+\sigma_{n+1}^2)/2+\lambda_n[(1+L)\|x_{n+1}-x_n\|+a_{s_n}+\theta_n w_A(\|x_n-x_{n+1}\|)]\sigma_{n+1} \\
&\quad +\lambda_n\theta_n\langle Ay^*-y^*, J(x_{n+1}-y^*)\rangle+\lambda_n\theta_n[\sigma_{n+1}^2-\sigma_{n+1}\phi(\sigma_{n+1})].
\end{aligned}$$

It follows that

$$\begin{aligned}
\sigma_{n+1}^2 &\le \sigma_n^2+\frac{2\lambda_n}{\delta_n}[(1+L)\|x_{n+1}-x_n\|+a_{s_n}+\theta_n w_A(\|x_n-x_{n+1}\|))\sigma_{n+1} \\
&\quad +\theta_n\langle Ay^*-y^*, J(x_{n+1}-y^*)\rangle-\theta_n\sigma_{n+1}\phi(\sigma_{n+1})], \quad \forall n\in\mathbb{N}. \qquad (4.9)
\end{aligned}$$

Note $\{\sigma_n\}$ is bounded, it follows from (4.8) that there exists a constant $K_2>0$ such that $\|x_{n+1}-x_n\|\le\lambda_n K_2$ for all n in $\mathbb{N}$. Since for each m in $\mathbb{N}$, $\{z_{m,s}: s\in G\}$ is strongly convergent in C as $s\to\infty$, Lemma 4.2 gives us that $\limsup_{n\to\infty}\langle Ay^*-y^*, J(x_n-y^*)\rangle\le 0$ and hence there exists a positive null sequence $\{\Upsilon_n\}$ such that $\langle Ay^*-y^*, J(x_n-y^*)\rangle\le\Upsilon_n$ for all $n=1,2,\ldots$. Set $K_3=\sup_{n\in\mathbb{N}}\sigma_n$. Then, from (4.9), we have

$$\begin{aligned}
\sigma_{n+1}^2 &\le \sigma_n^2-\frac{2\lambda_n\theta_n}{\delta_n}\sigma_{n+1}\phi(\sigma_{n+1})+\frac{2\lambda_n}{\delta_n}[(1+L)K_2\lambda_n+a_{s_n} \\
&\quad +\theta_n w_A(\lambda_n K_2)]K_3+\frac{2\lambda_n\theta_n}{\delta_n}\langle Ay^*-y^*, J(x_{n+1}-y^*)\rangle \qquad (4.10)
\end{aligned}$$

for all n in $\mathbb{N}$. Note $a_{s_n}/\theta_n\to 0$, $\lambda_n/\theta_n\to 0$, $w_A(\lambda_n K_2)\to 0$ and $\sum_{n=1}^{\infty}\lambda_n\theta_n=\infty$. It then follows from (4.10) and Lemma 2.7 that $\sigma_n\to 0$. □

Theorem 4.4 *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space X with a uniformly Gâteaux differentiable norm, and $A: C\to C$ be a uniformly continuous ϕ-strongly-pseudo-contractive mapping with a bounded range $A(C)$. Let $\mathcal{T}=\{T_s: s\in G\}$ be a semigroup of continuous pseudo-contractive and nearly uniformly L-Lipschitzian mappings from C into itself associated with the net $\{a_s\}$. Suppose $F(\mathcal{T})\neq\emptyset$ and $\mathcal{T}$ has property ($\mathscr{A}$). Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0,1]$ satisfying the conditions $(S1)$, $(S2)$ and $(S3)$. Let $\{s_n\}$ be a sequence in G such that $\lim_{n\to\infty}s_n=\infty$. Assume the sequence $\{x_n\}$ generated by Algorithm 4.1 is bounded. Then $\{x_n\}$ converges strongly to the unique solution of $GVI(4.1)[C, F(\mathcal{T}), \mathcal{F}, \phi]$.*

