A GENERALIZED HYBRID STEEPEST-DESCENT METHOD FOR VARIATIONAL INEQUALITIES IN BANACH SPACES

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ABSTRACT. The hybrid steepest descent method introduced by Yamada [Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, D. Butnariu, Y. Censor, and S. Reich, eds., North-Holland, Amsterdam, Holland, 2001, pp. 473-504] is an algorithmic solution to the variational inequality problem over the fixed point set of nonlinear mapping and applicable to broad range of convexly constrained nonlinear inverse problems in real Hilbert spaces. Lehdili and Moudafi [*Optimization*, 37(1996), 239-252] introduced the new prox-Tikhonov regularization method for proximal point algorithm to generate a strongly convergent sequence and established a convergence property for it by using the technique of variational distance in Hilbert spaces. In this paper, motivated by Yamada's hybrid steepest-descent and Lehdili and Moudafi's algorithms, a generalized hybrid steepest-descent algorithm for computing the solutions of variational inequality problem over the common fixed point set of sequence of nonexpansive type mappings in the framework of Banach space is proposed. The strong convergence for the proposed algorithm to the solution is guaranteed under some assumptions. Our strong convergence theorems extend and improve certain corresponding results in recent literature.

1. INTRODUCTION

Let $\mathscr{H}$ be a real Hilbert space with inner product $\langle .,.\rangle$ and norm $\|.\|$, respectively. Let C be a nonempty closed convex subset of $\mathscr{H}$ and D a nonempty closed convex subset of C.

It is well known that the standard smooth convex optimization problem ([35]), given a convex, Fréchet differentiable function $f : \mathscr{H} \to \mathbb{R}$ and a closed convex subset C of $\mathscr{H}$, find a point $x^* \in C$ such that

$$f(x^*) = \min\{x \in C : f(x)\}$$

can be formulated equivalently as the variational inequality problem $VIP(\nabla f, \mathscr{H})$ over C (see [13, 17]):

$$\langle \nabla f x^*, v - x^* \rangle \geq 0 \text{ for all } v \in C,$$

where $\nabla f : \mathscr{H} \to \mathscr{H}$ is the gradient of f.

In general for a nonlinear mapping $\mathcal{F} : \mathscr{H} \to \mathscr{H}$ over C, the variational inequality problem $VIP(\mathcal{F}, C)$ over D is to find a point $x^* \in D$ such that

$$(1.1) \qquad\qquad \langle \mathcal{F} x^*, v - x^* \rangle \geq 0 \text{ for all } v \in D.$$

It is important to note that the theory of variational inequalities has been playing an important role in the study of many diverse disciplines, for instance, partial differential equations, optimal control, optimization, mathematical programming, mechanics, finance, etc., see, for example, [12, 13, 20, 33, 35] and references therein.

It is also known that if $\mathcal{F}$ is Lipschitzian and strongly monotone, then for small $\mu > 0$, the mapping $P_C(I - \mu\mathcal{F})$ is a contraction, where P_C is the metric projection from $\mathscr{H}$ onto C (see section 2.3). In this case, the Banach contraction principle guarantees that $VIP(\mathcal{F}, C)$ has unique solution x^* and the sequence of Picard iteration process, given by,

$$(1.2) \qquad x_{n+1} = P_C(I - \mu\mathcal{F})x_n \text{ for all } n \in \mathbb{N}$$

converges strongly to x^*. This simplest iterative method for approximating the unique solution of $VIP(\mathcal{F}, C)$ over C is called the *projected gradient method* ([35]). It has been used widely in many practical problems, due partially to its fast convergence.

The gradient projection method was first proposed by Goldstein [9] and Levitin and Polyak [16] for solving convexly constrained minimization problems. This method is regarded as an extension of the steepest descent or Cauchy algorithm for solving unconstrained optimization problems. It now has many variants in different setting, and supplies a prototype for various more advanced projection methods. In [22], the first author introduced the normal S-iteration process and studied an iterative method for approximating the unique solution of $VIP(\mathcal{F}, C)$ over C as follows:

$$(1.3) \qquad x_{n+1} = P_C(I - \mu\mathcal{F})[(1 - \alpha_n)x_n + \alpha_n P_C(I - \mu\mathcal{F})x_n] \text{ for all } n \in \mathbb{N}.$$

Note that the rate of convergence of iterative method (1.3) is faster than projected gradient method (1.2), see [22].

The projected gradient method requires repetitive use of P_C, although the closed form expression of P_C is not always known in many situations. In order to reduce the complexity probably caused by the projection mapping P_C, Yamada (see [33]) introduced a hybrid steepest descent method for solving the problem $VIP(\mathcal{F}, \mathscr{H})$. Here is the idea. Suppose T (e.g., $T = P_C$) is a mapping from a Hilbert space $\mathscr{H}$ into itself with a nonempty fixed point set $F[T]$, and $\mathcal{F}$ is a Lipschitzian and strongly monotone over $\mathscr{H}$. Starting with an arbitrary initial guess x_1 in $\mathscr{H}$, one generates a sequence $\{x_n\}$ by the following algorithm:

$$(1.4) \qquad x_{n+1} := T[x_n - \lambda_n \mathcal{F}(x_n)] \text{ for all } n \in \mathbb{N},$$

where $\{\lambda_n\}$ is a slowly diminishing sequence. Yamada [33, Theorem 3.3, p.486] proved that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a unique solution of $VIP(\mathcal{F}, \mathscr{H})$ over $F[T]$.

Let X be a real Banach space with dual space X^*. We denote by J the normalized duality mapping from X into 2^{X^*} defined by

$$J(x) := \{f^* \in X^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\}, \ x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that the normalized duality mapping is single-valued if X smooth, see [1]. Let C be a nonempty subset of a real Banach space X. A mapping $T : X \to X$ is said to be

(1) *pseudocontractive over C* if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq ||x - y||^2,$$

(2) *δ-strongly accretive over C* if for each $x, y \in C$, there exist a constant $\delta > 0$ and $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \geq \delta ||x - y||^2.$$

We consider the following general variational inequality problem over the fixed point set of nonlinear mapping in the framework of Banach space.

Problem 1.1. (General Variational Inequality Problem Over the Fixed Point Set of Nonlinear Mapping) Let C be a nonempty closed convex subset of a real smooth Banach space X. Let $T : C \to C$ be a (possibly nonlinear) mapping of which fixed point set $F[T]$ is a nonempty closed convex set. Then for a given strongly accretive operator $\mathcal{F} : X \to X$ over C, the general variational inequality problem $VIP(\mathcal{F}, C)$ over $F[T]$ is

$$\text{find a point } x^* \in F[T] \text{ such that } \langle \mathcal{F}x^*, J(v - x^*) \rangle \geq 0 \text{ for all } v \in F[T].$$

Recently, the method (1.4) has been applied successfully to signal processing, inverse problems, and so on [24, 25, 32]. This situation induces a natural question:

Qustion 1.2. *Does sequence $\{x_n\}$ defined by (1.4) converges strongly a solution to a general variational inequality problem in the Banach space setting; i.e., Problem 1.1 in a case where $T : C \to C$ is given as such a nonexpansive mapping?*

We now consider the following variational inclusion problem:

$$\text{find } \ z \in C \ \text{ such that } 0 \in Az, \qquad\qquad\qquad (P)$$

in the framework of Banach space X, where $A : X \to 2^X$ is a multi-valued operator acting on $C \subseteq X$. In the sequel we assume that $\mathbb{S} = A^{-1}(0)$, the set of solutions of Problem (P) is nonempty.

The Problem (P) can be regarded as a unified formulation of several important problems. For an appropriate choice of the operator A, Problem (P) covers a wide range of mathematical applications; for example, variational inequalities, complementarity problems, and non-smooth convex optimization. Problem (P) has applications in physics, economics, and in several areas of engineering. In particular, if $\psi : \mathscr{H} \to \mathbb{R} \cup \{\infty\}$ is a proper, lower semi-continuous convex function, its subdifferential $\partial\psi = A$ is a maximal monotone operator, and a point $z \in \mathscr{H}$ minimizes ψ if and only if $0 \in \partial\psi(z)$.

