Some Modified Extragradient Methods For Common Solutions of Generalized Equilibrium Problems and Fixed Points of Nonexpansive mappings[1]

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Abstract

In this paper, we introduce some new iterative schemes based on the extragradient method (and the hybrid method) for finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a family of nonexpansive mappings, and the set of solutions of the variational inequality for a monotone, Lipschitz continuous mapping in Hilbert spaces. We obtain some strong convergence theorems and weak convergence theorems. The results in this paper generalize, improve and unify some well-known convergence theorems in the literature.

Key words. Generalized equilibrium problem; Extragradient method; Hybrid method; Nonexpansive mapping; Strong convergence; Weak convergence.

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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let F be a bifunction from $C \times C$ into the real line $\mathbb{R}$ and let $B : C \to H$ be a nonlinear mapping. Moudafi [5], Moudafi and Thera [6], Peng and Yao [11–13], and Takahashi and Takahashi [18] considered the following *generalized equilibrium problem*:

$$\text{Find } x \in C \text{ such that } F(x,y) + \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GEP(C, F, B)$.

If $B = 0$, the generalized equilibrium problem (1.1) reduces to the so-called *equilibrium problem*. If $F = 0$, then (1.1) becomes the *variational inequality problem*, i.e., to find $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \quad \forall y \in C.$$

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The set of solutions of the variational inequality problem is denoted by $VI(C, B)$.

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games; see for instance, [1, 5, 6, 11–13, 18] and the references therein.

Recall that a mapping $S : C \to H$ is *nonexpansive* if there holds that

$$\|Sx - Sy\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote the set of fixed points of S by Fix(S).

Several algorithms have been proposed for finding the solution of problem (1.1). Moudafi [5] introduced an iterative scheme for finding a solution of problem (1.1), which is also a fixed point of a nonexpansive mapping, and proved a weak convergence theorem. Moudafi and Thera [6] introduced an auxiliary scheme for finding a solution of problem (1.1) and obtained a weak convergence theorem. Peng and Yao [11–13] introduced some iterative schemes for finding a common solution of problem (1.1) and the variational inequality for a monotone, Lipschitz-continuous mapping, which is also a fixed point of a family of nonexpansive mappings. Takahashi and Takahashi [18] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space, and proved a strong convergence theorem. Some methods also have been proposed to solve the equilibrium problem when $B = 0$ in (1.1) ; see, for instance, [2–4, 10, 14–17, 20] and the references therein.

Recently Nakajo, Shimoji and Takahashi [8], and Takahashi, Takeuchi and Kubota [19] introduced and studied some iterative methods for finding a common fixed point of a family of nonexpansive mappings satisfying the so-called NST-condition (I), and obtained some strong convergence theorems in a Banach space or a Hilbert space.

Inspired by the ideas in the [2–6, 8, 10–20] and the references therein, we introduce some new iterative schemes based on the extragradient method (and the hybrid method) for finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a family of nonexpansive mappings and the set of solutions of the variational inequality for a monotone, Lipschitz-continuous mapping. We obtain both strong convergence theorems and weak convergence theorems for the sequences generated by the corresponding processes. The results in this paper generalize, improve and unify some well-known convergence theorems in the literatures.

2   Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \text{for all } y \in C.$$

The mapping P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping from H onto C. Moreover,

$$z = P_C(x) \quad \text{if and only if} \quad \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

A mapping $A : C \longrightarrow H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \text{for all } x, y \in C;$$

A is called α-*inverse strongly monotone* if $\alpha > 0$ and

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \text{for all } x, y \in C;$$

A is called k-*Lipschitz continuous* if $k > 0$ and

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \text{for all } x, y \in C.$$