Proof. Let $m\in\mathbb{N}$. Define $S_{m,t}: C\to C$ by

$$S_{m,t}x=(1-\lambda_m)x+\lambda_m T_t x \quad \text{for all } x \text{ in } C \text{ and } t \text{ in } G.$$

Note for each t in G, the mapping $S_{m,t}$ is pseudocontractive and $y^* \in \bigcap_{t\in G} F(S_{m,t})$. Let $\{b_s\}$ be a net in $(0,1)$ with $\lim_{s\to\infty} b_s = 0$. l

By Proposition 3.1, there exists a point $z_{m,t}$ in C such that

$$z_{m,t} = b_t A z_{m,t} + (1-b_t) S_{m,t} z_{m,t} \quad \text{for each } t \text{ in } G.$$

By Theorem 3.4, $\{z_{m,t} : t \in G\}$ converges strongly to an element of C. Thus, $\{z_{m,t} : t \in G\}$ is eventually bounded. It then follows from (4.5) that $\{S_{m,t} z_{m,t} : t \in G\}$ is also eventually bounded. By the boundedness of $A(C)$, we have $\|z_{m,t} - S_{m,t} z_{m,t}\| = b_t \|A z_{m,t} - S_{m,t} z_{m,t}\| \to 0$ as $t \to \infty$ and $\|z_{m,t} - T_t z_{m,t}\| = \frac{\|z_{m,t} - S_{m,t} z_{m,t}\|}{\lambda_m} \to 0$ as $t \to \infty$. Now we can follow the proof of Theorem 4.3. □

We remark that in the previous theorems, if all T_s are nonexpansive then the sequence $\{x_n\}$ generated by Algorithm 4.1 is automatically bounded. Indeed, from (4.2), we have

$$\begin{aligned}
\|x_{n+1} - y^*\| &\le (1-\lambda_n(1+\theta_n))\|x_n - y^*\| + \lambda_n \|T_{s_n} x_n - T_{s_n} y^*\| + \lambda_n \theta_n \|A x_n - y^*\| \\
&\le (1-\lambda_n(1+\theta_n))\|x_n - y^*\| + \lambda_n \|x_n - y^*\| + \lambda_n \theta_n \mathrm{dist}(y^*, A(C)) \\
&\le \max\{\|x_n - y^*\|, \mathrm{dist}(y^*, A(C))\} \\
&\le \max\{\|x_1 - y^*\|, \mathrm{dist}(y^*, A(C))\}, \quad \forall n = 1, 2, \cdots .
\end{aligned}$$

Corollary 4.5 *Let C be a nonempty closed convex subset of a Banach space X with a uniformly Gâteaux differentiable norm, and $A : C \to C$ be a uniformly continuous ϕ-strongly-pseudo-contractive mapping with a bounded range $A(C)$. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0,1]$ satisfying the conditions $(S1)$ and $(S2)$. For m in $\mathbb{N}$ and t in $(0,1)$, let $S_m = (1-\lambda_m)I + \lambda_m T$ and $z_{m,t}$ be a unique point in C described by $z_{m,t} = tAz_{m,s} + (1-t)S_m z_{m,t}$. Suppose for each m in $\mathbb{N}$, $\{z_{m,t}\}$ is strongly convergent in C as $t \to 0^+$. Then the sequence $\{x_n\}$ generated by*

$$x_{n+1} := (1-\lambda_n(1+\theta_n))x_n + \lambda_n T x_n + \lambda_n \theta_n A x_n, \quad \forall n \in \mathbb{N}, \tag{4.11}$$

converges strongly to the unique solution of $GVI(4.1)[C, F(T), \mathcal{F}, \phi]$.

