Strong Convergence Theorems by a Hybrid Extragradient-like Approximation Method for Asymptotically Nonexpansive Mappings in the Intermediate Sense in Hilbert Spaces

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Abstract. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be an asymptotically nonexpansive map in the intermediate sense with the fixed point set $F(S)$. Let $A : C \to H$ be a Lipschitz continuous map, and $VI(C, A)$ be the set of solutions $u \in C$ of the variational inequality

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

The purpose of this work is to introduce a hybrid extragradient-like approximation method for finding a common element in $F(S)$ and $VI(C, A)$. We establish some strong convergence theorems for sequences produced by our iterative method.

Keywords: Asymptotically nonexpansive mapping in the intermediate sense; Variational inequality; Hybrid extragradient-like approximation method; Monotone mapping; Fixed point; Strong convergence.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and let P_C be the metric projection from H onto C. A mapping $A : C \to H$ is called *monotone* [7,8,9] if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in C;$$

and A is called *k-Lipschitz continuous* if there exists a positive constant k such that

$$\|Au - Av\| \leq k\|u - v\|, \quad \forall u, v \in C.$$

Let S be a mapping of C into itself. Denote by $F(S)$ the set of fixed points of S; that is $F(S) = \{u \in C : Su = u\}$. Recall that S is *nonexpansive* if

$$\|Su - Sv\| \leq \|u - v\|, \quad \forall u, v \in C;$$

and S is *asymptotically nonexpansive* [4] if there exists a null sequence $\{\gamma_n\}$ in $[0, +\infty)$ such that

$$\|S^n u - S^n v\| \leq (1 + \gamma_n)\|u - v\|, \quad \forall u, v \in C \text{ and } n \geq 1.$$

We call S an *asymptotically nonexpansive mapping in the intermediate sense* [10] if there exists two null sequences $\{\gamma_n\}$ and $\{c_n\}$ in $[0, +\infty)$ such that

$$\|S^n x - S^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + c_n, \quad \forall x, y \in C, \forall n \geq 1.$$

Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping. The variational inequality problem [3] is to find the elements $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. The idea of an extragradient iterative process was first introduced by Korpelevich in [6]. When $S : C \to C$ is a uniformly continuous asymptotically nonexpansive mapping in the intermediate sense, a hybrid extragradient-like approximation method was proposed by Ceng, Sahu and Yao to ensure the *weak* convergence of some algorithms for finding a member of $F(S) \cap VI(C, A)$ [2, Theorem 1.1]. Meanwhile, assuming S is *nonexpansive*, Ceng, Hadjisavvas and Wong in [1] introduced an iterative process and proved its strong convergence to a member of $F(S) \cap VI(C, A)$.

It is known that an asymptotically nonexpansive mapping in the intermediate sense is not necessarily nonexpansive. Extending both [2, Theorem 1.1] and [1, Theorem 5], the main result, Theorem 1, of this paper provides a technical method to show the strong convergence of an iterative scheme to an element of $F(S) \cap VI(C, A)$, under the weaker assumption on the asymptotical nonexpansivity in the intermediate sense of S.

2. Strong convergence theorems

Let C be a nonempty closed convex subset of a real Hilbert space H. For any x in H there exists a unique element in C, which is denoted by P_Cx, such that $\|x - P_Cx\| \leq \|x - y\|$ for all y in C. We call P_C the *metric projection* of H onto C. It is well-known that P_C is a nonexpansive mapping from H onto C, and

$$\langle x - P_Cx, P_Cx - y\rangle \geq 0, \quad \text{for all } x \in H, y \in C; \tag{1}$$

see for example [5]. It is easy to see that (1) is equivalent to

$$\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2, \quad \text{for all } x \in H,\ y \in C. \tag{2}$$

Let A be a monotone mapping of C into H. In the context of variational inequality problems, the characterization of the metric projection (1) implies that

$$u \in VI(C, A) \quad \Longleftrightarrow \quad u = P_C(u - \lambda Au) \quad \text{for some } \lambda > 0.$$