One of the most interesting and important problems in the theory of maximal monotone operators is to find an efficient iterative algorithm to compute approximately zeroes of maximal monotone operators. One method for solving zeros of maximal monotone operators is *proximal point algorithm*. Let A be a maximal monotone operator in a Hilbert space $\mathscr{H}$. The proximal point algorithm generates, for starting $x_1 \in \mathscr{H}$, a sequence $\{x_n\}$ in $\mathscr{H}$ by

$$(1.5) \qquad\qquad x_{n+1} = J_{c_n} x_n \text{ for all } n \in \mathbb{N},$$

where $J_{c_n} := (I + c_n A)^{-1}$ is the resolvent operator associated with the operator A, and $\{c_n\}$ is a regularization sequence in $(0, \infty)$. This iterative procedure is based on the fact that the proximal map J_{c_n} is single-valued and nonexpansive. This algorithm was first introduced by Martinet [19]. If $\psi : \mathscr{H} \to \mathbb{R} \cup \{\infty\}$ is a proper lower semicontinuous convex function, then the algorithm reduces to

$$x_{n+1} = \text{argmin}_{y \in \mathscr{H}} \left\{ \psi(y) + \frac{1}{2c_n} \|x_n - y\|^2 \right\} \text{ for all } n \in \mathbb{N}.$$

Rockafellar [21] studied the proximal point algorithm in the framework of Hilbert space and he proved the following:

Theorem 1.3. *Let $\mathscr{H}$ be a Hilbert space and $A \subset \mathscr{H} \times \mathscr{H}$ a maximal monotone operator. Let $\{x_n\}$ be a sequence in $\mathscr{H}$ defined by (1.5), where $\{c_n\}$ is a sequence in $(0, \infty)$ such that $\liminf_{n\to\infty} c_n > 0$. If $\ \mathbb{S} \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element of $\mathbb{S}$.*

Such weak convergence is global; i.e., the result just announced holds in fact for any $x_1 \in \mathscr{H}$.

Further, Rockafellar [21] posed an open question of whether the sequence generated by (1.5) converges strongly or not. This question was solved by Güler [10], who constructed an example for which the sequence generated by (1.5) converges weakly but not strongly. This brings us a natural question how to modify the proximal point algorithm so that strongly convergent sequence is guaranteed. The *Tikhonov method* which generates a sequence $\{\widetilde{x}_n\}$ by the rule

$$(1.6) \qquad \widetilde{x}_n = J_{\mu_n}^A u \text{ for all } n \in \mathbb{N},$$

where $u \in \mathscr{H}$ and $\mu_n > 0$ such that $\mu_n \to \infty$ is studied by several authors (see, e.g., Takahashi [27] and Wong, Sahu, Yao [30]) to answer the above question.

In [15], Lehdili and Moudafi combined the technique of the proximal map and the Tikhonov regularization to introduce the *prox-Tikhonov method* which generates the sequence $\{x_n\}$ by the algorithm

$$(1.7) \qquad x_{n+1} = J_{\lambda_n}^{A_n} x_n \text{ for all } n \in \mathbb{N},$$

where $A_n = \mu_n I + A$, $\mu_n > 0$ is viewed as a Tikhonov regularization of A. Note that A_n is strongly monotone, i.e., $\langle x - x', y - y' \rangle \geq \mu_n \|x - x'\|^2$ for all $(x, y), (x', y') \in G(A_n)$, where $G(A_n)$ is graph of A_n.

Using the technique of variational distance, Lehdili and Moudafi [15] were able to prove strong convergence of the algorithm (1.7) for solving Problem (P) when A is maximal monotone operator on $\mathscr{H}$ under certain conditions imposed upon the sequences $\{\lambda_n\}$ and $\{\mu_n\}$.

It should be also noted that A_n is now a maximal monotone operator and hence $\{J_{\lambda_n}^{A_n}\}$ is a sequence of nonexpansive mappings.

The main objective of this article is to solve the proposed Problem 1.1. To achieve this goal, we present an existence theorem for Problem 1.1. Motivated by Yamada's hybrid steepest-descent and Lehdili and Moudafi's algorithms (1.4) and (1.7), we also present an iterative algorithm and investigate the convergence theory of the proposed algorithm for solving Problem 1.1. The outline of this paper is as follows. In section 2, we present some theoretical tools which are needed in the sequel. In section 3, we present (Theorem 3.3) the existence and uniqueness of solution of Problem 1.1 in a case when $T : C \to C$ is not necessarily nonexpansive mapping. In section 4, we propose an iterative algorithm (Algorithm 4.1), as a generalization of Yamada's hybrid steepest-descent and Lehdili and Moudafi's algorithms (1.4) and (1.7), for computing to a unique solution of the variational inequality $VIP(\mathcal{F}, C)$ over $\cap_{n \in \mathbb{N}} F[T_n]$ in the framework of Banach space. In section 5, we apply our result to the problem of finding a common fixed point of a countable family of nonexpansive mappings and the solution of Problem (P). Our strong convergence theorems extend and improve corresponding results of Ceng, Anasri and Yao [5]; Ceng, Xu, Yao [6]; Lehdili and Moudafi [15]; Sahu [22]; and Yamada [33].

2. Preliminaries and notations

2.1. Derivatives of functionals.
Let X be a real Banach space. In the sequel, we always use S_X to denote the unit sphere $S_X = \{x \in X : \|x\| = 1\}$. Then X is said to be

  (i) *strictly convex* if $x, y \in S_X$ with $x \neq y \Rightarrow \|(1 - \lambda)x + \lambda y\| < 1$ for all $\lambda \in (0, 1)$;
  (ii) *smooth* if the limit $\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$ exists for each x and y in S_X. In this case, the norm of X is said to be *Gâteaux differentiable*.

The norm of X is said to be *uniformly Gâteaux differentiable* if for each $y \in S_X$, this limit is attained uniformly for $x \in S_X$.

It is well known that every uniformly smooth space (e.g., L_p space, $1 < p < \infty$) has uniformly Gâteaux differentiable norm (see e.g., [1]).

Let U be an open subset of a real Hilbert space $\mathscr{H}$. Then, a function $\Theta : \mathscr{H} \to \mathbb{R} \cup \{\infty\}$ is called Gâteaux differentiable [34, p. 135] on U if for each $u \in U$, there exists $a(u) \in \mathscr{H}$ such that

$$\lim_{t \to 0} \frac{\Theta(u + th) - \Theta(u)}{t} = \langle a(u), h \rangle \text{ for all } h \in \mathscr{H}.$$

Then, $\Theta' : U \to \mathscr{H} : u \to a(u)$ is called the *Gâteaux derivative* of Θ on U.

Example 2.1. (see [33]) Suppose that $h \in \mathscr{H}$, $\beta \in \mathbb{R}$ and $Q : \mathscr{H} \to \mathscr{H}$ is a bounded linear, self-adjoint, i.e. $\langle Q(x), y \rangle = \langle x, Q(y) \rangle$ for all $x, y \in \mathscr{H}$, and strongly positive mapping, i.e., $\langle Q(x), x \rangle \geq \alpha \|x\|^2$ for all $x \in \mathscr{H}$ and for some $\alpha > 0$. Define the quadratic function $\Theta : \mathscr{H} \to \mathbb{R}$ by

$$\Theta(x) := \frac{1}{2} \langle Q(x), x \rangle - \langle h, x \rangle + \beta \text{ for all } x \in \mathscr{H}.$$

Then, the Gâteaux derivative $\Theta'(x) = Q(x) - \beta$ is $\|Q\|$ -Lipschitzian and α-strongly monotone on $\mathscr{H}$.

2.2. **Lipschitzian type mappings.** Let C be nonempty subset of a real Banach space X and let $S_1, S_2 : C \to X$ be two mappings. We denote $\mathscr{B}(C)$, the collection of all bounded subsets of C. The deviation between S_1 and S_2 on $B \in \mathscr{B}(C)$, denoted by $\mathscr{D}_B(S_1, S_2)$, is defined by

$$\mathscr{D}_B(S_1, S_2) = \sup\{\|S_1 x - S_2 x\| : x \in B\}.$$

A mapping $T : C \to X$ is said to be

(1) *L-Lipschitzian* if there exists a constant $L \in (0, \infty)$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$;

(2) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

(3) *strongly pseudocontractive* if for each $x, y \in C$, there exist a constant $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2,$$

(4) λ-*strictly pseudocontractive* (see [3]) if for each $x, y \in C$, there exist a constant $\lambda > 0$ and $j(x - y) \in J(x - y)$ such that

(2.1) $\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda\|x - y - (Tx - Ty)\|^2.$

The inequality (2.1) can be restated as

(2.2) $\langle x - y - (Tx - Ty), j(x - y) \rangle \geq \lambda\|x - y - (Tx - Ty)\|^2.$

In Hilbert spaces, (2.1) (and so (2.2)) is equivalent to the following inequality

(2.3) $\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2,$

where $k = 1 - 2\lambda$. From (2.2) one can prove that if T is λ-strict pseudo-contractive, then T is Lipschitz continuous with the Lipschitz constant L = $\frac{1+\lambda}{\lambda}$ (see, Proposition 3.1). Throughout the paper, we assume that $L_{\lambda,\delta} := \sqrt{\frac{1-\delta}{\lambda}}$.

Fact 2.2. [1, Corollary 5.7.15] Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a continuous strongly pseudocontractive mapping. Then T has a unique fixed point in C.

Fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$ and let $\{T_n\}$ be a sequence of mappings from C into X. Then $\{T_n\}$ is called a sequence of asymptotically nonexpansive mappings if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$\|T_n x - T_n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

Motivated by notion of nearly nonexpansive mappings (see [1, 23]), we say $\{T_n\}$ is a sequence of nearly nonexpansive mappings if

$$\|T_n x - T_n y\| \leq \|x - y\| + a_n \text{ for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

Remark 2.3. If $\{T_n\}$ is a sequence of asymptotically nonexpansive mappings with bounded domain, then $\{T_n\}$ is a sequence of nearly nonexpansive mappings . To see this, let $\{T_n\}$ be a sequence of asymptotically nonexpansive mappings with sequence $\{k_n\}$ defined on a bounded set C with diameter $diam(C)$. Fix $a_n := (k_n - 1)diam(C)$. Then,

$$\|T_n x - T_n y\| \leq \|x - y\| + (k_n - 1)\|x - y\| \leq \|x - y\| + a_n$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

We prove the following proposition:

Proposition 2.4. *Let C be a closed bounded set of a Banach space X and $\{T_n\}$ a sequence of nearly nonexpansive self-mappings of C with sequence $\{a_n\}$ such that $\sum_{n=1}^{\infty} \mathscr{D}_C(T_n, T_{n+1}) < \infty$. Then, for each $x \in C$, $\{T_n x\}$ converges strongly to some point of C. Moreover, if T is a mapping of C into itself defined by $Tz = \lim_{n \to \infty} T_n z$ for all $z \in C$, then T is nonexpansive and $\lim_{n \to \infty} \mathscr{D}_C(T_n, T) = 0$.*

Proof. The assumption $\sum_{n=1}^{\infty} \mathscr{D}_C(T_n, T_{n+1}) < \infty$ implies that $\sum_{n=1}^{\infty} \|T_n x - T_{n+1} x\| < \infty$ for all $z \in C$. Hence $\{T^n z\}$ is a Cauchy sequence for each $z \in C$. Hence, for $x \in C$, $\{T^n x\}$ converges strongly to some point in C. Let T be a mapping of C into itself defined by $Tz = \lim_{n \to \infty} T_n z$ for all $z \in C$. It is easy to see that T is nonexpansive. For $z \in C$ and $m, n \in \mathbb{N}$ with $m > n$, we have

$$
\begin{aligned}
\|T_n x - T_m x\| &\leq \sum_{k=n}^{m-1} \|T_k x - T_{k+1} x\| \\
&\leq \sum_{k=n}^{m-1} \mathscr{D}_C(T_k, T_{k+1}) \\
&\leq \sum_{k=n}^{\infty} \mathscr{D}_C(T_k, T_{k+1}).
\end{aligned}
$$

Then

$$\|T_n x - Tx\| = \lim_{m \to \infty} \|T_n x - T_m x\| \leq \sum_{k=n}^{\infty} \mathscr{D}_C(T_k, T_{k+1}) \text{ for all } x \in C \text{ and } n \in \mathbb{N},$$

which implies that

$$\mathscr{D}_C(T_n, T) \leq \sum_{k=n}^{\infty} \mathscr{D}_C(T_k, T_{k+1}) \text{ for all } n \in \mathbb{N}.$$

Therefore, $\lim_{n \to \infty} \mathscr{D}_C(T_n, T) = 0$. $\square$

2.3. **Nonexpansive mapping and fixed point.** A closed convex subset C of a Banach space X is said to have the fixed-point property for nonexpansive self-mappings if every nonexpansive mapping of a nonempty closed convex bounded subset M of C into itself has a fixed point in M.

A closed convex subset C of a Banach space X is said to have normal structure if for each closed convex bounded subset of D of C which contains at least two points, there exists an element $x \in D$ which is not a diametral point of D. It is well known that a closed convex subset of a uniformly smooth Banach space has normal structure, see [1] for more details.

The following result was proved by Kirk [14].

Fact 2.5. (Kirk [14]) *Let X be a reflexive Banach space and let C be a nonempty closed convex bounded subset of X which has normal structure. Let T be a nonexpansive mapping of C into itself. Then $F[T]$ is nonempty.*

A subset C of a Banach space X is called a *retract* of X if there exists a continuous mapping P from X onto C such that $Px = x$ for all x in C. We call such P a *retraction* of X onto C. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A retraction P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each x in X and $t \geq 0$. If a sunny retraction P is also nonexpansive, then C is said to be a *sunny nonexpansive retract* of X.

Let C be a nonempty subset of a Banach space X and let $x \in X$. An element $y_0 \in C$ is said to be a *best approximation* to x if $\|x - y_0\| = d(x, C)$, where $d(x, C) = \inf_{y \in C} \|x - y\|$. The set of all best approximations from x to C is denoted by

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}.$$

This defines a mapping P_C from X into 2^C and is called the *nearest point projection mapping* (*metric projection mapping*) onto C. It is well known that if C is a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$, then the nearest point projection P_C from $\mathscr{H}$ onto C is the unique sunny nonexpansive retraction of $\mathscr{H}$ onto C. It is also known that $P_C x \in C$ and

$$(2.4) \qquad \langle x - P_C x, P_C x - y \rangle \geq 0 \text{ for all } x \in \mathscr{H}, y \in C.$$

Let $\mathcal{F}$ be a monotone mapping of $\mathscr{H}$ into $\mathscr{H}$ over $C \subseteq \mathscr{H}$. In the context of the variational inequality problem the characterization of projection (2.4) implies

$$x^* \in VIP(\mathcal{F}, C) \iff x^* = P_C(x^* - \mu A x^*) \text{ for all } \mu > 0.$$

We know the following fact concerning nonexpansive retraction.

Fact 2.6. (Goebel and Reich [8, Lemma 13.1]). *Let C be a convex subset of a real smooth Banach space X, D a nonempty subset of C and P a retraction from C onto D. Then the following are equivalent:*

   (a) *P is a sunny and nonexpansive.*
   (b) *$\langle x - Px, J(z - Px) \rangle \leq 0 \quad$ for all $x \in C, z \in D$.*
   (c) *$\langle x - y, J(Px - Py) \rangle \geq \|Px - Py\|^2$ for all $x, y \in C$.*

Fact 2.7. (Wong, Sahu and Yao [30, Proposition 6.1]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X and let $\lambda_i > 0$ $(i = 1, 2, \cdots, N)$ such that $\sum_{i=1}^{N} \lambda_i = 1$. Let $T_1, T_2, \cdots, T_N : C \to C$ be nonexpansive mappings with $\cap_{i=1}^{N} F(T_i) \neq \emptyset$ and let $T = \sum_{i=1}^{N} \lambda_i T_i$. Then T is nonexpansive from C into itself and $F(T) = \cap_{i=1}^{N} F(T_i)$.*

Fact 2.8. (Bruck [4]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{S_k\}$ be a sequence nonexpansive mappings of C into itself with $\cap_{k=1}^{\infty} F[S_k] \neq \emptyset$ and $\{\beta_k\}$ sequence of positive real numbers such that $\sum_{k=1}^{\infty} \beta_k = 1$. Then the mapping $T = \sum_{k=1}^{\infty} \beta_k S_k$ is well defined on C and $F[T] = \cap_{k=1}^{\infty} F[S_k]$.*

2.4. **Accretive operators and zero.** Let X be a real Banach space X. For an operator $A : X \to 2^X$, we define its domain, range and graph as follows:

$$D(A) = \{x \in X : Ax \neq \emptyset\}, \ R(A) = \bigcup\{Az : z \in D(A)\}$$

and

$$G(T) = \{(x, y) \in X \times X : x \in D(A), y \in Ax\},$$

respectively. Thus, we write $A : X \to 2^X$ as follows: $A \subset X \times X$. The inverse A^{-1} of A is defined by

$$x \in A^{-1}y \text{ if and only if } y \in Ax.$$

The operator A is said to be *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there is $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. An accretive operator A is said to be *maximal accretive* if there is no proper accretive extension of A and *m-accretive* if $R(I + A) = X$. (It follows that $R(I + rA) = X$ for all $r > 0$.) If A is *m*-accretive, then it is maximal accretive (see Fact 2.10), but the converse is not true in general. If A is accretive, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda : R(1 + \lambda A) \to D(A)$ by $J_\lambda = (I + \lambda A)^{-1}$. It is called the *resolvent* of A. An accretive operator A defined on X is said to satisfy the *range condition* if $\overline{D(A)} \subset R(1 + \lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. It is well known that for an accretive operator A which satisfies the range condition, $A^{-1}(0) = F(J_\lambda^A)$ for all $\lambda > 0$. We also define the *Yosida approximation* A_r by $A_r = (I - J_r^A)/r$. We know that $A_r x \in A J_r^A x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq |Ax| = \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know the following [28]: For each $\lambda, \mu > 0$ and $x \in R(I + \lambda A) \cap R(I + \mu A)$, it holds that

$$(2.5) \qquad \|J_\lambda x - J_\mu x\| \leq \frac{|\lambda - \mu|}{\lambda} \|x - J_\lambda x\|.$$

Let f be a continuous linear functional on ℓ_∞. We use $f_n(x_{n+m})$ to denote

$$f(x_{m+1}, x_{m+2}, x_{m+3}, \cdots, x_{m+n}, \cdots),$$

for $m = 0, 1, 2, \ldots$. A continuous linear functional j on l_∞ is called a *Banach limit* if $\|j\|_* = j(1) = 1$ and $j_n(x_n) = j_n(x_{n+1})$ for each $x = (x_1, x_2, \cdots)$ in l_∞.

