Necessary conditions for multiobjective optimal control problems with state constraints *

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Abstract. Necessary conditions of optimality are derived for multiobjective optimal control problems with pathwise state constraints, in which the dynamics constrain is modeled as a differential inclusion. The obtained result extends results of [2] and [29].

1 Introduction

The derivation of necessary conditions for multiojective optimal control problems in which the dynamic constraint is modeled as a differential inclusion has been an area research recently. Problems of multiobjective optimal control naturally arise, for example, in economics (see [6]), in chemical engineering (see [3]) and in multiobjective control design (see [27]). Let us assume that $\prec$ is a preference in R^m. We are interested in deriving necessary conditions for the problem with state constraints

(P) Minimize $g(x(a), x(b))$
over arcs $x \in W^{1,1}([a,b], R^n)$ which satisfy
$\dot{x}(t) \in F(t, x(t))$, a.e.,
$(x(a), x(b)) \in C$,
$h(t, x(t)) \leq 0$ for all $t \in [a,b]$,

where $g : R^n \times R^n \to R^m$ and $h : [a,b] \times R^n \to R$ are given functions, $F : [a,b] \times R^n \rightrightarrows R^n$ is a given multifunction, C is a closed set in $R^n \times R^n$ and $W^{1,1}([a,b], R^n)$ is the space of

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absolutely continuous functions $x : [a, b] \to R^n$ with the norm $\|x\|_{1,1} := |x(a)| + \int_a^b |\dot{x}(t)|dt$, in which $|\cdot|$ denotes the norm in R^n.

An arc $x \in W^{1,1}([a, b], R^n)$ is called a feasible trajectory for (P) if it holds $\dot{x}(t) \in F(t, x(t))$ a.e.,$t \in [a, b]$, $(x(a), x(b)) \in C$ and $h(t, x(t) \leq 0$ for all $t \in [a, b]$. We say that a feasible trajectory $\overline{x}$ is a local solution of (P) if there do not exist any feasible trajectory x with $\|x - x_*\|_{1,1} \leq \epsilon$ such that $g(x(a), x(b)) \prec g(\overline{x}(a), \overline{x}(b))$ for some $\epsilon > 0$.

In the scalar case (m=1), there are several papers dealing with necessary conditions of the Euler-Lagrange type for (P). The generalized Euler-Lagrange condition was first established by Mordukhovich [16] for problems governed by nonconvex, compact-valued and Lipschitzian differential inclusions on the fixed time interval, where the notion $W^{1,1}$ local minimizer was studied under the name *intermediate local minimizers*, which are different from the classical notions of weak and strong local minimizer in variational and optimal control problems. This result was extended later by [15] to free-time problems. Further extensions for unbounded differential inclusions were given by Ioffe [9], Loewen and Rockafellar [11], Vinter and Zheng [26] for problems with unbounded differential inclusions on the fixed time interval and then by Vinter and Zheng [24] and Vinter [23] for free-time problems.

Recently, Zhu [29] had established a result on the Hamiltonian necessary conditions for a nonsmooth multiojective optimal control problem with endpoint constraints involving regular preferences. This result was extended by Bellaassali and Jourani [2]. Based on an analysis of Ioffe's scheme [9], as it was mentioned, [2] obtained a interesting result on necessary conditions for multiobjective optimal control problems. However, [2] and [29] considered only optimal problems, where state constraints are free.

The aim of this paper is to derive necessary conditions for (P) in the presence of state constraints. In order to obtain the necessary conditions for (P), we will use a variant of Ioffe's scheme to reduce the problem to the scalar case. We then apply the Ekeland principle and necessary conditions of the Bolza problem in the same way as in [2] and [25] to derive necessary conditions for (P). Our obtained result extends results of [2] and [29] in the vector case, and some preceding result for the scalar case.

The rest of the paper contains two sections. In Section 2 we present some notions and auxiliary results involving our problem. Section 3 is devoted to the main theorem where a detailed proof is provided.

2 Preliminaries and auxiliary results

Throughout this paper R_∞ stands for $R \cup \{+\infty\}$ and B stands for the closed unit ball in R^n.

Let $\Gamma : R^n \rightrightarrows R^n$ be a set-valued mapping. The notation

$$\limsup_{x \to \overline{x}} \Gamma(x) := \{x^* \in R^n : \exists x_k \to \overline{x}, x_k^* \to x^* \text{with } x_k^* \in \Gamma(x_k)\}$$

signifies the sequential Painlevé-Kuratowski upper limit of Γ at a point $\overline{x} \in R^n$. The set

$$\mathrm{Gph}\Gamma := \{(x, y) \in R^n \times R^n : y \in \Gamma(x)\}$$

is called the graph of Γ.

Take a closed set $A \subset R^n$ and point $x \in A$. The set

$$\hat{N}_A(x) := \{x^* \in R^n : \limsup_{u \xrightarrow{A} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq 0\}$$

is called the Fréchet normal cone to A at x. Let $\overline{x} \in A$, the set

$$N_A(\overline{x}) := \limsup_{x \to \overline{x}} \hat{N}_A(x)$$

is the limiting normal cone to A at $\overline{x}$.

Given a lower semicontinuos function $f : R^n \to R_\infty$ and a point $x \in R^n$ such that $f(x) < \infty$, the limiting subdifferential of f at x is the set

$$\partial f(x) = \{x^* : (x^*, -1) \in N_{epif}(x, f(x))\}.$$

It is well know that if f is Lipschitz continuous around x with rank K then for any $x^* \in \partial f(x)$, one has $\|x^*\| \leq K$. The limiting normal cone and limiting subdifferential were introduced by Mordukhovich [19]. We refer the reader to Chapter 1 in [13] for comprehensive comenmentaries. Further properties of limiting normal cone and limiting subdifferential can be founded in [13] and [4].