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is *monotone*, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, the scalar function $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Motivated by Nakajo, Shimoji and Takahashi [8] and Takahashi, Takeuchi and Kubota [19], we give the following definitions: Let $\{S_n\}$ and Γ be two families of nonexpansive mappings of C into itself such that $\emptyset \neq \text{Fix}(\Gamma) = \cap_{n=i}^{\infty}\text{Fix}(S_n)$, where $\text{Fix}(\Gamma)$ is the set of all common fixed points of mappings in Γ. Then, $\{S_n\}$ is said to *satisfy the NST-condition (I) with* Γ if for each bounded sequence $\{t_n\} \subseteq C$,

$$\lim_{n \to \infty} \|t_n - S_n t_n\| = 0 \quad \text{implies that} \quad \lim_{n \to \infty} \|t_n - T t_n\| = 0 \text{ for all } T \in \Gamma.$$

In particular, if $\Gamma = \{T\}$, i.e., Γ consists of exactly one mapping T, then $\{S_n\}$ is said to satisfy the NST-condition (I) with T.

3 Main results

We now present the strong convergence of an iterative algorithm based on extragradient method and hybrid method which solves the problem of finding a common element of the set of solutions of a generalized equilibrium problem, the fixed point set of a family of nonexpansive mappings and the set of solutions of the variational inequality for a monotone, Lipschitz continuous mapping in a Hilbert space.

In the following, we always assume that C is a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4), let A be a monotone

and k-Lipschitz continuous mapping of C into H, and let B be an α-inverse strongly monotone mapping of C into H. Let $\{S_n\}$ and Γ be families of nonexpansive mappings of C into itself such that

$$\Omega = \cap_{i=1}^{\infty} \text{Fix}(S_i) \cap VI(C, A) \cap GEP(C, F, B) \neq \emptyset$$

and $\cap_{i=1}^{\infty} \text{Fix}(S_i) = \text{Fix}(\Gamma)$. Assume also that $\{S_n\}$ satisfies the NST-condition (I) with Γ.

Theorem 3.1 *Suppose $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [0, c]$ for some $c \in (0, 1)$, and $\{r_n\} \subset [\gamma, e]$ for some $\gamma, e \in (0, 2\alpha)$. Pick any $x_1 = x \in C$ and set $C_1 = C$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the scheme*

$$\begin{cases} F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ y_n = P_C(u_n - \lambda_n Au_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S_n P_C(u_n - \lambda_n Ay_n), \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x \end{cases} \tag{3.1}$$

for every $n = 1, 2, \ldots$. Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $w = P_\Omega(x)$.

Proof. First we note that under assumptions (A1)-(A4), it is held that for any $r > 0$ and $x \in H$ there is a unique q in C such that

$$F(u, y) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \quad \forall y \in C$$

(see, e.g., [1, 18]). In particular, if we put $r = r_n$ and $x = x_n - r_n Bx_n$ then we can solve for u_n.

It is obvious that C_n is closed for every $n = 1, 2, \ldots$. Since

$$C_{n+1} = \{z \in C_n : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z \rangle \leq 0\},$$

we also have that C_n is convex for every $n = 1, 2, \ldots$.

Next, we show by induction that $\Omega \subseteq C_i$ for $i = 1, 2, \ldots$. From $C_1 = C$, we have $\Omega \subseteq C_1$. Suppose that $\Omega \subseteq C_n$ for some positive integer number n. Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2 in [17]. As $u \in GEP(C, F, B)$, we have $u = T_{r_n}(u - r_n Bu)$, and as $u \in VI(C, A)$, we have $u = P_C(u - \lambda_n Au)$. Putting $t_n = P_C(u_n - \lambda_n Ay_n)$ for every $n = 1, 2, \ldots$. From $u_n = T_{r_n}(x_n - r_n Bx_n) \in C$ and the proof of Theorem 3.1 in [13], we have

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2 \leq \|x_n - u\|^2, \tag{3.2}$$