Theorem 1. *Let C be a nonempty closed convex subset of a real Hilbert spaces H. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping. Let $S : C \to C$ be a uniformly continuous asymptotically nonexpansive mapping in the intermediate sense with nonnegative null sequences $\{\gamma_n\}$ and $\{c_n\}$. Suppose that $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $F(S) \cap VI(C, A)$ is nonempty and bounded.*

Assume that
(i) $0 < \mu \leq 1$, *and* $0 < a < b < \frac{3}{8k\mu}$*;*
(ii) $a \leq \lambda_n \leq b$, $\alpha_n \geq 0$, $\beta_n \geq 0$, $\alpha_n + \beta_n \leq 1$, *and* $3/4 < \delta_n \leq 1$, *for all* $n \geq 0$*;*
(iii) $\lim_{n\to\infty} \alpha_n = 0$*;*
(iv) $\liminf_{n\to\infty} \beta_n > 0$*;*
(v) $\lim_{n\to\infty} \delta_n = 1$.
Set, for all $n \geq 0$,

$$\begin{aligned}
\Delta_n &= \sup\{\|x_n - u\| : u \in F(S) \cap VI(C, A)\},\\
d_n &= 2b(1 - \mu)\alpha_n\Delta_n,\\
w_n &= b^2\mu\alpha_n + 4b^2\mu^2\beta_n(1 - \delta_n)(1 + \gamma_n),\\
v_n &= b^2(1 - \mu)\alpha_n + 4b^2(1 - \mu)^2\beta_n(1 - \delta_n)(1 + \gamma_n), \text{ and}\\
\vartheta_n &= \beta_n\gamma_n\Delta_n^2 + \beta_n c_n.
\end{aligned}$$

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the algorithm:

$$\begin{cases}
x_0 \in C \text{ chosen arbitrarily},\\
y_n = (1 - \delta_n)x_n + \delta_n P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1 - \mu)Ay_n),\\
z_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n y_n + \beta_n S^n P_C(x_n - \lambda_n Ay_n),\\
C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + d_n\|Ay_n\| + w_n\|Ax_n\|^2 + v_n\|Ay_n\|^2 + \vartheta_n\}\\
Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}\\
x_{n+1} = P_{C_n\cap Q_n}(x_0), \quad \forall n \geq 0.
\end{cases} \tag{3}$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in (3) are well-defined and converge strongly to the same point $q = P_{F(S)\cap VI(C,A)}(x_0)$.

Proof. First note that $\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} c_n = 0$. We will see that $\{\Delta_n\}$ is bounded, and thus $\lim_{n\to\infty} d_n = \lim_{n\to\infty} w_n = \lim_{n\to\infty} v_n = \lim_{n\to\infty} \vartheta_n = 0$.

We divide the proof into several steps.

Step 1. We claim that the following statements hold:

(a) C_n is closed and convex for all $n \in \mathbb{N}$;

(b) $\|z_n - u\|^2 \le \|x_n - u\|^2 + d_n\|Ay_n\| + w_n\|Ax_n\|^2 + v_n\|Ay_n\|^2 + \vartheta_n$ for all $n \ge 0$ and $u \in F(S) \cap VI(C,A)$;

(c) $F(S) \cap VI(C,A) \subset C_n$ for all $n \in \mathbb{N}$.

It is obvious that C_n is closed for all $n \in \mathbb{N}$. On the other hand, the defining inequality in C_n is equivalent to the inequality

$$\langle 2(x_n - z_n), z\rangle \le \|x_n\|^2 - \|z_n\|^2 + d_n\|Ay_n\| + w_n\|Ax_n\|^2 + v_n\|Ay_n\|^2 + \vartheta_n,$$

which is affine in z. Therefore, C_n is convex.