We now assume that Γ has closed values and define the function $\rho_\Gamma : R^n \times R^n \to R$ by

$$\rho_\Gamma(x, y) = d(y, \Gamma(x)) := \inf_{v \in \Gamma(x)} \|y - v\|.$$

The following property of the subdifferential of ρ_F was first established by [22], will be needed in the next section.

Lemma 2.1 *Assume that* GphF *is closed and* $(\overline{x}, \overline{y}) \in \mathrm{Gph}\Gamma$. *Then one has*

$$N_{\mathrm{Gph}\Gamma}(\overline{x}, \overline{y}) = \bigcup_{\lambda \geq 0} \lambda \partial \rho_\Gamma(\overline{x}, \overline{y}).$$

Moreover, if $\rho_\Gamma(x, y) > 0$ *and* $v \in \partial_y \rho_\Gamma(x, y)$ *then there exists a point* $z \in \Pi_{\Gamma(x)}(y)$ *such that* $v = \frac{y - z}{\|y - z\|}$. *Here* $\Pi_{\Gamma(x)}(y)$ *is the set of metric projections of* y *onto* $\Gamma(x)$.

The proof of Lemma 2.1 can also be found in [9], [13] and [25].

Let $C[a,b]$ be the space of continuous functions on $[a,b]$. A linear function μ on $C[a,b]$ is called a positive Radon measure if $\langle \mu, x\rangle \geq 0$ for all $x \in C[a,b]$ satisfying $x \geq 0$. The set of all positive Radon measures will be denoted by $C^{\oplus}[a,b]$. It is clear that from the Radon-Riesz theorem (see [8, Theorem 3.4]) we can identify a positive Radon measure μ with a Borel measure. Recall that a sequence $\{\mu_n\}$ of positive Radon measures converges weakly* to μ if $\int_a^b x(t)d\mu_n \to \int_a^b x(t)d\mu$ for all $x \in C[a,b]$ with $x \geq 0$. We have the following familiar property of positive Radon measures.

Lemma 2.2 *Let $\{\mu_n\}$ be a sequence of positive Radon measures. Assume that there exists a constant $M > 0$ such that $\|\mu_n\| \leq M$ for all n. Then there exists a subsequence $\{\mu_{n_k}\}$ which converges weakly* to a positive Radon measure μ.*

The rest of this section is destined for some notion of preferences in R^m.

The concept of a preference first appeared in the value theory of economics. In the area of multiobjective optimization and optimal control much research has been devoted to the weak Pareto solution and its generalizations. The preference relation between vectors $x, y \in R^m$ in the sense of weak Pareto is defined by $x \prec y$ if and only if $x_i \leq y_i$ for $i = 1,..,m$ and at least one of the inequalities is strict. In other word $x \prec y$ if and only if $x - y \in K := \{z \in R^m : z \leq 0\}$ and $x \neq y$. In this paper we will use more general preference relations for which necsseary conditions of the weak Pareto solution and its generalization can be derived and refined from our necessary conditions.

Let $\prec$ be a preference in R^m and $r \in R^m$. We will call the set $\mathcal{L}[r] := \{s \in R^m : s \prec r\}$ a level set at r and $\overline{\mathcal{L}}\ [r]$ is the closure of $\mathcal{L}[r]$.

We shall use the following definition (see [13, Dedinition 5.55] and [29]).

Definition 2.3 *A preference $\prec$ is closed provided that*
(a) for any $r \in R^n$, $r \in \overline{\mathcal{L}}\ [r]$;
(b) for any $r \prec s$, $t \in \overline{\mathcal{L}}[r]$ implies that $t \prec s$.

We say that $\prec$ is regular at $\overline{r}$ (in the sense of [29]) provided that
(c)

$$\limsup_{r,\theta\to\overline{r}} N_{\overline{\mathcal{L}}[r]}(\theta) \subset N_{\overline{\mathcal{L}}[\overline{r}]}(\overline{r}).$$

It is noted that the regularity notion for preference was introduced by [18] under the name of *normal semicontinuity* under which it is studied in Chapter 5 of [13]. In the above definition, the regularity is somewhat different from that in Definition 5.69 of [13], where a preference $\prec$ is regular at $(\overline{\theta}, \overline{r}) \in \mathrm{Gph}\mathcal{L}$ if

$$\limsup_{(r,\theta)\xrightarrow{\mathrm{Gph}\mathcal{L}}(\overline{\theta},\overline{r})} \hat{N}_{\mathcal{L}[r]}(\theta) = N_{\mathcal{L}[\overline{\theta}]}(\overline{r}).$$

Let us give some examples for Definition 2.3.

Example 2.4 (single objective problem). When $m = 1$ the relation $r \prec s$ becomes $r < s$. It is obvious that this relation satisfies conditions $(a)-(c)$. Therefore necessary conditions for (P) are true generalizations of necessary conditions for single objective optimal control (see Corollary 3.3).

Example 2.5 (weak Pareto optimal control problem). In a weak Pareto optimal control problem we define the preference by $r \prec s$ iff $r_i \leq s_i$, $i = 1, 2, ..., m$, and at least one of the inequalities is strict. It is easy to check that this $\prec$ satisfies (a) and (b) at any $r \in R^n$. Moreover, for any $r \in R^m$, $\mathcal{L}[r] = r + R^m_-$, where $R^n_- := \{s \in R^m : s_i \leq 0, i = 1, 2, ..., m\}$. It follows that $N_{\overline{\mathcal{L}[r]}}(\theta) \subset R^m_+ = N_{\overline{\mathcal{L}[r]}}(r)$ for all r and θ. Hence (c) also satisfied. Thus the necessary conditions for (P) with respect to $\prec$, are true for weak Pareto optimal control problems (see Corollary 3.4).