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n \langle Bx_n - Bu, x_n - u_n \rangle - r_n^2 \|Bx_n - Bu\|^2, \tag{3.3}$$

and

$$\begin{aligned} \|t_n - u\|^2 &\leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k \|u_n - y_n\| \|t_n - y_n\| \\ &\leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2 k^2 \|u_n - y_n\|^2 + \|t_n - y_n\|^2 \\ &= \|u_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2. \end{aligned} \tag{3.4}$$

Therefore from (3.2), (3.4), $z_n = \alpha_n x_n + (1 - \alpha_n) S_n t_n$ and $u = S_n u$, we have

$$\|z_n - u\|^2 \leq \alpha_n \|x_n - u\|^2 + (1 - \alpha_n)\|S_n t_n - u\|^2$$

$$\leq \alpha_n \|x_n - u\|^2 + (1 - \alpha_n)\|t_n - u\|^2$$
$$\leq \alpha_n \|x_n - u\|^2 + (1 - \alpha_n)[\|u_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2]$$
$$\leq \|x_n - u\|^2 + (1 - \alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2$$
$$\leq \|x_n - u\|^2, \tag{3.5}$$

for every $n = 1, 2, \ldots$, and hence $u \in C_{n+1}$. So, $\Omega \subset C_i$ for every $i = 1, 2, \ldots$.

Let $l_0 = P_\Omega x$. From $x_n = P_{C_n} x$ and $l_0 \in \Omega \subset C_n$, we have

$$\|x_n - x\| \leq \|l_0 - x\| \tag{3.6}$$

for every $n = 1, 2, \ldots$. Therefore, $\{x_n\}$ is bounded. From (3.2), (3.4) and (3.5), we also obtain that $\{u_n\}$, $\{t_n\}$ and $\{z_n\}$ are bounded. Since $x_{n+1} \in C_{n+1} \subseteq C_n$ and $x_n = P_{C_n} x$, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\|$$

for every $n = 1, 2, \ldots$. Therefore, $\lim_{n \to \infty} \|x_n - x\|$ exists.

Since
$$\|x_{n+1} - x_n\|^2 = \|x_{n+1} - P_{C_n} x\|^2 \leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$
for every $n = 1, 2, \ldots$. This implies that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_{n+1}$, we have $\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$ and hence

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2\|x_n - x_{n+1}\|$$

for every $n = 1, 2, \ldots$. From $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$, we have $\|x_n - z_n\| \to 0$.

For $u \in \Omega$, from (3.5) we obtain

$$\|u_n - y_n\|^2 \leq \frac{1}{(1 - \alpha_n)(1 - \lambda_n^2 k^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|. \tag{3.7}$$

Since $\|x_n - z_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|u_n - y_n\| \to 0$. By the same process as in (3.4), we also have

$$\|t_n - u\|^2 \leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|t_n - y_n\|$$
$$\leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \|u_n - y_n\|^2 + \lambda_n^2 k^2 \|t_n - y_n\|^2$$
$$= \|u_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - t_n\|^2.$$

Then,
$$\|z_n - u\|^2 \leq \alpha_n \|x_n - u\|^2 + (1 - \alpha_n)\|t_n - u\|^2$$
$$\leq \alpha_n \|x_n - u\|^2 + (1 - \alpha_n)[\|u_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - t_n\|^2]$$
$$\leq \|x_n - u\|^2 + (1 - \alpha_n)(\lambda_n^2 k^2 - 1)\|y_n - t_n\|^2.$$

and, rearranging as in (3.7), we obtain

$$\|t_n - y_n\|^2 \leq \frac{1}{(1-\alpha_n)(1-\lambda_n{}^2 k^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.$$

Since $\|x_n - z_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|t_n - y_n\| \to 0$. As A is k-Lipschitz continuous, we have $\|Ay_n - At_n\| \to 0$. From $\|u_n - t_n\| \leq \|u_n - y_n\| + \|y_n - t_n\|$, we also have $\|u_n - t_n\| \to 0$.