Let $t_n = P_C(x_n - \lambda_n Ay_n)$ for all $n \ge 0$. Assume that $u \in F(S) \cap VI(C,A)$ is arbitrary. In view of (3), the monotonicity of A, and the fact $u \in VI(C,A)$, we conclude that

$$\begin{aligned}
&\|t_n - u\|^2 \\
&\le \|x_n - \lambda_n Ay_n - u\|^2 - \|x_n - \lambda_n Ay_n - t_n\|^2 \\
&= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Ay_n, u - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n[\langle Ay_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle] \\
&\le \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle.
\end{aligned} \tag{4}$$

Now, using

$$y_n = (1-\delta_n)x_n + \delta_n P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n),$$

we estimate the last term

$$\begin{aligned}
&\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle \\
= \;&\langle x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n - y_n, t_n - y_n\rangle + \lambda_n\mu\langle Ax_n - Ay_n, t_n - y_n\rangle \\
\le \;&\langle x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n - (1-\delta_n)x_n - \delta_n P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n), t_n - y_n\rangle \\
&\quad + \lambda_n\mu\|Ax_n - Ay_n\|\|t_n - y_n\| \\
\le \;&\delta_n\langle x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n - P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n), t_n - y_n\rangle \\
&\quad - (1-\delta_n)\lambda_n\langle \mu Ax_n + (1-\mu)Ay_n, t_n - y_n\rangle + \lambda_n\mu k\|x_n - y_n\|\|t_n - y_n\|.
\end{aligned} \tag{5}$$

It follows from the properties (1) and (2) of the projection $P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n)$ that

$$
\begin{aligned}
& \langle x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n - P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n), t_n - y_n\rangle \\
= \ & \langle x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n - P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n), \\
& \qquad t_n - (1-\delta_n)x_n - \delta_n P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n)\rangle \\
= \ & (1-\delta_n)\langle x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n - P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n), t_n - x_n\rangle \\
& + \delta_n\langle x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n - P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n), \\
& \qquad t_n - P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n)\rangle \\
\leq \ & (1-\delta_n)\langle x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n - P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n), t_n - x_n\rangle \\
\leq \ & (1-\delta_n)\|x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n - P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n)\|\|t_n - x_n\| \\
\leq \ & (1-\delta_n)\|\lambda_n\mu Ax_n + \lambda_n(1-\mu)Ay_n)\|\|t_n - x_n\| \\
\leq \ & (1-\delta_n)\lambda_n(\mu\|Ax_n\| + (1-\mu)\|Ay_n\|)(\|t_n - y_n\| + \|y_n - x_n\|).
\end{aligned} \tag{6}
$$

In view of (4)–(6), $\lambda_n \leq b$, and the inequalities $2\alpha\beta \leq \alpha^2 + \beta^2$ and $(\alpha+\beta)^2 \leq 2\alpha^2 + 2\beta^2$, we conclude that