Example 2.6 The preference determined by the lexicographical order $\prec$, is defined by $x \prec y$ if there exists an integer $k \in \{0, 1, 2, ..., m-1\}$ such that $x_i = y_i$ for all $i = 1, 2, .., k$ and $x_{k+1} < y_{k+1}$. This preference is not closed.

3 The main result

We now return to problem (P). Fixing a feasible trajectory $x_* \in W^{1,1}$, we impose assumptions on the components of the problem which involve numbers $\epsilon > 0$ and $\beta > 0$:

(H1) g is Lipschitz continuous on $(x_*(a), x_*(b)) + \epsilon(B \times B)$.
(H2) Graph of $F(t, \cdot)$ is closed for a.e. t.
(H3) F is integrable sub-Lipschitz, that is, there exists an integrable function $k(t)$ such that for any $N > 0$, one has

$$F(t, x) \cap (\dot{x}_*(t) + NB) \subset F(t, x') + (k(t) + \beta N)|x - x'|B$$

for all x, x' with $|x - x_*(t)|, |x' - x_*(t)| \leq \epsilon$ and a.e. $t \in [a, b]$.
(H4) h is u.s.c. near $(t, x_*(t))$ for all t and there exists a constant k_h such that

$$|h(t, x) - h(t, x')| \leq k_h|x - x'|$$

for all $t \in [a, b]$ and $x, x' \in x_*(t) + \epsilon B$.

In what follows $H(t, x, p) := \sup\{\langle p, v\rangle : v \in F(t, x)\}$. The following theorem is our main result.

Theorem 3.1 *Let x_* be a $W^{1,1}$ local minimizer of (P) with respect to preference $\prec$ in R^m. Assume that $\prec$ is regular at $g(x_*(a), x_*(b))$ and assumptions* (H1)- (H4) *are satisfied. Then there exist an arc $p \in W^{1,1}$, a non-negative constant λ, a positive Radon measure*

μ, a μ-integrable function $\gamma : [a,b] \to R^n$ and $w \in N_{\overline{\mathcal{L}}[g(x_*(a),x_*(b))]}(g(x_*(a),x_*(b))$ with $|w| = 1$ such that
(i) $\lambda + \|p\|_\infty + \|\mu\| = 1$,
(ii) $\dot p(t) \in \mathrm{co}\{\eta : (\eta, q(t)) \in N_{\mathrm{Grph}F(t,\cdot)}(x_*(t), \dot x_*(t))\}$ *a.e., where* $q(t) := p(t) + \int_{[a,t)} \gamma(s)\mu(ds)$,
(iii) $(p(a), -q(b)) \in \lambda\partial\langle w, g(x_*(a), x_*(b))\rangle + N_C(x_*(a), x_*(b))$,
(iv) $\langle q(t), \dot x_*(t)\rangle = H(t, x_*(t), q(t))$ *a.e.,*
(v) $\gamma(t) \in \partial_x^> h(t, x_*(t))$ *$\mu-$a.e. and supp*$\{\mu\} \subset \{t : h(t, x_*(t)) = 0\}$, *where*
$\partial_x^> h(t,x) := co\{\lim_i \xi_i : \exists t_i \to t, x_i \to x \text{ such that } h(t_i, x_i) > 0 \text{ and } \xi_i \in \partial_x h(t_i, x_i)\}$.

An important step in the proof of Theorem 3.1 is to use necessary conditions of a finite Bolza problem. We restate necessary conditions of the following Bolza problem.

(BP) Minimize $J(x) := l(x(a), x(b)) + \int_a^b L(t, x(t), \dot x(t))dt$
over arcs $x \in W([a,b], R^n)$ which satisfy $h(t, x(t)) \le 0$ for all t.

Here $l : R^n \times R^n \to R_\infty$, $L : [a,b] \times R^n \times R^n \to R$ and $h : [a,b] \times R^n \to R$ are given functions.

Recall that an arc $x_* \in W^{1,1}$ is said to be a feasible trajectory of (BP) if $h(t, x(t)) \le 0$ for all t. A feasible trajectory x_* is called a local solution of (BP) if there exists $\epsilon > 0$ such that $J(x_*) \le J(x)$ for all feasible trajectories x satisfying $\|x - x_*\|_{1,1} \le \epsilon$.

We fix a feasible trajectory x_* of (BP) and assume the following assumptions:

(BH1) l is Lipschitz continuous around $(x_*(a), x_*(b))$.
(BH2) $L(\cdot, x, \cdot)$ is $\mathcal{L} \times \mathcal{B}$ measurable for each x, where $\mathcal{L}$ and $\mathcal{B}$ denote the Lebesgue subset of $[a,b]$ and the Borel subsets of R^n respectively.
(BH3) For every N there exist $\epsilon > 0$ and $k \in L^1$ such that

$$|L(t,x,v) - L(t,x',v')| \le k(t)(|x - x'| + |v - v'|),\ L(t, x_*(t), v) \ge -k(t)$$

for all $x, x' \in x_*(t) + \epsilon B$ and $v, v' \in \dot x_*(t) + NB$, a.e. $t \in [a,b]$.
(BH4) There exist $k_l > 0$ and $\epsilon > 0$ such that

$$|l(x,y) - l(x',y')| \le k_l(|x - y| + |x' - y'|)$$

for all $x, x' \in x_*(a) + \epsilon B$ and $y, y' \in x_*(b) + \epsilon B$.
(BH5) h is u.s.c. near $(t, x_*(t))$ for all t and there exist constants k_h and $\epsilon > 0$ such that

$$|h(t,x) - h(t,x')| \le k_h|x - x'|$$

for all $t \in [a,b]$ and $x, x' \in x_*(t) + \epsilon B$.

We have the following result on necessary conditions for (BP).