Corollary 4.6 *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and C a nonempty closed convex subset of X. Let $A : C \to C$ be a uniformly continuous ϕ-strongly pseudo-contractive mapping with a bounded range $A(C)$, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0,1]$ satisfying the conditions $(S1)$ and $(S2)$. Then $\{x_n\}$ generated by (4.11) converges strongly to the unique solution of $GVI(4.1)[C, F(T), \mathcal{F}, \phi]$.*

In light of Remark 2.1, we derive the following

Corollary 4.7 *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and C a nonempty closed convex subset of X. Let $A : C \to C$ be a uniformly continuous ϕ-strongly pseudo-contractive mapping with a bounded range $A(C)$. Let $\mathcal{T} = \{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings from C into itself with $F(\mathcal{T}) \neq \emptyset$ and property $(\mathscr{A})$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0,1]$ satisfying the conditions $(S1)$ and $(S2)$. Then $\{x_n\}$ generated by*

$$x_{n+1} := (1-\lambda_n(1+\theta_n))x_n + \lambda_n T_n x_n + \lambda_n \theta_n A x_n, \quad \forall n \in \mathbb{N},$$

converges strongly to the unique solution of $GVI(4.1)[C, F(\mathcal{T}), \mathcal{F}, \phi]$.

Remark 4.8 (I) We have already shown that there are some nonexpansive mappings which are not necessarily regular and also that if $\mathcal{T} = \{T\}$ is singleton, then it automatically satisfies property $(\mathscr{A})$. Thus, uniform asymptotic regularity becomes an extra condition when $\mathcal{T}$ is singleton. In this aspect, Theorem 4.4 is an improvement upon all the results concerning with uniformly asymptotically regular semigroups (see, e.g. [25, 30, 31, 32, 42] and the references therein).

(II) Corollary 4.7 extends and unifies a number of results (see, e.g., Takahashi [36, Theorem 5.1]) for approximating of common fixed points of a sequence of nonexpansive self-mappings.

(III) It is well known that L_p spaces ($1 < p < \infty$, $p \neq 2$) do not possess weakly sequentially continuous duality mappings and hence Song [30, Theorem 3.3] and Song and Chen [31, Theorem 3.2] cannot be applied to these spaces.

(IV) Corollary 4.5 is an important improvement and a significant generalization of the results of Shioji and Takahashi [29] and Suzuki [33, Theorem 3], since in our results, $\{\theta_n\}$ is not assumed to be constant.

Following Theorem 4.3, we are able to establish the next convergence result.

Corollary 4.9 *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a closed convex subset of X, and $\mathcal{T} = \{T_s : s \in G\}$ a semigroup of demicontinuous, pseudo-contractive, locally c-pseudo-contractive and nearly uniformly L-Lipschitzian mappings from C into itself associated with the net $\{a_s\}$, having property $(\mathscr{A})$ and $F(\mathcal{T}) \neq \emptyset$. Let $\{s_n\}$ be a sequence in G such that $\lim_{n\to\infty} s_n = \infty$ and let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0,1]$ satisfying the conditions $(S1)$, $(S1)$ and $(S3)$. For any given u in C, if the sequence $\{x_n\}$ generated by Algorithm 4.1 is bounded, then $\{x_n\}$ converges strongly to $Qu \in F(\mathcal{T})$ as $s \to \infty$, where Q so defined is a sunny nonexpansive retraction from C onto $F(\mathcal{T})$.*

We should remark that Theorem 3.7 and Corollary 4.9 appear to be new results for demicontinuous pseudo-contractive self-mappings. Corollary 4.9 improves various known results established concerning pseudo-contractive mappings in Hilbert and Banach spaces. In particular, Corollary 4.9 improves the convergence result of Bruck [6] without the acceptably paired assumption in the Banach space setting.

5 Applications

Let C be a nonempty closed convex subset of a Banach space X and $A : C \to C$ a uniformly continuous ϕ-strongly pseudo-contractive mapping with a bounded range $A(C)$. Let $T_1, T_2, \cdots, T_N : C \to C$ be nonexpansive mappings with $\cap_{i=1}^N F(T_i) \neq \emptyset$. We now propose a parallel algorithm for finding solution of $GVI(4.1)[C, \cap_{i=1}^N F(T_i), \mathcal{F}, \phi]$ and to remove the assumption (1.3).