Fix any Banach limit and denote it by LIM. Note that $\|LIM\|_* = 1$,

$$\liminf_{n \to \infty} t_n \leq LIM_n t_n \leq \limsup_{n \to \infty} t_n,$$

and

$$LIM_n t_n = LIM_n t_{n+1}, \quad \forall (t_n) \in l_\infty.$$

The following facts will be needed in the sequel for the proof of our main results:

Fact 2.9. (Ha and Jung [11, Lemma 1]). *Let X be a Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X and $\{x_n\}$ a bounded sequence in X. Let LIM be a Banach limit and $y \in C$ such that $LIM_n\|y_n - y\|^2 = \inf_{x \in C} LIM_n\|y_n - x\|^2$. Then $LIM_n\langle x - y, J(x_n - y)\rangle \leq 0$ for all $x \in C$.*

Fact 2.10. (Cioranescu [7]). Let X be a Banach space and let $A : X \to 2^X$ be an *m*-accretive operator. Then A is maximal accretive. If $\mathscr{H}$ is a Hilbert space, then $A : \mathscr{H} \to 2^{\mathscr{H}}$ is maximal accretive if and only if it is *m*-accretive.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF $VIP(\mathcal{F}, C)$

In this section we deal with the existence and uniqueness of solution of Problem 1.1 in a case where $T : C \to C$ is given as such a pseudocontractive mapping.

The following propositions will be used frequently throughout the paper.

Proposition 3.1. *Let C be a nonempty subset of a real smooth Banach space X and $\mathcal{F} : X \to X$ an operator over C. Then:*

    (a) *If $\mathcal{F}$ is λ-strictly pseudocontractive, then $\mathcal{F}$ is Lipschitzian with constant $1 + \frac{1}{\lambda}$.*

    (b) *If $\mathcal{F}$ is both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$, then $I - \mathcal{F}$ is contraction with Lipschitz constant $L_{\lambda,\delta}$.*

    (c) *If $\tau \in (0,1)$ is a fixed number and $\mathcal{F}$ is both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$, then $I - \tau\mathcal{F} : C \to C$ is a contraction mapping with Lipschitz constant $1 - (1 - L_{\lambda,\delta})\tau$.*

Proof. (a) Let $x, y \in C$. From (2.2), we have

$$
\begin{aligned}
\lambda \|x - y - (\mathcal{F}x - \mathcal{F}y)\|^2 &\leq \langle x - y - (\mathcal{F}x - \mathcal{F}y), J(x - y) \rangle \\
&\leq \|x - y - (\mathcal{F}x - \mathcal{F}y)\| \|x - y\|,
\end{aligned}
$$

which gives us

$$
\|x - y - (\mathcal{F}x - \mathcal{F}y)\| \leq \frac{1}{\lambda} \|x - y\|.
$$

Thus,

$$
\|\mathcal{F}x - \mathcal{F}y\| \leq \|x - y\| + \|x - y - (\mathcal{F}x - \mathcal{F}y)\| \leq \left(1 + \frac{1}{\lambda}\right) \|x - y\|.
$$

Hence $\mathcal{F}$ is Lipschitzian with constant $1 + \frac{1}{\lambda}$.

(b) Let $x, y \in C$. Further, from (2.2), we have

$$
\begin{aligned}
\lambda \|x - y - (\mathcal{F}x - \mathcal{F}y)\|^2 &\leq \|x - y\|^2 - \langle \mathcal{F}x - \mathcal{F}y, J(x - y) \rangle \\
&\leq (1 - \delta) \|x - y\|^2.
\end{aligned}
$$

Observe that

$$
\lambda + \delta > 1 \Leftrightarrow L_{\lambda,\delta} \in (0,1).
$$

Hence

$$
\|x - y - (\mathcal{F}x - \mathcal{F}y)\| \leq \sqrt{\frac{1 - \delta}{\lambda}} \|x - y\| = L_{\lambda,\delta} \|x - y\|.
$$

(c) Let $x, y \in C$ and fixed a number $\tau \in (0,1)$. Assume that $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$. Since $I - \mathcal{F}$ is contraction with Lipschitz constant $L_{\lambda,\delta}$, we have

$$
\begin{aligned}
\|(I - \tau\mathcal{F})x - (I - \tau\mathcal{F})y\| &\leq \|x - y - \tau(\mathcal{F}x - \mathcal{F}y)\| \\
&= \|(1 - \tau)(x - y) + \tau[(I - \mathcal{F})x - (I - \mathcal{F})y]\| \\
&\leq (1 - \tau)\|x - y\| + \tau\|(I - \mathcal{F})x - (I - \mathcal{F})y\| \\
&\leq (1 - (1 - L_{\lambda,\delta})\tau)\|x - y\|.
\end{aligned}
$$

Therefore, $I - \tau\mathcal{F} : C \to C$ is a contraction mapping with Lipschitz constant $1 - (1 - L_{\lambda,\delta})\tau$. $\square$

Proposition 3.2. *Let C be a nonempty closed convex subset of a real smooth Banach space X. Let $T : C \to C$ be a continuous pseudocontractive mapping and let $\mathcal{F} : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. Assume that C has the fixed-point property for nonexpansive self-mappings. Then we have the following:*

(a) *For each $t \in (0,1)$, we choose a number $\mu_t \in (0,1)$ arbitrarily, there exists a unique point v_t of C defined by*

$$(3.1) \qquad\qquad v_t = (1-t)Tv_t + t(I - \mu_t \mathcal{F})v_t.$$

(b) *If $F[T] \neq \emptyset$ and v_t is a unique point of C defined by (3.1), then*
    (i) *$\{v_t\}$ is bounded,*
    (ii) *$\langle \mathcal{F}(v_t), J(v_t - v) \rangle \leq 0$ for all $v \in F[T]$.*

Proof. (a) For each $t \in (0,1)$, we choose a number $\mu_t \in (0,1)$ arbitrarily and then the mapping $G_t : C \to C$ defined by

$$G_t v = (1-t)Tv + t(I - \mu_t \mathcal{F})v \text{ for all } v \in C$$

is continuous strongly pseudocontractive with constant $1 - (1 - L_{\lambda,\delta})t\mu_t$. Indeed, for all $x, y \in C$, by Proposition 3.1 we have

$$
\begin{aligned}
\langle G_t x - G_t y, J(x-y) \rangle &= (1-t)\langle Tx - Ty, J(x-y) \rangle + t\langle (I - \mu_t F)x - (I - \mu_t \mathcal{F})y, J(x-y) \rangle \\
&\leq (1-t)\|x-y\|^2 + t\|(I - \mu_t \mathcal{F})x - (I - \mu_t \mathcal{F})y\|\|x-y\| \\
&= [1 - (1 - L_{\lambda,\delta})t\mu_t]\|x-y\|^2.
\end{aligned}
$$

By Fact 2.2, there exists a unique fixed point v_t of G_t in C defined by

$$v_t = (1-t)Tv_t + t(I - \mu_t \mathcal{F})v_t.$$

(b) Assume that $F[T] \neq \emptyset$. Take any $p \in F[T]$. Using (3.1), we have

$$
\begin{aligned}
\langle v_t - [(1-t)p &+ t(I - \mu_t \mathcal{F})v_t], J(v_t - p) \rangle \\
&= \langle (1-t)Tv_t + t(I - \mu_t \mathcal{F})v_t - [(1-t)p + t(I - \mu_t \mathcal{F})v_t], J(v_t - p) \rangle \\
&= (1-t)\langle Tv_t - p, J(v_t - p) \rangle \\
&\leq (1-t)\|v_t - p\|^2.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\langle v_t - [(1-t)p &+ t(I - \mu_t \mathcal{F})v_t], J(v_t - p) \rangle \\
&= \langle (1-t)(v_t - p) + t[v_t - (I - \mu_t \mathcal{F})v_t], J(v_t - p) \rangle \\
&= (1-t)\|v_t - p\|^2 + t\mu_t \langle \mathcal{F}(v_t), J(v_t - p) \rangle.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(1-t)\|v_t - p\|^2 &+ t\mu_t \langle \mathcal{F}(v_t), J(v_t - p) \rangle \\
&= \langle v_t - [(1-t)p + t(I - \mu_t \mathcal{F})v_t], J(v_t - p) \rangle \\
&\leq (1-t)\|v_t - p\|^2,
\end{aligned}
$$