Lemma 3.2 ([25, Theorem 3]) *Let x_* be a $W^{1,1}$ local minimizer of the Bolza problem, for which $J(x_*) < \infty$. Assume that* (BH1)-(BH5) *are satisfied. Then there exist an arc $p \in W^{1,1}$, a non-negative constant λ, a positive Radon measure μ and a μ-integrable function $\gamma : [a,b] \to R^n$ such that*
(i) $\lambda + \|p\| + \|\mu\| = 1$,
(ii) $\dot{p}(t) \in \mathrm{co}\{\eta : (\eta, p(t) + \int_{[a,t)} \gamma(s)\mu(ds)) \in \lambda\partial L(t, x_*(t), \dot{x}_*(t))\}$ *a.e.*,
(iii) $(p(a), -[p(b) + \int_a^b \gamma(s)\mu(ds)]) \in \lambda\partial l(x_*(a), x_*(b))$,
(iv) $\langle p(t) + \int_{[a,t)} \gamma(s)\mu(ds)), \dot{x}_*(t)\rangle - \lambda L(t, x_*(t), \dot{x}_*(t)) \geq \langle p(t) + \int_{[a,t)} \gamma(s)\mu(ds)), v\rangle - \lambda L(t, x_*(t), v)$ *for all $v \in R^n$, a.e.*,
(v) $\gamma(t) \in \partial_x^> h(t, x_*(t))$ *$\mu-$a.e. and* $\mathrm{supp}\{\mu\} \subset \{t : h(t, x_*(t)) = 0\}$. *Here* $\partial_x^> h(t,x) := \mathrm{co}\{\lim_i \xi_i : \exists t_i \to t, x_i \to x$ *such that* $h(t_i, x_i) > 0$ *and* $\xi_i \in \partial_x h(t_i, x_i)\}$.

Proof of Theorem 3.1. In the proof we use some techniques from [2] and [25].

By reducing the size of ϵ we can arrange that x_* is minimizing in the relation to arcs x satisfying $\|x - x_*\| \leq \epsilon$ and (H1)-(H4) also satisfy for chosen ϵ. Put

$$W_\epsilon = \{x \in W^{1,1} : h(t, x(t)) \leq 0, \|x - x_*\|_{1,1} \leq \epsilon\},$$

$$S_\epsilon = \{x \in W_\epsilon : \dot{x}(t) \in F(t, x(t)) \text{ a.e.}, (x(a), x(b)) \in C, \},$$

$$\rho_F(t, x(t), \dot{x}(t)) = d(\dot{x}(t), F(t, x(t)).$$

It is clear that W_ϵ is a complete metric space with the distance induced by the norm $\|\cdot\|_{1,1}$ and S_ϵ is a closed set in W_ϵ.

According to [9], there are two following possible situation:

(a) There exist $\epsilon' \in (0, \epsilon)$ and $K > 0$ such that for any $x \in W_{\epsilon'}$ one has

$$d(x, S_\epsilon) \leq K[\int_a^b \rho_F(t, x(t), \dot{x}(t))dt + d_C(x(a), x(b))] \tag{1}$$

(b) There exist a sequence of arcs $\overline{x}_k$ such that $h(t, \overline{x}_k(t)) \leq 0$ for all $t \in [a,b]$ and $\overline{x}_k \to x_*$ in $W^{1,1}$, and

$$d(\overline{x}_k, S_\epsilon) > 2k[\int_a^b \rho_F(t, \overline{x}_k(t), \dot{\overline{x}}_k(t))dt + d_C(\overline{x}_k(a), \overline{x}_k(b))]. \tag{2}$$

Case(a). Since $g(x_*(a), x_*(b)) \in \overline{\mathcal{L}}[g(x_*(a), x_*(b))]$, there exists a sequence η_n such that

$$\eta_n \in \mathcal{L}[g(x_*(a), x_*(b))], \eta_n \to g(x_*(a), x_*(b)).$$

Hence for each k, there exists n_k such that $|\eta_{n_k} - g(x_*(a), x_*(b))| \leq 1/k^2$. Putting $\theta_k = \eta_{n_k}$, we have $|\theta_k - g(x_*(a), x_*(b))| \leq 1/k^2$. Put $\Omega_k = \overline{\mathcal{L}}[\theta_k]$ and define the function

$$\varphi(x,\theta)=\begin{cases}|g(x(a),x(b))-\theta| & \text{if } (x,\theta)\in S_{\epsilon'}\times\Omega_k\\ +\infty & \text{otherwise.}\end{cases}$$

Sine $S_{\epsilon'}\times\Omega_k$ is a closed set in $W_{\epsilon'}\times R^m$, we see that φ is l.s.c. on $W_{\epsilon'}\times\Omega_k$. Since $\varphi(x,\theta)\geq 0$, one has

$$\varphi(x_*,\theta_k)\leq \inf_{(x,\theta)\in W_{\epsilon'}\times\Omega_k}\varphi(x,\theta)+1/k^2.$$

By the Ekeland principle (see, for instance [4]), there exists $(x_k,\xi_k)\in W_{\epsilon'}\times\Omega_k$ such that

$$\varphi(x_k,\xi_k)\leq\varphi(x_*,\theta_k)<\frac{1}{k^2}, \tag{3}$$

$$\|x_k-x_*\|_{1,1}+|\xi_k-\theta_k|\leq 1/k, \tag{4}$$

$$\varphi(x_k,\xi_k)\leq\varphi(x,\theta)+\frac{1}{k}(\|x-x_k\|_{1,1}+|\theta-\xi_k|)\ \forall(x,\theta)\in W_{\epsilon'}\times\Omega_k. \tag{5}$$