Algorithm 5.1 Given x_1 in C and $t_1, t_2, \ldots, t_N > 0$ such that $\sum_{i=1}^N t_i = 1$, and two sequences $\{\lambda_n\}$ and $\{\theta_n\}$ in $(0,1]$ satisfying the conditions $(S1)$ and $(S2)$, a sequence $\{x_n\}$ in C is constructed as follows.

$$x_{n+1} := (1 - \lambda_n(1 + \theta_n))x_n + \lambda_n \sum_{i=1}^N t_i T_i x_n + \lambda_n \theta_n A x_n \quad \forall n \in \mathbb{N}. \tag{5.1}$$

Theorem 5.2 *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a closed convex subset of X and $A : C \to C$ a uniformly continuous ϕ-strongly pseudo-contractive mapping with a bounded range $A(C)$. Let $T_1, T_2, \cdots, T_N : C \to C$ be nonexpansive mappings with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Then $\{x_n\}$ generated by (5.1) converges strongly to the unique solution of $GVI(4.1)[C, \bigcap_{i=1}^{N} F(T_i), \mathcal{F}, \phi]$.*

Proof. Let $T = \sum_{i=1}^{N} t_i T_i$. Proposition 2.6 implies that T is nonexpansive from C into itself and $F(T) = \bigcap_{i=1}^{N} F(T_i)$. Hence the result follows from Corollary 4.6. □

We remark that Theorem 5.2 is a significant improvement and unification of many existing results concerning approximation of the solutions of a variational inequality, which are also common fixed points of a family of nonexpansive mappings (see, e.g. [7, 39, 40]) in the following senses:

(1) Theorem 5.2 holds for reflexive and strictly convex Banach spaces.

(2) The assumption (1.3) is not needed.

(3) The domains of the mappings $\mathcal{F}$ and T_i's are not necessarily the whole space.

(4) The domain of $\mathcal{F}$ is independent of the ranges of T_i's.

In Theorem 5.2, no metric projection mapping is used. However, metric projection mappings have wide applications in various disciplines, for example, image recovery. Recall that the so-called problem of image recovery is essentially to find a common element of finitely many nonexpansive retracts $C_1, C_2, \ldots, C_N$ of C with $\bigcap_{i=1}^{N} C_i \neq \emptyset$. It is easy to see that every nonexpansive retraction P_i of C onto C_i is a nonexpansive mapping of C into itself. Therefore, the image recovery problem can be thought of finding a common fixed point of finitely many nonexpansive mappings $P_1, \ldots, P_N$ of C into itself. Therefore, Theorem 5.2 should improve a number of results connected to the problem of image recovery.

Applying Proposition 2.6, we obtain

Corollary 5.3 *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, and $A : C \to C$ a uniformly continuous ϕ-strongly pseudo-contractive mapping with a bounded range $A(C)$. Let $\mathcal{T} = \{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings from C into itself such that $F(\mathcal{T}) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,1) such that $\sum_{n=1}^{\infty} \alpha_n = 1$, and define $Tx = \sum_{n=1}^{\infty} \alpha_n T_n x$ for all x in C. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0, 1]$ satisfying the conditions $(S1)$ and $(S2)$. Then $\{x_n\}$ generated by*

$$x_{n+1} := (1 - \lambda_n(1 + \theta_n))x_n + \lambda_n T x_n + \lambda_n \theta_n A x_n, \quad \forall n \in \mathbb{N},$$

converges strongly to the unique solution of $GVI(4.1)[C, F(\mathcal{T}), \mathcal{F}, \phi]$.