$$
\begin{aligned}
\|t_n - u\|^2 \ & \leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle \\
& \leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
& \quad + 2\lambda_n[\delta_n(1-\delta_n)(\mu\|Ax_n\| + (1-\mu)\|Ay_n\|)(\|t_n - y_n\| + \|y_n - x_n\|) \\
& \qquad - 2(1-\delta_n)\lambda_n\langle \mu Ax_n + (1-\mu)Ay_n, t_n - y_n\rangle + 2\lambda_n\mu k\|x_n - y_n\|\|t_n - y_n\|] \\
& \leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
& \quad + 2\delta_n(1-\delta_n)b(\mu\|Ax_n\| + (1-\mu)\|Ay_n\|)(\|t_n - y_n\| + \|y_n - x_n\|) \\
& \quad + 2(1-\delta_n)b(\mu\|Ax_n\| + (1-\mu)\|Ay_n\|)\|t_n - y_n\| + 2b\mu k\|x_n - y_n\|\|t_n - y_n\| \\
& = \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
& \quad + 2\delta_n(1-\delta_n)(b^2\mu^2\|Ax_n\|^2 + b^2(1-\mu)^2\|Ay_n\|^2 + \|t_n - y_n\|^2 + \|y_n - x_n\|^2) \\
& \quad + (1-\delta_n)(b^2\mu^2\|Ax_n\|^2 + b^2(1-\mu)^2\|Ay_n\|^2 + 2\|t_n - y_n\|^2) \\
& \quad + b\mu k(\|x_n - y_n\|^2 + \|t_n - y_n\|^2) \\
& = \|x_n - u\|^2 - \|x_n - y_n\|^2(1 - 2\delta_n(1-\delta_n) - bk\mu) \\
& \quad - \|t_n - y_n\|^2(2\delta_n^2 - \delta_n - bk\mu) \\
& \quad + 2(1-\delta_n^2)b^2\mu^2\|Ax_n\|^2 + 2(1-\delta_n^2)b^2(1-\mu)^2\|Ay_n\|^2.
\end{aligned} \tag{7}
$$

Since $\frac{3}{4} < \delta_n \leq 1$ and $b < \frac{3}{8k\mu}$, we have from (7) for all $n \in \mathbb{N}$,

$$
\|t_n - u\|^2 \ \leq \|x_n - u\|^2 + 4(1-\delta_n)b^2\mu^2\|Ax_n\|^2 + 4(1-\delta_n)b^2(1-\mu)^2\|Ay_n\|^2. \tag{8}
$$

In view of the fact that $u \in VI(A, C)$ and properties of P_C, we obtain

$$
\begin{aligned}
\|y_n - u\|^2 &= \|(1-\delta_n)(x_n - u) + \delta_n(P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n) - u)\|^2 \\
&\leq (1-\delta_n)\|x_n - u\|^2 + \delta_n\|P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n) - P_C(u)\|^2 \\
&\leq (1-\delta_n)\|x_n - u\|^2 + \delta_n\|x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n - u\|^2 \\
&= (1-\delta_n)\|x_n - u\|^2 + \delta_n[\|x_n - u\|^2 - 2\langle\lambda_n\mu Ax_n + \lambda_n(1-\mu)Ay_n, x_n - u\rangle \\
&\quad + \|\lambda_n\mu Ax_n + \lambda_n(1-\mu)Ay_n\|^2] \\
&= (1-\delta_n)\|x_n - u\|^2 + \delta_n[\|x_n - u\|^2 - 2\lambda_n\mu\langle Ax_n, x_n - u\rangle - 2\lambda_n(1-\mu)\langle Ay_n, x_n - u\rangle \\
&\quad + \|\lambda_n\mu Ax_n + \lambda_n(1-\mu)Ay_n\|^2] \\
&\leq (1-\delta_n)\|x_n - u\|^2 + \delta_n[\|x_n - u\|^2 + 2\lambda_n(1-\mu)\|Ay_n\|\|x_n - u\| \\
&\quad + \lambda_n^2\mu\|Ax_n\|^2 + \lambda_n^2(1-\mu)\|Ay_n\|^2] \\
&\leq (1-\delta_n)\|x_n - u\|^2 + \delta_n[\|x_n - u\|^2 + 2b(1-\mu)\Delta_n\|Ay_n\| \\
&\quad + b^2\mu\|Ax_n\|^2 + b^2(1-\mu)\|Ay_n\|^2] \\
&\leq \|x_n - u\|^2 + \delta_n[2b(1-\mu)\Delta_n\|Ay_n\| + b^2\mu^2\|Ax_n\|^2 + b^2(1-\mu)^2\|Ay_n\|^2] \\
&\leq \|x_n - u\|^2 + 2b(1-\mu)\Delta_n\|Ay_n\| + b^2\mu\|Ax_n\|^2 + b^2(1-\mu)\|Ay_n\|^2.
\end{aligned} \tag{9}
$$