From (5) we have

$$\varphi(x_k,\xi_k)\leq\varphi(x_k,\theta)+\frac{1}{k}|\theta-\xi_k|\ \forall\theta\in\Omega_k. \tag{6}$$

and

$$\varphi(x_k,\xi_k)\leq\varphi(x,\xi_k)+\frac{1}{k}\|x-x_k\|_{1,1}\ \forall x\in W_{\epsilon'}. \tag{7}$$

From (3) and (4) one has $x_k\in S_{\epsilon'}$ and $x_k\to x_*$ in $W^{1,1}$. We claim that $\xi_k\neq g(x_k(a),x_k(b))$. Indeed, suppose that $\xi_k=g(x_k(a),x_k(b))$. Since $\prec$ is closed, the relation $\xi_k\in\overline{\mathcal{L}}[\xi_k]$ and $\xi_k\prec g(x_*(a),x_*(b))$ imply $g(x_k(a),x_k(b))\prec g(x_*(a),x_*(b))$. This contradicts the fact that x_* is a minimizer.

Put $w_k=\frac{\xi_k-g(x_k(a),x_k(b))}{|\xi_k-g(x_k(a),x_k(b))|}$. We can assume that $w_k\to w$ with $|w|=1$. From (6) we obtain $0\in\partial(\varphi(x_k,\cdot)+\frac{1}{k}|\cdot-\xi_k|)(\xi_k)+N_{\Omega_k}(\xi_k)$. This implies that $w_k\in\frac{1}{k}B+N_{\Omega_k}(\xi_k)$. Hence $w\in\lim_{k\to\infty}N_{\Omega_k}(\xi_k)\subset N_{\overline{\mathcal{L}}[g(x_*(a),x_*(b))]}(g(x_*(a),x_*(b))$. Taking any $x\in W_{\epsilon'}$ we obtain from (1) and (7) that

$$\varphi(x_k,\xi_k)\leq\varphi(x,\xi_k)+\frac{1}{k}\|x-x_k\|_{1,1}+\int_a^b\rho_F(t,x(t),\dot{x}(t))dt+d_C(x(a),x(b)).$$

This is equivalent to

$$\begin{aligned}|g(x_k(a),x_k(b))-\xi_k|\leq&\int_a^b(\rho_F(t,x(t),\dot{x}_k(t))+\frac{1}{k}|\dot{x}(t)-\dot{x}_k(t)|)dt+\\&+|g(x(a),x(b))-\xi_k|+\frac{1}{k}|x(a)-x_k(a)|+d_C(x(a),x(b))\end{aligned}$$

for all $x\in W_{\epsilon'}$. Hence x_k is a solution of the following Bolza problem:

Minimize $\quad J(x):=l(x(a),x(b))+\int_a^b L(t,x(t),\dot{x}(t))dt$

over arcs $x \in W_{\epsilon'}$,

where

$$l(u,v) = |g(u,v) - \xi_k| + \frac{1}{k}|u - x_k(a)| + d_C(u,v)$$

and

$$L(t,u,v) = \rho_F(t,u,v) + \frac{1}{k}|v - \dot{x}_k(t)|.$$

By lemma 7 in [25], we see that l and L satisfy all conditions of Lemma 3.2. According to this lemma, there exist an arc $p_k \in W^{1,1}$, a non-negative constant λ_k, a positive Radon measure μ_k and a μ_k-integrable function $\gamma_k : [a,b] \to R^n$ such that

(A) $\lambda_k + \|p_k\|_\infty + \|\mu_k\| = 1$,
(B) $\dot{p}_k(t) \in \mathrm{co}\{\eta : (\eta, p_k(t) + \int_{[a,t)} \gamma(s)\mu(ds)) \in \lambda_k \partial L(t, x_k(t), \dot{x}_k(t))\}$ a.e.,
(C) $(p_k(a), -[p_k(b) + \int_a^b \gamma_k(s)\mu_k(ds)]) \in \lambda_k \partial l(x_k(a), x_k(b))$,
(D) $\langle p_k(t) + \int_{[a,t)} \gamma_k(s)\mu(ds)), \dot{x}_k(t)\rangle \geq \langle p_k(t) + \int_{[a,t)} \gamma_k(s)\mu_k(ds)), v\rangle - \lambda_k \rho_F(t, x_k(t), v) - \frac{\lambda_k}{k}|v - \dot{x}_k(t)|$ for all $v \in R^n$, a.e.,
(E) $\gamma_k(t) \in \partial_x^{>} h(t, x_k(t))$ μ_k-a.e. and $\mathrm{supp}\{\mu_k\} \subset \{t : h(t, x_k(t)) = 0\}$.

By Lemma 2.1 we have

$$\begin{aligned}\lambda_k \partial L(t, x_k(t), \dot{x}_k(t)) &\subset \lambda_k \partial \rho_F(t, x_k(t), \dot{x}_k(t)) + \frac{1}{k}(\{0\} \times B)\\ &\subset N_{\mathrm{GraphF(t,\cdot)}}(x_k(t), \dot{x}_k(t)) + \frac{1}{k}(\{0\} \times B).\end{aligned}$$

Recalling lemma 7 in [25], it follows from (B) that

$$|\dot{p}_k(t)| \leq \lambda_k[(1 + \beta\epsilon)k(t) + 2\beta|\dot{x}_k(t) - \dot{x}_*(t)|].$$

But $x_k \to x_*$ in $W^{1,1}$ (and so $\dot{x}_k \to \dot{x}_*$ in L^1) and $\|p_k\| \leq 1$. It follows that $p_k \to p$ uniformly and $\dot{p}_k \xrightarrow{L^1} \dot{p}$ for some $p \in W^{1,1}$. By Lemma 2.2, we can assume that $\mu_k \xrightarrow{*} \mu$ and $\lambda_k \to \lambda$. Since $x_k \to x_*$ and by Proposition 9.2.1 in [23], $\gamma_k(s)\mu_k(ds) \to \gamma(s)\mu(ds)$ for some μ-integrable γ such that

$$\gamma(t) \in \partial_x^{>} h(t, x_*(t)) \ \mu - \text{a.e.}, \ \mathrm{supp}\{\mu\} \subset \{t : h(t, x_*(t)) = 0\}.$$