¿From (3.5) and (3.2), we have

$$\|z_n - u\|^2 \leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|u_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2]$$

$$\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|u_n - u\|^2$$

$$\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|x_n - u\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2]$$

$$= \|x_n - u\|^2 + (1-\alpha_n)r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2.$$

Hence, we have

$$(1-c)\gamma(2\alpha - e)\|Bx_n - Bu\|^2$$

$$\leq (1-\alpha_n)r_n(2\alpha - r_n)\|Bx_n - Bu\|^2$$

$$\leq \|x_n - u\|^2 - \|z_n - u\|^2$$

$$\leq (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.$$

Since $\|x_n - z_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|Bx_n - Bu\| \to 0$.

Then, by (3.5), (3.4) and (3.3), we get

$$\|z_n - u\|^2 \leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2$$

$$\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|u_n - u\|^2$$

$$\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|x_n - u\|^2 - \|x_n - u_n\|^2$$

$$+ 2r_n\langle Bx_n - Bu, x_n - u_n\rangle - r_n^2\|Bx_n - Bu\|^2]$$

$$\leq \|x_n - u\|^2 - (1-\alpha_n)\|x_n - u_n\|^2 + (1-\alpha_n)2r_n\|Bx_n - Bu\|\|x_n - u_n\|.$$

Hence,

$$(1-c)\|x_n - u_n\|^2 \leq (1-\alpha_n)\|x_n - u_n\|^2$$

$$\leq \|x_n - u\|^2 - \|z_n - u\|^2 + (1-\alpha_n)2r_n\|Bx_n - Bu\|\|x_n - u_n\|$$

$$\leq (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\| + (1-\alpha_n)2r_n\|Bx_n - Bu\|\|x_n - u_n\|.$$

Since $\|x_n - z_n\| \to 0$, $\|Bx_n - Bu\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|x_n - u_n\| \to 0$. ¿From $\|z_n - t_n\| \leq \|z_n - x_n\| + \|x_n - u_n\| + \|u_n - t_n\|$ we have $\|z_n - t_n\| \to 0$. From $\|t_n - x_n\| \leq \|t_n - u_n\| + \|x_n - u_n\|$ we also have $\|t_n - x_n\| \to 0$.

Since $z_n = \alpha_n x_n + (1-\alpha_n)S_n t_n$, we have $(1-\alpha_n)(S_n t_n - t_n) = \alpha_n(t_n - x_n) + (z_n - t_n)$. Then

$$(1-c)\|S_n t_n - t_n\| \leq (1-\alpha_n)\|S_n t_n - t_n\| \leq \alpha_n\|t_n - x_n\| + \|z_n - t_n\|$$

and hence $\|S_n t_n - t_n\| \to 0$. Since $\{S_n\}$ satisfies the NST-condition (I) with Γ, we have for all $T \in \Gamma$,

$$\lim_{n\to\infty} \|Tt_n - t_n\| = 0.$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$, we obtain that $u_{n_i} \rightharpoonup w$. From $\|u_n - t_n\| \to 0$, we also obtain that $t_{n_i} \rightharpoonup w$. Since $\{u_{n_i}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

Now, we show that $w \in \mathrm{Fix}(\Gamma)$. Assume $w \notin \mathrm{Fix}(\Gamma)$. Since $t_{n_i} \rightharpoonup w$ and $w \neq Tw$ for some $T \in \Gamma$, from the Opial condition (see [9]) we have

$$\liminf_{i\to\infty} \|t_{n_i} - w\| < \liminf_{i\to\infty} \|t_{n_i} - Tw\|$$
$$\leq \liminf_{i\to\infty} \{\|t_{n_i} - Tt_{n_i}\| + \|Tt_{n_i} - Tw\|\}$$
$$\leq \liminf_{i\to\infty} \|t_{n_i} - w\|.$$

This is a contradiction. So, we get $w \in \mathrm{Fix}(\Gamma) = \cap_{i=1}^{\infty} \mathrm{Fix}(S_i)$. By exactly the same argument in the proof of Theorem 3.1 in [13] we can show $w \in GEP(C, F, B)$ and $w \in VI(C, A)$, which implies $w \in \Omega$.