Recall that an accretive operator $\mathbb{A}$ in a Banach space X is said to satisfy the *range condition* if $\overline{D(\mathbb{A})} \subset R(1 + \lambda\mathbb{A})$ for all $\lambda > 0$. Here, $D(\mathbb{A})$ is the domain of $\mathbb{A}$ and $R(1 + \lambda\mathbb{A})$ is the range of $1 + \lambda\mathbb{A}$. If $\mathbb{A}$ is accretive, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda^{\mathbb{A}} : R(1 + \lambda\mathbb{A}) \to D(\mathbb{A})$ by $J_\lambda^{\mathbb{A}} = (I + \lambda\mathbb{A})^{-1}$. It is called the *resolvent* of $\mathbb{A}$. It is well known that for an accretive operator $\mathbb{A}$ which satisfies the range condition, $\mathbb{A}^{-1}(0) = F(J_\lambda^{\mathbb{A}})$ for all $\lambda > 0$. We also define the *Yosida approximation* $\mathbb{A}_r := (I - J_r^{\mathbb{A}})/r$. We know that $\mathbb{A}_r x \in \mathbb{A} J_r^{\mathbb{A}} x$ for all x in $R(I + r\mathbb{A})$ and $\|\mathbb{A}_r x\| \leq |\mathbb{A}x| = \inf\{\|y\| : y \in \mathbb{A}x\}$ for all x in $D(\mathbb{A}) \cap R(I + r\mathbb{A})$.

The following result in an improvement of Wong, Sahu and Yao [37, Theorem 6.3] and Zegeye and Shahzad [43, Theorem 3.3].

Corollary 5.4 *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, and $A : C \to C$ a uniformly continuous ϕ-strongly pseudo-contractive mapping with a bounded range $A(C)$. Let $\mathbb{A}_i \subset X \times X$ $(i = 1, 2, \cdots, N)$ be accretive operators with resolvent $J_t^{\mathbb{A}_i}$ for $t > 0$ such that $\cap_{i=1}^N \mathbb{A}_i^{-1} 0 \neq \emptyset$ and $\overline{D(\mathbb{A}_i)} \subset C \subset \bigcap_{t>0} R(I + t\mathbb{A}_i)$. Let $t_1, t_2, \ldots, t_N > 0$ such that $\sum_{i=1}^N t_i = 1$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0, 1]$ satisfying the conditions $(S1)$ and $(S2)$. Then $\{x_n\}$ generated by*

$$x_{n+1} := (1 - \lambda_n(1 + \theta_n))x_n + \lambda_n \sum_{i=1}^{N} t_i J_{t_i}^{\mathbb{A}_i} x_n + \lambda_n \theta_n A x_n, \quad \forall n \in \mathbb{N},$$

converges strongly to the unique solution of $GVI(4.1)[C, \cap_{i=1}^N \mathbb{A}_i^{-1} 0, \mathcal{F}, \phi]$.

Proof. Note that each $J_t^{\mathbb{A}_i}$ is nonexpansive for each $i = 1, 2, \cdots, N$ and $t > 0$. Set $T := \sum_{i=1}^N t_i J_t^{\mathbb{A}_i}$. Proposition 2.6 implies that T is nonexpansive from C into itself and $F(T) = \cap_{i=1}^N \mathbb{A}_i^{-1} 0$. Hence the result follows from Corollary 4.6. □

Corollary 5.5 *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, and $A : C \to C$ a uniformly continuous ϕ-strongly pseudo-contractive mapping with a bounded range $A(C)$. Let $\mathfrak{A} = \{\mathbb{A}_n : n \in \mathbb{N}\}$ be a sequence of accretive operators with resolvent $J_t^{\mathbb{A}_n}$ for $t > 0$ such that $\cap_{n\in\mathbb{N}} \mathbb{A}_n^{-1} 0 \neq \emptyset$ and $\overline{D(\mathbb{A}_n)} \subset C \subset \bigcap_{t>0} R(I + t\mathbb{A}_n)$ for all n in $\mathbb{N}$. Let $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^\infty \alpha_n = 1$, and define $Sx = \sum_{n=1}^\infty \alpha_n J_{\alpha_n}^{\mathbb{A}_n} x$ for all x in C. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0, 1]$ satisfying the conditions $(S1)$ $(S2)$. Then $\{x_n\}$ generated by*

$$x_{n+1} := (1 - \lambda_n(1 + \theta_n))x_n + \lambda_n S x_n + \lambda_n \theta_n A x_n, \quad \forall n \in \mathbb{N},$$

converges strongly to the unique solution of $GVI(4.1)[C, \cap_{n\in\mathbb{N}} \mathbb{A}_n^{-1} 0, \mathcal{F}, \phi]$.

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