Since S is asymptotically nonexpansive in the intermediate sense, in view of $S^n u = u$, we conclude that

$$
\begin{aligned}
\|z_n - u\|^2 &= \|(1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n S^n t_n - u\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|x_n - u\|^2 + \alpha_n\|y_n - u\|^2 + \beta_n\|S^n t_n - u\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|x_n - u\|^2 \\
&\quad + \alpha_n[\|x_n - u\|^2 + 2b(1-\mu)\Delta_n\|Ay_n\| + b^2\mu\|Ax_n\|^2 + b^2(1-\mu)\|Ay_n\|^2] \\
&\quad + \beta_n[(1+\gamma_n)\|t_n - u\|^2 + c_n] \\
&\leq (1-\alpha_n-\beta_n)\|x_n - u\|^2 \\
&\quad + \alpha_n[\|x_n - u\|^2 + 2b(1-\mu)\Delta_n\|Ay_n\| + b^2\mu\|Ax_n\|^2 + b^2(1-\mu)\|Ay_n\|^2] \\
&\quad + \beta_n(1+\gamma_n)[\|x_n - u\|^2 + 2(1-\delta_n)b^2\mu^2\|Ax_n\|^2 + 2(1-\delta_n)b^2(1-\mu)^2\|Ay_n\|^2] \\
&\quad + \beta_n c_n \\
&\leq \|x_n - u\|^2 + \beta_n\gamma_n\Delta_n^2 + 2b(1-\mu)\alpha_n\Delta_n\|Ay_n\| \\
&\quad + (b^2\mu\alpha_n + 2b^2\mu^2\beta_n(1-\delta_n)(1+\gamma_n))\|Ax_n\|^2 \\
&\quad + (b^2(1-\mu)\alpha_n + 2b^2(1-\mu)^2\beta_n(1-\delta_n)(1+\gamma_n))\|Ay_n\|^2 \\
&\quad + \beta_n c_n.
\end{aligned} \tag{10}
$$

This implies that $u \in C_n$. Therefore, $F(S) \cap VI(C, A) \subset C_n$.

Step 2. We prove that the sequence $\{x_n\}$ is well-defined and $F(S) \cap VI(C, A) \subset C_n \cap Q_n$ for all $n \geq 0$.

We prove this assertion by mathematical induction. For $n = 0$ we get $Q_0 = C$. Hence by step 1 we deduce that $F(S) \cap VI(C, A) \subset C_1 \cap Q_1$. Assume that x_k is defined and $F(S) \cap VI(C, A) \subset C_k \cap Q_k$ for some $k \geq 1$. Then y_k, z_k are well-defined elements of C. We notice that C_k is a closed convex subset of C since

$$C_k = \{z \in C : \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - z\rangle \leq d_n\|Ay_n\| + w_n\|Ax_n\|^2 + v_n\|Ay_n\|^2 + \vartheta_n\}.$$

It is easy to see that Q_k is closed and convex. Therefore, $C_k \cap Q_k$ is a closed and convex subset of C, since by the assumption we have $F(S) \cap VI(C,A) \subset C_k \cap Q_k$. This means that $P_{C_k \cap Q_k} x_0$ is well-defined.

By the definition of x_{k+1} and of Q_{k+1}, we deduce that $C_k \cap Q_k \subset Q_{k+1}$. Hence, $F(S) \cap VI(C,A) \subset Q_{k+1}$. Exploiting Step 1 we conclude that $F(S) \cap VI(C,A) \subset C_{k+1} \cap Q_{k+1}$.