which implies that

$$\langle \mathcal{F}(v_t), J(v_t - p) \rangle \leq 0.$$

Since $\mathcal{F}$ is δ-strongly accretive, we have

$$
\begin{aligned}
\delta\|v_t - p\|^2 &\leq \langle \mathcal{F}(v_t) - \mathcal{F}(p), J(v_t - p) \rangle \\
&= \langle \mathcal{F}(v_t), J(v_t - p) \rangle - \langle \mathcal{F}(p), J(v_t - p) \rangle \\
(3.2) \qquad &\leq \langle -\mathcal{F}(p), J(v_t - p) \rangle \\
&\leq \|\mathcal{F}(p)\|\|v_t - p\|,
\end{aligned}
$$

which implies that

$$\delta\|v_t - p\| \leq \|\mathcal{F}(p)\|.$$

It shows that $\{v_t\}$ is bounded. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we are ready to present main result of this section.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real reflexive Banach space X with a uniformly Gâteaux differentiable norm. Let $T : C \to C$ be a continuous pseudocontractive mapping with $F[T] \neq \emptyset$ and let $\mathcal{F} : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. Assume that C has the fixed-point property for nonexpansive self-mappings. Then $\{v_t\}$ converges strongly as $t \to 0^+$ to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $F[T]$.*

Proof. By Proposition 3.2, $\{v_t : t \in (0,1)\}$ is bounded. Since $\mathcal{F}$ is a Lipschitzian mapping, it follows that $\{\mathcal{F}v_t : t \in (0,1)\}$ is bounded. From (3.1), we have

$$(3.3) \qquad\qquad Tv_t = v_t + \frac{t\mu_t}{1-t}\mathcal{F}(v_t) \text{ for all } t \in (0,1).$$

and hence

$$\|Tv_t\| \leq \|v_t\| + \frac{t\mu_t}{1-t}\|\mathcal{F}(v_t)\| \leq \|v_t\| + \frac{t}{1-t}\|\mathcal{F}(v_t)\| \text{ for all } t \in (0,1).$$

Noticing that $\lim_{t\to 0^+}\frac{t}{1-t} = 0$, there exists $t_0 \in (0,1)$ that $\{Tv_t : t \in (0, t_0]\}$ is bounded. This implies from (3.3) that $\|v_t - Tv_t\| \to 0$ as $t \to 0^+$. The key is to show that $\{v_t : t \in (0, t_0]\}$ is relatively compact as $t \to 0^+$. We may choose a sequence $\{t_n\}$ in $(0, t_0]$ such that $\lim_{n\to\infty} t_n = 0$. Set $v_n := v_{t_n}$. We will show that $\{v_n\}$ contains a subsequence converging strongly to an element of C. Define the function $\varphi : C \to \mathbb{R}^+$ by $\varphi(x) := LIM_n\|v_n - x\|^2$, $x \in C$ and let

$$(3.4) \qquad\qquad M := \{y \in C : \varphi(y) = \inf_{x \in C}\varphi(x)\}.$$

Since X is reflexive, $\varphi(x) \to \infty$ as $\|x\| \to \infty$, and φ is continuous convex function. By Barbu and Precupanu [2, Theorem 1.2, p. 79], we have that the set M is nonempty. By Takahashi [28], we see that M is also closed, convex and bounded.

From [18, Theorem 6] we know that the mapping $2I - T$ has a nonexpansive inverse, denoted by g, which maps C into itself with $F[T] = F[g]$. Note $\lim_{n\to\infty}\|v_n - Tv_n\| = 0$ implies that $\lim_{n\to\infty}\|v_n - gv_n\| = 0$. Moreover, M is invariant under g, i.e., $R(g) \subseteq M$. In fact, for each $y \in M$, we have

$$\varphi(gy) = LIM_n\|v_n - gy\|^2 \leq LIM_n\|gv_n - gy\|^2 \leq LIM_n\|v_n - y\|^2 = \varphi(y)$$

and hence $gy \in M$. By assumption, we have $M \cap F[g] \neq \emptyset$. Let $y^* \in M \cap F[g]$. By Fact 2.9, we have

$$LIM_n\langle z - y^*, J(v_n - y^*)\rangle \leq 0 \text{ for all } z \in C.$$

In particular, by taking $z = y^* - \mathcal{F}(y^*)$, we have

$$(3.5) \qquad\qquad LIM_n\langle -\mathcal{F}(y^*), J(v_n - y^*)\rangle \leq 0.$$

Using (3.2) and (3.5), we have

$$\delta LIM_n\|v_n - y^*\|^2 \leq LIM_n\langle -\mathcal{F}(y^*), J(v_n - y^*)\rangle \leq 0.$$

Thus, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} \to y^*$.

Assume that $\{v_{n_j}\}$ is another subsequence of $\{v_n\}$ such that $v_{n_j} \to z^* \neq y^*$. It is easy to see that $z^* \in F[T]$. Since $v_{n_i} \to y^*$ and J is norm to weak* uniform continuous, we obtain from Proposition 3.2(b) that

$$\langle \mathcal{F}(y^*), J(y^* - z^*)\rangle \leq 0.$$

Similarly, we have
$$\langle \mathcal{F}(z^*), J(z^* - y^*)\rangle \leq 0.$$
Adding above two inequalities yield
$$\langle \mathcal{F}(y^*) - \mathcal{F}(z^*), J(y^* - z^*)\rangle \leq 0,$$
which implies that
$$\delta \|y^* - z^*\|^2 \leq \langle \mathcal{F}(y^*) - \mathcal{F}(z^*), J(y^* - z^*)\rangle \leq 0,$$
a contradiction. Hence $\{v_{t_n}\}$ converges strongly to y^*.

To see that the entire net $\{v_t\}$ actually converges strongly as $t \to 0^+$, we assume that there is another sequence $\{s_n\}$ with $s_n \in (0, t_0]$ and $s_n \to 0$ as $n \to \infty$ such that $v_{s_n} \to z$ as $n \to \infty$. Then $z \in F[T]$. From Proposition 3.2(b), we conclude that $z = y^*$. Therefore, $\{v_t\}$ converges strongly as $t \to 0^+$ to $y^* \in F[T]$. Noticing that $y^* \in F[T]$ is a solution of $VIP(\mathcal{F}, C)$ over $F[T]$. Indeed, from Proposition 3.2(b), we have
$$\langle \mathcal{F}(y^*), J(y^* - v)\rangle \leq 0 \text{ for all } v \in F[T].$$
One can easily see that y^* is the unique solution of $VIP(\mathcal{F}, C)$ over $F[T]$. $\square$

As the domain of operators considered in Theorem 3.3 is not necessarily the entire space X, therefore, Theorem 3.3 is more general in nature. It improves Ceng, Anasri and Yao [5, Proposition 4.3] significantly and provides solutions of Problem 1.1.

We now replace the fixed point property assumption, mentioned in Theorem 3.3 by imposing strict convexity on the underlying space.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real strictly convex reflexive Banach space X with a uniformly Gâteaux differentiable norm. Let $T : C \to C$ be a continuous pseudocontractive mapping with $F[T] \neq \emptyset$ and let $\mathcal{F} : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0, 1)$. Then $\{v_t\}$ converges strongly as $t \to 0^+$ to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $F[T]$.*

Proof. To be able to use the argument of the proof of Theorem 3.3, we just need to show that the set M defined by (3.4) has a fixed point of g. Since $F[g] \neq \emptyset$, let $v \in F[g]$. Since X is strictly convex, it follows from [1, Proposition 2.1.10] that the set M_0 defined by $M_0 = \{u \in M : \|u - v\| = \inf_{x \in M} \|x - v\|\}$ is a singleton. Let $M_0 = \{u_0\}$ for some $u_0 \in M$. Observe that
$$\|gu_0 - v\| = \|gu_0 - gv\| \leq \|u_0 - v\| = \inf_{x \in M} \|x - v\|.$$
Therefore, $gu_0 = u_0$. $\square$

4. Generalized hybrid steepest-descent algorithm

Motivated by Yamada's hybrid steepest-descent and Lehdili and Moudafi's algorithms (1.4) and (1.7), we introduce the following generalized hybrid steepest-descent algorithm for computing to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $\cap_{n \in \mathbb{N}} F[T_n]$:

Algorithm 4.1. *Let C be a nonempty closed convex subset of a real smooth Banach space X and $\mathcal{F} : X \to X$ an accretive operator over C such that $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0, 1)$. Assume that $\{T_n\}$ is a sequence of nearly nonexpansive mappings form C into itself with sequence $\{a_n\}$ such that $\cap_{n \in \mathbb{N}} F[T_n] \neq \emptyset$. Starting with an arbitrary initial guess $x_1 \in C$, a sequence $\{x_n\}$ in C is generated via the following iterative scheme:*

(4.1) $$x_{n+1} = T_n[x_n - \alpha_n \mathcal{F}(x_n)] \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$.