Note that $\lim_{k\to\infty} \|\mu_k\| = \lim_{k\to\infty} \mu_k([a,b]) = \lim_{k\to\infty} \int_a^b d\mu_k = \int_a^b d\mu = \|\mu\|$. Hence from (A) we obtain $\lambda + \|p\|_\infty + \|\mu\| = 1$. By letting $k \to \infty$, from (B) we get

$$\dot{p}(t) \in \mathrm{co}\{\eta : (\eta, p(t) + \int_{[a,t)} \gamma(s)\mu(ds)) \in N_{\mathrm{Grph}F(t,\cdot)}(x_*(t), \dot{x}_*(t))\} \text{ a.e. } t.$$

Since

$$\begin{aligned}\lambda_k \partial l(x_k(a), x_k(b)) &\subset \lambda_k(\partial|g(u,v) - \xi_k|(x_k(a), x_k(b))) + \frac{\lambda_k}{k}(B \times \{0\}) + N_C(x_k(a), x_k(b))\\ &\subset \partial\langle w_k, g(x_k(a), x_k(b))\rangle + \frac{\lambda_k}{k}(B \times \{0\}) + N_C(x_k(a), x_k(b)),\end{aligned}$$

(C) implies

$$(p(a), -[p(b) + \int_a^b \gamma(s)\mu(ds)]) \in \lambda\partial\langle w, g(x_*(a), x_*(b))\rangle + N_C(x_*(a), x_*(b)).$$

For k sufficiently large $\rho_F(t, \cdot, \cdot)$ is Lipschitz continuous near $(x_k(t), \dot{x}_k(t))$, in view of lemma 7 in [25]. Taking any $v \in F(t, x_*(t))$, from (D) we have

$$\langle p_k(t)+\int_{[a,t)} \gamma_k(s)\mu(ds)), \dot{x}_k(t)\rangle \geq \langle p_k(t)+\int_{[a,t)} \gamma_k(s)\mu_k(ds)), v\rangle - \lambda_k\rho_F(t, x_k(t), v) - \frac{\lambda_k}{k}|v-\dot{x}_k(t)|.$$

Note that we may assume that $\dot{x}_k(t) \to x_*(t)$ almost everywhere. By passing to the limit we get

$$\langle p(t) + \int_{[a,t)} \gamma(s)\mu(ds)), \dot{x}_*(t)\rangle \geq \langle p(t) + \int_{[a,t)} \gamma(s)\mu(ds)), v\rangle \ \forall v \in F(t, x_*(t)) \ a.e.$$

Thus we obtain the conclusion of the theorem.

Case (b). Putting $J(x) = \int_a^b \rho_F(t, x(t), \dot{x}(t))dt + d_C(x(a), x(b))$, we can write (b) in the form

$$J(\overline{x}_k) < \frac{1}{2k} d(\overline{x}_k, S_\epsilon) < \inf_{x \in W_\epsilon} J(x) + \frac{1}{2k} a_k, \tag{8}$$

where $a_k = d(\overline{x}_k, S_\epsilon)$. Note that $0 < a_k \leq \|x_k - x_*\| \to 0$. We claim that J is l.s.c. on W_ϵ. In fact, assume that $z_k \xrightarrow{W^{1,1}} x$, then $z_k \to x$ uniformly and $\dot{z}_k \xrightarrow{L^1} \dot{x}$. Consequently, in view of Lemma 7 in [25], we have

$$\begin{aligned} |J(z_k) - J(x)| &\leq \int_a^b |\rho_F(t, z_k(t), \dot{z}_k(t) - \rho_F(t, x(t), \dot{x}(t)|dt + |d_C(z_k(a), z_k(b)) - d_C(x(a), x(b))| \\ &\leq ((1+\beta)\|k\|_{L^1} + 2\beta\|\dot{z}_k - \dot{x}\|_{L^1})\|z_k - x\|_\infty + \|\dot{z}_k - \dot{x}\|_{L^1} + 2\|z_k - x\|_\infty. \end{aligned}$$

The right side converges to 0 as $k \to \infty$ and so J is l.s.c.

According to the Ekeland principle, it follows from (8) that for each k there exists $x_k \in W_\epsilon$ such that

$$\|x_k - \overline{x}_k\| \leq \frac{a_k}{2} \tag{9}$$

and x_k is a minimizer of the problem

$$J(x) + \frac{1}{k}\|x - x_k\|_{1,1} \to \inf. \tag{10}$$

From (9), it follow that $x_k \notin S_\epsilon$. Hence $(x_k(a), x_k(b)) \notin C$ or $\dot{x}_k(t) \notin F(t, x_k(t))$ on a set of positive measure. Rewrite

$$J(x)+\frac{1}{k}\|x-x_k\|_{1,1} = \int_a^b (\rho_F(t, x(t), \dot{x}(t))+\frac{1}{k}|\dot{x}(t)-\dot{x}_k(t)|)dt+\frac{1}{k}|x(a)-x_k(a)|+d_C(x(a), x(b)).$$

Thus x_k is a minimizer of the Bolza problem

$\tilde{J}(x) = l(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))dt \to \inf$
over arcs $x \in W_\epsilon$,

where $L(t,u,v) = \rho_F(t,u,v) + \frac{1}{k}|v - \dot{x}_k(t)|,\ l(u,v) = d_C(u,v) + \frac{1}{k}|u - x_k(a)|$. Using lemma 7 in [25] again, we see that all assumptions of Lemma 3.2 are satisfied. By this lemma, there exist λ_k, p_k, μ_k and γ_k as in Case (a) such that