Step 3. We claim that the following assertions hold:
(d) $\lim_{n\to\infty} \|x_n - x_0\|$ exists and hence $\{x_n\}$, as well as $\{\Delta_n\}$, is bounded.
(e) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
(f) $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Let $u \in F(S) \cap VI(C,A)$. Since $x_{n+1} = P_{C_n \cap Q_n} x_0$ and $u \in F(S) \cap VI(C,A) \subset C_n \cap Q_n$, we conclude that

$$\|x_{n+1} - x_0\| \leq \|u - x_0\|, \quad \forall n \geq 0. \tag{11}$$

This means that $\{x_n\}$ is bounded, and so are $\{y_n\}$, Ax_n and $\{Ay_n\}$, because of the Lipschitz-continuity of A. On the other hand, we have $x_n = P_{Q_n} x_0$ and $x_{n+1} \in C_n \cap Q_n \subset Q_n$. This implies that

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \qquad \forall n \geq 0. \tag{12}$$

In particular, $\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$ hence $\lim_{n\to\infty} \|x_n - x_0\|$ exists. It follows from (12) that

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0. \tag{13}$$

Since $x_{n+1} \in C_n$, we obtain

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + d_n \|Ay_n\| + w_n \|Ax_n\|^2 + v_n \|Ay_n\|^2 + \vartheta_n.$$

In view of $\lim_{n\to\infty} \gamma_n = 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \delta_n = 1$ and from the boundedness of $\{Ax_n\}$ and $\{Ay_n\}$ we infer that $\lim_{n\to\infty}(x_{n+1} - z_n) = 0$. Combining with (13) we deduce that $\lim_{n\to\infty}(x_n - z_n) = 0$.

Step 4. We claim that the following assertions hold:
(g) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.
(h) $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$.

In view of (3), $z_n = (1 - \alpha_n - \beta_n) x_n + \alpha_n y_n + \beta_n S^n t_n$, and $S^n u = u$, we obtain from (9) and (8) that

$$
\begin{aligned}
&\|z_n - u\|^2 \\
&= \|(1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n S^n t_n - u\|^2 \\
&\le (1-\alpha_n-\beta_n)\|x_n-u\|^2 + \alpha_n\|y_n-u\|^2 + \beta_n\|S^n t_n - u\|^2 \\
&\le (1-\alpha_n-\beta_n)\|x_n-u\|^2 \\
&\quad + \alpha_n[\|x_n-u\|^2 + 2b(1-\mu)\Delta_n\|Ay_n\| + \ b^2\mu\|Ax_n\|^2 + b^2(1-\mu)\|Ay_n\|^2] \\
&\quad + \beta_n[(1+\gamma_n)\|t_n-u\|^2 + c_n] \\
&\le (1-\alpha_n-\beta_n)\|x_n-u\|^2 \\
&\quad + \alpha_n[\|x_n-u\|^2 + 2b(1-\mu)\Delta_n\|Ay_n\| + b^2\mu\|Ax_n\|^2 + b^2(1-\mu)\|Ay_n\|^2] \\
&\quad + \beta_n(1+\gamma_n)[\|x_n-u\|^2 - (1-2\delta_n(1-\delta_n)\|x_n-y_n\|^2 - bk\mu) \\
&\qquad\qquad - (2\delta_n^2 - 1 - bk\mu)\|t_n-y_n\|^2 + 4(1-\delta_n)b^2\mu^2\|Ax_n\|^2 \\
&\qquad\qquad + 4(1-\delta_n)b^2(1-\mu)^2\|Ay_n\|^2] + \beta_n c_n \\
&\le \|x_n-u\|^2 + \beta_n\gamma_n\Delta_n^2 + \beta_n c_n \\
&\quad + 2b(1-\mu)\alpha_n\Delta_n\|Ay_n\| + [b^2\mu\alpha_n + 4b^2\mu^2\beta_n(1+\gamma_n)(1-\delta_n)]\|Ax_n\|^2 \\
&\quad + [b^2(1-\mu)\alpha_n + 4b^2(1-\mu)^2\beta_n(1+\gamma_n)(1-\delta_n)]\|Ay_n\|^2 \\
&\quad - [\beta_n(1+\gamma_n)(1-2\delta_n(1-\delta_n) - bk\mu)]\|x_n-y_n\|^2 \\
&\quad - [\beta_n(1+\gamma_n)(2\delta_n^2 - \delta_n - bk\mu)]\|t_n-y_n\|^2.
\end{aligned} \tag{14}
$$