We shall study our Algorithm 4.1 under the conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and either $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty} \left|1 - \frac{\alpha_n}{\alpha_{n+1}}\right| = 0$,

(C2) either $\sum_{n=1}^{\infty} \mathscr{D}_D(T_n, T_{n+1}) < \infty$ or $\lim_{n\to\infty} \frac{\mathscr{D}_D(T_n, T_{n+1})}{\alpha_{n+1}} = 0$ for each $D \in \mathscr{B}(C)$,

(C3) $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$.

Now we are ready to prove the main theorem for computing solution of $VIP(\mathcal{F}, C)$ over $\cap_{n\in\mathbb{N}} F[T_n]$ in the framework of Banach space.

Theorem 4.2. *Let C be a nonempty closed convex subset of a reflexive Banach space X with a uniformly Gâteaux differentiable norm and $\{T_n\}$ a sequence of nearly nonexpansive mappings from C into itself with sequence $\{a_n\}$ such that $\cap_{n\in\mathbb{N}} F[T_n] \neq \emptyset$. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and let $\mathcal{F} : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. Assume that C has the fixed-point property for nonexpansive self-mappings. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by (4.1), where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying conditions (C1)~(C3). Then, $\{x_n\}$ converges strongly to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $\cap_{n\in\mathbb{N}} F[T_n]$.*

Proof. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$. It is clear that T is a nonexpansive mapping and $\cap_{n\in\mathbb{N}} F[T_n] \subseteq F[T]$. So, we have $F[T] \neq \emptyset$. For each $t \in (0,1)$, we choose a number $\mu_t \in (0,1)$ arbitrarily, let x_t be a unique point of C such that

$$x_t = (1-t)Tx_t + t(I - \mu_t\mathcal{F})(x_t).$$

It follows from Theorem 3.3 that $\{x_t\}$ converges strongly as $t \to 0^+$ to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $\cap_{n\in\mathbb{N}} F[T_n]$. Set $y_n := x_n - \alpha_n\mathcal{F}(x_n)$. We now proceed with the following steps:

Step 1. *$\{x_n\}$ and $\{y_n\}$ are bounded.*

From (4.1), we have

$$
\begin{aligned}
\|y_n - x^*\| &\leq \|x_n - x^*\| + \alpha_n\|\mathcal{F}(x_n)\| \\
&\leq \|x_n - x^*\| + \|\mathcal{F}(x_n) - \mathcal{F}(x^*)\| + \|F(x^*)\| \\
&\leq \left(1 + \frac{1}{\lambda}\right)\|x_n - x^*\| + \|\mathcal{F}(x^*)\| \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$
(4.2)

Invoking (4.2), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|T_n[x_n - \alpha_n\mathcal{F}(x_n)] - x^*\| \\
&\leq \|x_n - \alpha_n\mathcal{F}(x_n) - x^*\| + a_n \\
&\leq \|(I - \alpha_n F)x_n - (I - \alpha_n\mathcal{F})x^*\| + \alpha_n\|\mathcal{F}(x^*)\| + a_n \\
&\leq (1 - (1 - L_{\lambda,\delta})\alpha_n)\|x_n - x^*\| + \alpha_n\|\mathcal{F}(x^*)\| + a_n.
\end{aligned}
$$
(4.3)
(4.4)

Note $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$, so there exists a constant $K > 0$ such that

$$\frac{\alpha_n\|\mathcal{F}(x^*)\| + a_n}{\alpha_n} \leq K \text{ for all } n \in \mathbb{N}.$$

By (4.3), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1 - (1 - L_{\lambda,\delta})\alpha_n)\|x_n - x^*\| + \alpha_n K \\
&\leq \max\left\{\|x_n - x^*\|, \frac{K}{1 - L_{\lambda,\delta}}\right\} \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$

Hence $\{x_n\}$ is bounded and hence, from (4.2), $\{y_n\}$ is bounded.

Step 2. $\|y_n - Ty_n\| \to 0$ as $n \to \infty$.

Note that the condition $\lim_{n\to\infty} \alpha_n = 0$ implies that $\|y_n - x_n\| = \alpha_n\|F(x_n)\| \to 0$ as $n \to \infty$. Observe that

$$
\begin{aligned}
\|y_n - y_{n-1}\| &= \|(I - \alpha_n F)x_n - (I - \alpha_n \mathcal{F})x_{n-1} + (I - \alpha_n \mathcal{F})x_{n-1} - (I - \alpha_{n-1}\mathcal{F})x_{n-1}\| \\
&\leq (1 - (1 - L_{\lambda,\delta})\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\mathcal{F}(x_{n-1})\| \\
&\leq (1 - (1 - L_{\lambda,\delta})\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K_1
\end{aligned}
$$

for some constant $K_1 > 0$. Set $B := \{y_n\}$. Then $B \in \mathcal{B}(C)$. It follows from (4.1) that

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|T_n y_n - T_{n-1}y_{n-1}\| \\
&\leq \|T_n y_n - T_n y_{n-1}\| + \|T_n y_{n-1} - T_{n-1}y_{n-1}\| \\
&\leq \|y_n - y_{n-1}\| + \mathscr{D}_B(T_n, T_{n-1}) + a_n \\
&\leq (1 - (1 - L_{\lambda,\delta})\alpha_n)\|x_n - x_{n-1}\| + \mathscr{D}_B(T_n, T_{n-1}) + |\alpha_n - \alpha_{n-1}|K_1 + a_n.
\end{aligned}
$$

By conditions (C1)$\sim$(C3) and Xu [31, Lemma 2.5] we obtain that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Hence

$$
\begin{aligned}
\|x_{n+1} - T_n x_n\| &\leq \|x_n - x_{n+1}\| + \|T_n y_n - T_n x_n\| \\
&\leq \|x_n - x_{n+1}\| + \|y_n - x_n\| + a_n \to 0 \text{ as } n \to \infty
\end{aligned}
$$

and

$$
\|x_n - T_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| \to 0 \text{ as } n \to \infty.
$$

Moreover,

$$
\begin{aligned}
\|y_n - T_n y_n\| &\leq \|y_n - x_n\| + \|x_n - T_n x_n\| + \|T_n x_n - T_n y_n\| \\
&\leq 2\|y_n - x_n\| + \|x_n - T_n x_n\| + a_n \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

The definition of T implies that

$$
\begin{aligned}
\|Ty_n - y_n\| &\leq \|Ty_n - T_n y_n\| + \|x_{n+1} - x_n\| + \|x_n - y_n\| \\
&\leq \mathscr{D}_B(T, T_n) + \|x_{n+1} - x_n\| + \|x_n - y_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Step 3. $\limsup_{n\to\infty} \langle \mathcal{F}(x^*), J(x^* - y_n)\rangle \leq 0$.

Since $x_t - y_n = (1 - t)(Tx_t - y_n) + t[(I - \mu_t\mathcal{F})(x_t) - y_n]$, we have

$$
\begin{aligned}
\|x_t - y_n\|^2 &= (1 - t)\langle Tx_t - y_n, J(x_t - y_n)\rangle + t\langle (I - \mu_t\mathcal{F})(x_t) - y_n, J(x_t - y_n)\rangle \\
&\leq (1 - t)\langle Tx_t - Ty_n + Ty_n - y_n, J(x_t - y_n)\rangle \\
&\quad + t[\langle (I - \mu_t\mathcal{F})(x_t) - x_t, J(x_t - y_n)\rangle + \|x_t - y_n\|^2] \\
&\leq \|x_t - y_n\|^2 + (1 - t)\langle Ty_n - y_n, J(x_t - y_n)\rangle - t\mu_t\langle \mathcal{F}(x_t), J(x_t - y_n)\rangle \\
&\leq \|x_t - y_n\|^2 + (1 - t)\|Ty_n - y_n\|\|x_t - y_n\| - t\mu_t\langle \mathcal{F}(x_t), J(x_t - y_n)\rangle,
\end{aligned}
$$

which implies that

$$
(4.5) \qquad \langle \mathcal{F}(x_t), J(x_t - y_n)\rangle \leq \frac{1 - t}{t\mu_t}\|Ty_n - y_n\|\|x_t - y_n\|.
$$

Since $\{x_t\}$ and $\{y_n\}$ are bounded and $\|y_n - Ty_n\| \to 0$ as $n \to \infty$, taking the superior limit in (4.5), we obtain that

$$
(4.6) \qquad \limsup_{n\to\infty}\langle \mathcal{F}(x_t), J(x_t - y_n)\rangle \leq 0.
$$