(A') $\lambda_k + \|p_k\| + \|\mu_k\| = 1$,
(B') $\dot{p}_k(t) \in \mathrm{co}\{\eta : (\eta, \mathrm{q_k(t)}) \in \mathrm{\lambda_k \partial \rho_F(t, x_k(t), \dot{x}_k(t))} + \mathrm{\frac{\lambda_k}{k}(\{0\} \times B)}\}$ a.e., where $q_k(t) = p_k(t) + \int_{[a,t)} \gamma(s)\mu(ds)$.
(C') $(p_k(a), -q_k(b) \in \lambda_k \partial d_C(x_k(a), x_k(b)) + \frac{\lambda_k}{k}(B \times \{0\}$,
(D') $\langle q_k(t), \dot{x}_k(t)\rangle - \lambda_k \rho_F(t, x_k(t), \dot{x}_k(t)) \geq \langle q_k(t), v\rangle - \lambda_k \rho_F(t, x_k(t), v) - \frac{\lambda_k}{k}|v - \dot{x}_k(t)|$ for all $v \in R^n$, a.e.,
(E') $\gamma_k(t) \in \partial_x^> h(t, x_k(t))\ \mu_k$–a.e. and $\mathrm{supp}\{\mu_k\} \subset \{t : h(t, x_k(t)) = 0\}$.

By the similar arguments as in the proof of Case (a), we can assume that $p_k \to p$ uniformly and $\dot{p}_k \xrightarrow{L^1} \dot{p}$ for some $p \in W^{1,1}$,

$$\lambda_k \to \lambda',\ \mu_k \xrightarrow{*} \mu,\ \gamma_k(s)\mu_k(ds) \to \gamma(s)\mu(ds),$$

where γ is a μ-integrable which satisfies

$$\gamma(t) \in \partial_x^> h(t, x_*(t)) \text{ and } \mathrm{supp}\{\mu\} \subset \{t : h(t, x_*(t)) = 0\}.$$

From (A') we get $\lambda' + \|p\| + \|\mu\| = 1$. We now claim that $\|p\| + \|\mu\| > 0$. In fact, assume that $\|p\| + \|\mu\| = 0$. If $(x_k(a), x_k(b)) \notin C$ then (C') implies $(p_k(a) - \frac{\lambda_k}{k}b^*, -q_k(b)) \in \lambda_k d_C(x_k(a), x_k(b))$ for some $b^* \in B$. Hence $|p_k(a)| + |q_k(b)| \geq \lambda_k - \frac{\lambda_k}{k}$. Consequently, $|p(a)| + |q(b)| \geq 1$. This is impossible because $p = 0$ and $\mu = 0$. If $\dot{x}_k(t) \notin F(t, x_k(t))$ then (D') implies

$$|q_k(t)| \leq |p_k(t)| + \int_a^t |\gamma_k(s)|\mu_k(ds) \leq |p_k(t)| + k_h\|\mu_k\|.$$

Hence

$$\|p_k\| \geq \max_{[0,1]} |p_k(t)| \geq \lambda_k(1 - 1/k) - k_h\|\mu_k\|.$$

Since $\mu_k \to 0$, we obtain $\|p\| \geq 1$. But this is impossible sine $p = 0$. Our claim is proved.

By Lemma 2.1 and (B')-(D') we have

$$\dot{p}(t) \in \mathrm{co}\{\eta : (\eta, \mathrm{p(t)} + \int_{[a,t)} \mathrm{\gamma(s)\mu(ds)}) \in \mathrm{N_{GrphF(t,\cdot)}(x_*(t), \dot{x}_*(t))}\}, \text{for a.e. } t \in [a, b],$$

$$(p(a), -q(b)) \in N_C(x_*(a), x_*(b))$$

and
$$\langle p(t) + \int_{[a,t)} \gamma(s)\mu(ds)), \dot{x}_*(t)\rangle \geq \langle p(t) + \int_{[a,t)} \gamma(s)\mu(ds)), v\rangle$$
for all $v \in F(t, x_*(t))$, a.e., Since $\|p\| + \|\mu\| = 1 - \lambda' > 0$ we can scale the multipliers such that $\|p\| + \|\mu\| = 1$. Thus we obtain the conclusion of the theorem where $\lambda = 0$. The proof of the theorem is complete. $\square$

The rest of the paper is destined for some corollaries of Theorem 3.1.

When $m = 1$, (P) becomes a single objective optimal control problem. In this case, by putting $w = 1$, we have

Corollary 3.3 ([25, Theorem 4]) *Let x_* be a $W^{1,1}$ local minimizer of (P). Assume that assumptions* (H1)- (H4) *are satisfied. Then there exist an arc $p \in W^{1,1}$, a non-negative constant λ, a positive Radon measure μ and a μ-integrable function $\gamma : [a, b] \to R^n$ such that*
(i) $\lambda + \|p\|_\infty + \|\mu\| = 1$,
(ii) $\dot{p}(t) \in \mathrm{co}\{\eta : (\eta, q(t)) \in N_{\mathrm{Grph}F(t,\cdot)}(x_*(t), \dot{x}_*(t))\}$ *a.e.,*
(iii) $(p(a), -q(b)) \in \lambda\partial g(x_*(a), x_*(b)) + N_C(x_*(a), x_*(b))$,
(iv) $\langle q(t), \dot{x}_*(t)\rangle = H(t, x_*(t), q(t))$, *a.e.,*
(v) $\gamma(t) \in \partial_x^> h(t, x_*(t))$ *$\mu-$a.e. and supp*$\{\mu\} \subset \{t : h(t, x_*(t)) = 0\}$.