Thus we have

$$
\begin{aligned}
&\beta_n(1+\gamma_n)(1-2\delta_n(1-\delta_n) - bk\mu)\|x_n-y_n\|^2 \\
&\le \|x_n-u\|^2 - \|z_n-u\|^2 + \beta_n\gamma_n\Delta_n^2 + \beta_n c_n \\
&\quad + 2b(1-\mu)\alpha_n\Delta_n\|Ay_n\| + [b^2\mu\alpha_n + 4b^2\mu^2\beta_n(1+\gamma_n)(1-\delta_n)]\|Ax_n\|^2 \\
&\quad + [b^2(1-\mu)\alpha_n + 4b^2(1-\mu)^2\beta_n(1+\gamma_n)(1-\delta_n)]\|Ay_n\|^2 \\
&\le (\|x_n-u\| + \|z_n-u\|)\|x_n-z_n\| + \beta_n\gamma_n\Delta_n^2 + \beta_n c_n \\
&\quad + 2b(1-\mu)\alpha_n\Delta_n\|Ay_n\| + [b^2\mu\alpha_n + 4b^2\mu^2\beta_n(1+\gamma_n)(1-\delta_n)]\|Ax_n\|^2 \\
&\quad + [b^2(1-\mu)\alpha_n + 4b^2(1-\mu)^2\beta_n(1+\gamma_n)(1-\delta_n)]\|Ay_n\|^2.
\end{aligned}
$$

Since $bk\mu < 3/8$ and $3/4 \le \delta_n \le 1$ for all $n \ge 0$, we have

$$\lim_{n\to\infty} \|x_n - y_n\|^2 = 0.$$

In the same manner, from (14), we conclude that

$$\lim_{n\to\infty} \|t_n - y_n\|^2 = 0.$$

Since A is k-Lipschitz continuous, we obtain $\|Ay_n - Ax_n\| \to 0$. On the other hand,

$$\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|,$$

which implies that $\|x_n - t_n\| \to 0$. Since $z_n = (1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n S^n t_n$, we have

$$z_n - x_n = -\alpha_n x_n + \alpha_n y_n + \beta_n(S^n t_n - x_n).$$

From $\|z_n - x_n\| \to 0,\ \alpha_n \to 0,\ \liminf_{n\to 0}\beta_n > 0$ and the boundedness of $\{x_n, y_n\}$ we deduce that $\|S^n t_n - x_n\| \to 0$. Thus we get $\|t_n - S^n t_n\| \to 0$. By the triangle inequality, we obtain

$$\begin{aligned}\|x_n - S^n x_n\| &\le \|x_n - t_n\| + \|t_n - S^n t_n\| + \|S^n t_n - S^n x_n\| \\ &\le \|x_n - t_n\| + \|t_n - S^n t_n\| + \sqrt{(1+\gamma_n)\|t_n - x_n\| + c_n}\ .\end{aligned}$$

So $\|x_n - S^n x_n\| \to 0$. Since $\|x_n - x_{n+1}\| \to 0$, it follows from Lemma 2.7 of Sahu, Xu and Yao [10] that $\|x_n - Sx_n\| \to 0$. By the uniform continuity of S, we obtain $\|x_n - S^m x_n\| \to 0$ as $n \to \infty$ for all $m \ge 1$.

Step 5. We claim that $\omega_w(x_n) \subset F(S) \cap VI(C, A)$, where

$$\omega_w(x_n) := \{x \in H : x_{n_j} \to x \text{ weakly for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

The proof of this step is similar to that of [2, Theorem 1.1, step 5] and we omit it.