Further, since $x_t \to x^*$ as $t \to 0^+$, the set $\{x_t - y_n\}$ is bounded and the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X, it follows that

$$
\begin{aligned}
| \langle \mathcal{F}(x^*), J(y_n - x^*) \rangle &- \langle \mathcal{F}(x_t), J(y_n - x_t) \rangle | \\
&= | \langle \mathcal{F}(x^*), J(y_n - x^*) - J(y_n - x_t) \rangle + \langle \mathcal{F}(x^*) - \mathcal{F}(x_t), J(y_n - x_t) \rangle | \\
&\leq | \langle \mathcal{F}(x^*), J(y_n - x^*) - J(y_n - x_t) \rangle | \\
&\quad + \| \mathcal{F}(x^*) - \mathcal{F}(x_t) \| \, \|y_n - x_t\| \to 0 \text{ as } t \to 0^+.
\end{aligned}
$$

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$
\langle \mathcal{F}(x^*), J(x^* - y_n) \rangle < \langle \mathcal{F}(x_t), J(x_t - y_n) \rangle + \varepsilon \text{ for all } n \in \mathbb{N} \text{ and } t \in (0, \delta).
$$

Using (4.6), we get

$$
\begin{aligned}
\limsup_{n \to \infty} \langle \mathcal{F}(x^*), J(x^* - y_n) \rangle &\leq \limsup_{n \to \infty} \langle \mathcal{F}(x_t), J(x^* - y_n) \rangle + \varepsilon \\
&\leq \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, we obtain that $\limsup_{n \to \infty} \langle \mathcal{F}(x^*), J(x^* - y_n) \rangle \leq 0$.

Step 4. $\{x_n\}$ *converges strongly to* x^*.

Observe that

$$
\begin{aligned}
\|y_n - x^*\|^2 &= \langle (I - \alpha_n \mathcal{F})x_n - (I - \alpha_n \mathcal{F})x^* + (I - \alpha_n \mathcal{F})x^* - x^*, J(y_n - x^*) \rangle \\
&\leq (1 - (1 - L_{\lambda,\delta})\alpha_n)\|x_n - x^*\| \|y_n - x^*\| - \alpha_n \langle \mathcal{F}(x^*), J(y_n - x^*) \rangle \\
&\leq (1 - (1 - L_{\lambda,\delta})\alpha_n)[\|x_n - x^*\|^2 + \|y_n - x^*\|^2]/2 - \alpha_n \langle \mathcal{F}(x^*), J(y_n - x^*) \rangle.
\end{aligned}
$$

Hence

$$
\|y_n - x^*\|^2 \leq (1 - (1 - L_{\lambda,\delta})\alpha_n)\|x_n - x^*\|^2 - 2\alpha_n \langle \mathcal{F}(x^*), J(y_n - x^*) \rangle.
$$

From (4.1), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 = \|T_n y_n - x^*\|^2 &\leq (\|y_n - x^*\| + a_n)^2 \\
&\leq \|y_n - x^*\|^2 + K_2 a_n.
\end{aligned}
$$

for some $K_2 \geq 0$. Thus, we obtain

$$
\|x_{n+1} - x^*\|^2 \leq (1 - (1 - L_{\lambda,\delta})\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n \langle \mathcal{F}(x^*), J(x^* - y_n) \rangle + K_2 a_n
$$

for all $n \in \mathbb{N}$. Note $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \to \infty} \frac{a_n}{\alpha_n} = 0$ and $\limsup_{n \to \infty} \langle \mathcal{F}(x^*), J(x^* - y_n) \rangle \leq 0$. Therefore, we conclude from Xu [31, Lemma 2.5] that $\{x_n\}$ converges strongly to x^*. $\square$

Corollary 4.3. *Let C be a nonempty closed convex subset of a strictly convex reflexive Banach space X with a uniformly Gâteaux differentiable norm and $\{T_n\}$ a sequence of nonexpansive mappings from C into itself such that $\cap_{n \in \mathbb{N}} F[T_n] \neq \emptyset$. Let T be a mapping of C into itself defined by $Tz = \lim_{n \to \infty} T_n z$ for all $z \in C$ and let $\mathcal{F} : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$ and $R(I - \tau \mathcal{F}) \subseteq C$ for each $\tau \in (0, 1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by (4.1), where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying conditions (C1)$\sim$(C2). Then, $\{x_n\}$ converges strongly to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $\cap_{n \in \mathbb{N}} F[T_n]$.*

Corollary 4.4. *Let C be a nonempty closed convex subset of a real strictly convex reflexive Banach space X with a uniformly Gâteaux differentiable norm and T a nonexpansive mapping from C into itself such that $F[T] \neq \emptyset$. Let $\mathcal{F} : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$ and $R(I - \tau \mathcal{F}) \subseteq C$ for each $\tau \in (0, 1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$
(4.7) \qquad x_{n+1} = T[x_n - \alpha_n \mathcal{F}(x_n)] \text{ for all } n \in \mathbb{N},
$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying condition (C1). Then, $\{x_n\}$ converges strongly to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $F[T]$.

Remark 4.5. $a_n := 1/(n+1)^a$ (for all $n \in \mathbb{N}$ and $a \in (0, 1]$) satisfies the condition (C1).

Corollary 4.6. *Let C be a nonempty closed convex subset of a reflexive Banach space X with a uniformly Gâteaux differentiable norm and T a nonexpansive mapping from C into itself such that $F[T] \neq \emptyset$. Let $\mathcal{F} : C \to C$ be both κ-strongly pseudocontractive and λ-strictly pseudocontractive with $\lambda > \kappa$. Assume that C has the fixed-point property for nonexpansive self-mappings. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$(4.8) \qquad x_{n+1} = T[(1 - \alpha_n)x_n + \alpha_n \mathcal{F}(x_n)] \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying condition (C1). Then, $\{x_n\}$ converges strongly to a unique solution x^* of $VIP(I - \mathcal{F}, C)$ over $F[T]$.

Corollary 4.6 is an improvement upon Sahu [22, Theorem 5.6] in a Banach space without uniform convexity.

5. Applications

5.1. **Applications to the common fixed point problems for nonexpansive mappings.**

Theorem 5.1. *Let C be a nonempty closed convex subset of a strictly convex reflexive Banach space X with a uniformly Gâteaux differentiable norm. Let $\lambda_i > 0$ $(i = 1, 2, \cdots, N)$ such that $\sum_{i=1}^{N} \lambda_i = 1$ and let $T_1, T_2, \cdots, T_N : C \to C$ be nonexpansive mappings with $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\mathcal{F} : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0, 1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = \sum_{i=1}^{N} \lambda_i T_i[x_n - \alpha_n \mathcal{F}(x_n)] \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying condition (C1). Then, $\{x_n\}$ converges strongly to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $\cap_{i=1}^{N} F(T_i)$.

Proof. Define $T = \sum_{i=1}^{N} \lambda_i T_i$. Then T is nonexpansive from C into itself and hence, from Fact 2.7, we have $F(T) = \cap_{i=1}^{N} F(T_i)$. Therefore, Theorem 5.1 follows form Corollary 4.4. $\qquad \square$

Theorem 5.2. *Let C be a nonempty closed convex subset of a strictly convex reflexive Banach space X with a uniformly Gâteaux differentiable norm. Let $\{S_n\}$ be a sequence of nonexpansive mappings form C into itself such that $\cap_{n \in \mathbb{N}} F[S_n] \neq \emptyset$ and let $\mathcal{F} : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0, 1)$. Let $\{\beta_{n,k}\}$ be a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that*

(i) $\sum_{k=1}^{n} \beta_{n,k} = 1$ *for each $n \in \mathbb{N}$;*
(ii) $\lim_{n \to \infty} \beta_{n,k} > 0$ *for each $k \in \mathbb{N}$;*
(iii) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1,k} - \beta_{n,k}| < \infty$.

For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by

$$x_{n+1} = \sum_{k=1}^{n} \beta_{n,k} S_k[x_n - \alpha_n \mathcal{F}(x_n)] \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying conditions (C1)$\sim$(C2). Then, $\{x_n\}$ converges strongly to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $\cap_{n \in \mathbb{N}} F[S_n]$.