¿From $l_0 = P_\Omega x$, $w \in \Omega$ and (3.5), we have

$$\|l_0 - x\| \leq \|w - x\| \leq \liminf_{i\to\infty} \|x_{n_i} - x\| \leq \limsup_{i\to\infty} \|x_{n_i} - x\| \leq \|l_0 - x\|.$$

So, we obtain $w = l_0$ and

$$\lim_{i\to\infty} \|x_{n_i} - x\| = \|w - x\|.$$

From $x_{n_i} - x \rightharpoonup w - x$ we have $x_{n_i} - x \to w - x$ and hence $x_{n_i} \to w$. This implies that $x_n \to l_0$. It is easy to see $u_n \to l_0$, $y_n \to l_0$ and $z_n \to l_0$. The proof is now complete. $\square$

Combining the arguments in the proof of Theorem 3.1 and those in the proof of Theorem 3.1 in [12] and Theorem 3.1 in [13], respectively, we can easily obtain the following weak convergence theorem and strong convergence theorem for the corresponding iterative algorithms based on the extragradient method (and CQ method).

Theorem 3.2 *Assume $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\beta_n\} \subset [\delta, \varepsilon]$ for some $\delta, \varepsilon \in (0, 1)$, and $\{r_n\} \subset [d, e]$ for some $d, e \in (0, 2\alpha)$. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by the scheme*

$$\begin{cases} x_1 = x \in C, \\ \quad F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n Au_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n P_C(u_n - \lambda_n Ay_n) \end{cases} \tag{3.8}$$

for every $n = 1, 2, \ldots$. Then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w \in \Omega$, where $w = \lim_{n\to\infty} P_\Omega x_n$.

Theorem 3.3 *Assume $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1]$ and $\{r_n\} \subset [\gamma, e]$ for some $\gamma, e \in (0, 2\alpha)$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the*

scheme

$$\begin{cases} x_1 = x \in C, \\ \quad F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S_n P_C(u_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases} \qquad (3.9)$$

for every $n = 1, 2, \ldots$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_\Omega(x)$.

Remark 3.4. (i) It follows from Lemmas 3.1-3.12 in [8] and Lemmas 2.1-2.4 in [19] that the NST-condition (I) with Γ of $\{S_n\}$ contains many special cases. Hence, we can easily obtain many interesting results by using Theorems 3.1-3.3. For examples, let $S_n = S$ for all $n = 1, 2, \ldots$ in Theorem 3.3 and 3.2, respectively, by Lemma 2.1 in [19], we get Theorem 4.4 in [11] without the condition (B4) or (B2), and Theorem 3.1 in [12]. Let $S_n = S$ for all $n = 1, 2, \ldots$ and $B = 0$ in Theorems 3.3 and 3.2, respectively, we recover Theorems 3.1 and 4.1 in [16]. Let S_n be replaced by the W-mapping W_n generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$ in Theorems 3.3 and 3.2, respectively, by Lemma 3.6 in [8], we recover Theorems 3.1 and 4.1 in [13]. Let $F(x, y) = 0$ for all $x, y \in C$, $B = 0$ and $S_n = S$ for all $n = 1, 2, \ldots$ in Theorem 3.3, by Lemma 2.1 in [19], we recover Theorem 3.1 in [7].

(ii) Let $F(x, y) = 0$ for all $x, y \in C$, $A = B = 0$ in Theorems 3.1 and 3.2, respectively, we recover Theorems 3.3 and 3.4 in [19].

(iii) Since the α-inverse strongly monotonicity of A has been weakened by the monotonicity and Lipschitz continuity of A, Theorems 3.1-3.3 extend, generalize and improve Theorem 3.1 in [8], Theorem 3.1 in [15], Theorem 3.1 in [14], Theorem 3.1 in [3], and Theorem 3 in [4].

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