When (P) is a weak Pareto optimal control problem, from Example 2.5, we have

Corollary 3.4 *Assume that x_* is a weak Pareto solution of (P) and assumptions* (H1)- (H4) *are satisfied. Then there exist an arc $p \in W^{1,1}$, a non-negative constant λ, a positive Radon measure μ, a μ-integrable function $\gamma : [a, b] \to R^n$ and $w \in R_+^m$ with $\sum_{i=1}^m w_i = 1$ such that*
(i) $\lambda + \|p\|_\infty + \|\mu\| = 1$,
(ii) $\dot{p}(t) \in \mathrm{co}\{\eta : (\eta, q(t)) \in N_{\mathrm{Grph}F(t,\cdot)}(x_*(t), \dot{x}_*(t))\}$ *a.e.,*
(iii) $(p(a), -q(b)) \in \lambda\partial\langle w, g(x_*(a), x_*(b))\rangle + N_C(x_*(a), x_*(b))$,
(iv) $\langle q(t), \dot{x}_*(t)\rangle = \langle H(t, x_*(t), q(t))$ *a.e.,*
(v) $\gamma(t) \in \partial_x^> h(t, x_*(t))$ *$\mu-$a.e. and supp*$\{\mu\} \subset \{t : h(t, x_*(t)) = 0\}$.

Let us give an illustrative example for Theorem 3.1.

Example 3.5 Consider the weak Pareto optimal control problem

minimize $g(x(2)) = (x_1(2) - x_2(2), x_1(2))$

over arcs $x = (x_1, x_2) \in W^{1,1}([0,2], R^2)$ which satisfy

$$\begin{cases} (\dot{x}_1(t), \dot{x}_2(t)) \in F(t, x(t)), \\ x_2(t) \leq 0 \text{ for all } t \in [0,2], \\ (x_1(0), x_2(0)) = (0, -3), \end{cases}$$

where

$$F(t,x) := \begin{cases} [-1,1] \times \{1\} & \text{if } t \leq 1 \\ \{t, 1\} \times \{1\} & \text{if } t > 1. \end{cases}$$

Solution. Evidently, this is problem (P) with

$$C = \{(0,-3)\} \times R^2 \text{ and } h(x_1, x_2) = x_2.$$

For each $w = (w_1, w_2)$, $w_1 + w_2 = 1$ we have $\langle w, g(x(2))\rangle = x_1(2) - w_1 x_2(2)$. By a simple computation, we have

$$H(t, (x_1, x_2), (q_1, q_2)) = \begin{cases} |q_1| + q_2 & \text{if } t \leq 1 \\ \max\{tq_1, q_1\} + q_2 & \text{if } t > 1. \end{cases}$$

Assume that x is a solution of the problem. By Corollary 3.4, there exist $\lambda \geq 0$, p, μ, γ and $w = (w_1, w_2) \in R^2_+$, $w_1 + w_2 = 1$ such that assertions $(i) - (v)$ are satisfied. Since

$$\text{Grph}F(t, \cdot) = \begin{cases} R^2 \times ([-1,1] \times \{1\}) & \text{if } t \leq 1 \\ R^2 \times (\{t, 1\} \times \{1\}) & \text{if } t > 1, \end{cases}$$

we get

$$N_{\text{Grph}F(t,\cdot)}(x(t), \dot{x}(t)) = \begin{cases} \{(0,0)\} \times N_{[-1,1]\times\{1\}}(\dot{x}(t)) & \text{if } 0 \leq t \leq 1 \\ \{(0,0)\} \times N_{\{t,1\}\times\{1\}}(\dot{x}(t)) & \text{if } t > 1. \end{cases}$$

Hence (ii) implies that $\dot{p} = (0,0)$. Consequently, $p = (p_1, p_2)$, where p_1 and p_2 are constants. From (iii) we have

$$p(2) + \int_0^2 \gamma(s) d\mu = (-\lambda, \lambda w_1). \tag{11}$$

Since $h(t,x) = x_2$, from (v) we get $\gamma(t) = (0,1)$. Hence (11) implies

$$p_1 = -\lambda, \; p_2 + \mu[0,2] = \lambda w_1. \tag{12}$$

Since $\dot{x}_2 = 1$, $x_2 = t - 3$ and so $\text{supp}\mu \subset \{t \in [0,2] : t - 3 = 0\} = \emptyset$. Consequently, $\mu[0,2] = 0$. We now have from (iv) that

$$p_1\dot{x}_1 = \begin{cases} |p_1| & \text{if } 0 \le t \le 1 \\ \max\{tp_1, p_1\} & \text{if } 1 < t \le 2. \end{cases}$$

Since $p_1 = -\lambda \le 0$, one has

$$p_1\dot{x}_1 = \begin{cases} -p_1 & \text{if } 0 \le t \le 1 \\ p_1 & \text{if } 1 < t \le 2. \end{cases}$$

If $\lambda = 0$ then (12) implies $p_1 = 0$, $p_2 = p_2 + \mu[0,2] = 0$. But (i) implies $1 = \lambda + |p_1| + |p_2| + \mu[0,2] = |p_2|$ which is absurd. Hence we must have $p_1 = -\lambda \neq 0$. It follows that

$$\dot{x}_1 = \begin{cases} -1 & \text{if } 0 \le t \le 1 \\ 1 & \text{if } 1 < t \le 2 \end{cases}$$

and so

$$x_1 = \begin{cases} -t & \text{if } 0 \le t \le 1 \\ t-2 & \text{if } 1 < t \le 2. \end{cases}$$

Thus we showed that if $x_* = (x_{1*}, x_{2*})$ is a solution of the problem then $x_{2*} = t - 3$ and

$$x_{1*} = \begin{cases} -t & \text{if } 0 \le t \le 1 \\ t-2 & \text{if } 1 < t \le 2. \end{cases}$$

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