Proof. Define a sequence $\{T_n\}$ of mappings on C by $T_n x = \sum_{k=1}^n \beta_{n,k} S_k x$ for all $x \in C$ and $n \in \mathbb{N}$. It is easy to see, from the condition (i) and Fact 2.7, that each T_n is also a nonexpansive mapping from C into itself and $F[T_n] = \cap_{k=1}^n F[S_k]$. Note that $\cap_{k \in \mathbb{N}} F[S_k] \subseteq \cap_{n \in \mathbb{N}} F[T_n]$. Moreover, by (ii) we have that for every $k \in N$, there exists $n_0 \in \mathbb{N}$ such that $\beta_{n_0,k} > 0$. Thus, we have that $F[T_{n_0}] \subseteq F[S_k]$ for $k \in \mathbb{N}$ by Fact 2.8, which implies that $\cap_{n \in \mathbb{N}} F[T_n] \subseteq F[S_k]$ for all $k \in \mathbb{N}$. Therefore, we obtain that $\cap_{k \in \mathbb{N}} F[S_k] = \cap_{n \in \mathbb{N}} F[T_n] \neq \emptyset$. Now, let $B \in \mathscr{B}(C)$. The nonemptyness of $\cap_{k \in \mathbb{N}} F[S_k]$ implies that $\{S_k x : x \in B, k \in \mathbb{N}\}$ is bounded. By using argument of [26], we see that $Tz = \lim_{n \to \infty} T_n z$ for all $z \in C$. Hence Theorem 5.2 follows from Corollary 4.3. $\qquad \square$

5.2. **Applications to the zero point problems for accretive operators.** Consider C is a closed convex subset of a Banach space X and $A \subset X \times X$ is an accretive operator such that $\mathbb{S} \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. From Takahashi [28], we know that J_r^A is a nonexpansive mapping of C into itself and $F[J_r^A] = \mathbb{S}$ for each $r > 0$.

Motivated and inspired by two well-known methods, Yamada's hybrid steepest-descent method and Lehdili and Moudafi's method, we introduce the following algorithm which we call *prox-Tikhonov regularized hybrid steepest-descent algorithm*:

Algorithm 5.3. *For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$(5.1) \qquad x_{n+1} = J_{r_n}^A[x_n - \alpha_n \mathcal{F}(x_n)] \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ and $\{r_n\}$ is a regularization sequence in $(0, \infty)$.

One can easily see that the prox-Tikhonov regularized hybrid steepest-descent algorithm is an special case of generalized hybrid steepest-descent algorithm.

The following theorem gives sufficient conditions for strong convergence of the prox-Tikhonov regularized hybrid steepest-descent algorithm (5.1) to a solution of Problem (P).

Theorem 5.4. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm and C a nonempty closed convex subset of X which has the fixed-point property for nonexpansive self-mappings. Let $A \subset X \times X$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Let $\mathcal{F} : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive over C with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. For given $x_1 \in C$, let $\{x_n\}$ be a prox-Tikhonov regularized hybrid steepest-descent iterative sequence in C generated by (5.1), where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying condition (C1) and $\{r_n\}$ is a regularization sequence in $(0, \infty)$ such that $\inf_{n \in \mathbb{N}} r_n > 0$ and $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges strongly to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $A^{-1}0$.*

Proof. Set $T_n := J_{r_n}^A$. Then $\{T_n\}$ is a sequence of nonexpansive mappings from C into itself such that $F[T_n] = A^{-1}0 \neq \emptyset$ for every $n \in \mathbb{N}$. We first verify that $\sum_{n=1}^\infty \mathscr{D}_B(T_n, T_{n+1}) < \infty$ for every $B \in \mathscr{B}(C)$. Let $B \in \mathscr{B}(C)$. Since $F(T_n) = A^{-1}0 \neq \emptyset$ for every $n \in \mathbb{N}$, it follows that $\{T_n z : z \in B, n \in \mathbb{N}\}$ is bounded. Set $K_3 := \sup\{\|z - J_{r_{n+1}} z\| : z \in B, n \in \mathbb{N}\}$. By the assumptions for $\{r_n\}$, we may assume that $r_n \geq \varepsilon$ for all $n \in \mathbb{N}$ and $r_n \to r$ for some $r, \varepsilon > 0$. From (2.5), we have

$$
\begin{aligned}
\mathscr{D}_B(T_{n+1}, T_n) &= \sup\{\|J_{r_{n+1}} z - J_{r_n} z\| : z \in B\} \\
&\leq \sup\left\{ \frac{|r_{n+1} - r_n|}{r_{n+1}} \|z - J_{r_{n+1}} z\| : z \in B \right\} \\
&\leq \frac{|r_{n+1} - r_n|}{\varepsilon} K_3 \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$

Hence $\sum_{n=1}^{\infty} \mathscr{D}_B(T_n, T_{n+1}) < \infty$. Set $T := J_r$. Again, from (2.5), we have

$$\|Tx - T_n x\| \leq \frac{|r - r_n|}{r} \|x - Tx\| \text{ for all } x \in C,$$

which indicates that $Tx = \lim_{n \to \infty} T_n x$ for all $x \in C$. Therefore, by Theorem 4.2, $\{x_n\}$ converges strongly to a unique solution x^* of $VIP(\mathcal{F}, C)$ over $A^{-1}0$. $\qquad\square$

Corollary 5.5. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm and C a nonempty closed convex subset of X which has the fixed-point property for nonexpansive self-mappings. Let $A \subset X \times X$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Let $\mathcal{F} : C \to C$ be both κ-strongly pseudocontractive and λ-strictly pseudocontractive with $\lambda > \kappa$. For given $x_1 \in C$, let $\{x_n\}$ be a prox-Tikhonov regularized hybrid steepest-descent iterative sequence in C generated by*

$$(5.2) \qquad x_{n+1} = J_{r_n}^A[(1 - \alpha_n)x_n + \alpha_n \mathcal{F}(x_n)] \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying condition (C1) and $\{r_n\}$ is a regularization sequence in $(0, \infty)$ such that $\inf_{n \in \mathbb{N}} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges strongly to a unique solution x^* of $VIP(I - \mathcal{F}, C)$ over $A^{-1}0$.

6. Numerical results

In order to demonstrate the effectiveness, performance and convergence of the proposed algorithm, we discuss the following:

Example 6.1. Let $\mathscr{H} = \mathbb{R}$ and $C = [0, 1]$. Let $T, \mathcal{F} : C \to \mathscr{H}$ be two mappings defined by $Tx = 1 - x$ for all $x \in C$ and $\mathcal{F}x = x - 1$ for all $x \in C$. Clearly that T is nonexpansive and $\mathcal{F}$ is both 1-strongly accretive and λ-strictly pseudocontractive over C for each $\lambda > 0$. For each $\tau \in (0, 1)$, we have $(I - \tau\mathcal{F})x = x - \tau(x - 1) = (1 - \tau)x + \tau$ for all $x \in C$. Define $\{\alpha_n\}$ in $[0, 1]$ by $\alpha_n = n/(n+1)^a$ for all $n \in \mathbb{N}$, where $a \in (0, 1]$. Then the sequence $\{x_n\}$ defined by (4.7) is given by the relation

$$(6.1) \qquad x_{n+1} = (1 - \alpha_n)(1 - x_n) \text{ for all } n \in \mathbb{N}.$$

Choose $x_1 = 0 \in C$. For $a = 1$, the sequence $\{x_n\}$ defined by (6.1) can be explicitly written as

$$x_n = \begin{cases} \frac{1}{2}, & n = 2, 4, \cdots; \\ \frac{n-1}{2n}, & n = 3, 5 \cdots \end{cases}$$

and it converges to $\frac{1}{2} \in F[T]$. The corresponding iterations of sequence $\{x_n\}$ defined by (6.1) for $a = 1$, $a = 0.75$, and $a = 0.5$ are graphically represented in the following figure.

7. Concluding remarks

(I) Theorem 3.3, Theorem 4.2 and Theorem 5.4 apply to all uniformly convex and uniformly smooth Banach spaces and in particular, to all L_p spaces, $1 < p < \infty$.

(II) Theorem 4.2 appears to be a new result for solving Problem 1.1. In particular, Theorem 4.2 improves Yamada [33] in the framework of Banach space.

(III) The prox-Tikhonov algorithm (1.5) studied in Lehdili and Moudafi [15] deals essentially with a special case of the algorithm (5.2) in the framework of Hilbert space. In fact, if we set $\mathcal{F}x = 0$ for all $x \in \mathscr{H}$, then (5.2) becomes

$$(7.1) \qquad x_{n+1} = J_{c_n}^A((1 - \alpha_n)x_n) \text{ for all } n \in \mathbb{N}.$$

Setting $\lambda_n := \frac{c_n}{1-\alpha_n}$ and $\mu_n := \frac{\alpha_n}{c_n}$, (7.1) can be written as

$$(7.2) \qquad\qquad x_{n+1} = J_{\lambda_n}^{A_n} x_n \text{ for all } n \in \mathbb{N},$$

where $A_n = \mu_n I + A$. Thus, (7.2) is the prox-Tikhonov algorithm (1.5) considered in Lehdili and Moudafi [15]. The argument given in [15] depends heavily on the concept of the variational distance (see [29]) between two maximal monotone operators. Our argument is simple and more straightforward in the Banach space setting. Therefore, Corollary 5.5 improves and extends the convergence result presented in Lehdili and Moudafi [15] in the Banach space setting.

(IV) Our approach is simple and different from new iterative methods for finding solutions of problem 1.1 and zero of m-accretive operators proposed in Ceng, Anasri and Yao [5] and Ceng, Xu and Yao [6].

Acknowledgements

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