A similar argument as mentioned in [1, Theorem 5, Step 6] proves the following assertion.

Step 6. The sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to the same point $q = P_{F(S)\cap VI(C,A)}(x_0)$, which completes the proof. □

For $\alpha_n = 0,\ \beta_n = 1$ and $\delta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 1, we get the following corollary.

Corollary 2. *Let C be a nonempty closed convex subset of a real Hilbert spaces H. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and let $S : C \to C$ be a uniformly continuous asymptotically nonexpansive mapping in the intermediate sense with nonnegative null sequences $\{\gamma_n\}$ and $\{c_n\}$.*

Suppose that $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $F(S) \cap VI(C, A)$ is nonempty and bounded. Set $\vartheta_n = \gamma_n\Delta_n + c_n$. Let μ be a constant in $(0, 1]$, and let $\{\lambda_n\}$ be a sequence in $[a, b]$ with $a > 0$ and $b < \frac{3}{8k\mu}$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n), \\ z_n = S^n P_C(x_n - \lambda_n Ay_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + \vartheta_n\} \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \quad \forall n \ge 0. \end{cases} \tag{16}$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in (16) are well-defined and converge strongly to the same point $q = P_{F(S)\cap VI(C,A)}(x_0)$.

In Theorem 1, if we set $\alpha_n = 0$ and $\beta_n = 1$ for all $n \in \mathbb{N}$ then the following result concerning variational inequality problems holds.

Corollary 3. *Let C be a nonempty closed convex subset of a real Hilbert spaces H. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and let $S : C \to C$ be a uniformly continuous asymptotically nonexpansive mapping in the intermediate sense nonnegative null sequences $\{\gamma_n\}$ and $\{c_n\}$.*

Suppose that $\sum_{n=1}^{\infty}\gamma_n < \infty$ *and* $F(S)\cap VI(C,A)$ *is nonempty and bounded. Let* μ *be a constant in* $(0,1]$*, let* $\{\lambda_n\}$ *be a sequence in* $[a,b]$ *with* $a>0$ *and* $b<\frac{3}{8k\mu}$*, and let* $\{\delta_n\}$ *be a sequence in* $[0,1]$ *such that* $\lim_{n\to\infty}\delta_n = 1$ *and* $\delta_n > \frac{3}{4}$ *for all* $n\geq 0$*. Set* $\Delta_n = \sup\{\|x_n-u\| : u\in F(S)\cap VI(C,A)\}$*,* $w_n = 4b^2\mu^2(1+\gamma_n)(1-\delta_n)$*,* $\vartheta_n = \gamma_n\Delta_n + c_n$ *for all* $n\geq 0$*.*

Let $\{x_n\}$*,* $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_0\in C \qquad \text{chosen arbitrarily,}\\ y_n = (1-\delta_n)x_n + \delta_n P_C(x_n - \lambda_n\mu A x_n - \lambda_n(1-\mu)Ay_n),\\ z_n = S^n P_C(x_n - \lambda_n A y_n),\\ C_n = \{z\in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + w_n\|Ax_n\|^2 + \vartheta_n\}\\ Q_n = \{z\in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}\\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \quad \forall n\geq 0. \end{cases} \tag{17}$$

Then the sequences $\{x_n\}$*,* $\{y_n\}$ *and* $\{z_n\}$ *in (17) are well-defined and converge strongly to the same point* $q = P_{F(S)\cap VI(C,A)}(x_0)$*.*

The following theorem is yet an other easy consequence of Theorem 1.

Corollary 4. *Let* H *be a real Hilbert space. Let* $A : H\to H$ *be a monotone and* k*-Lipschitz continuous mapping and let* $S : H\to H$ *be a uniformly continuous asymptotically nonexpansive mapping in the intermediate sense nonnegative null sequences* $\{\gamma_n\}$ *and* $\{c_n\}$*.*

Suppose that $\sum_{n=1}^{\infty}\gamma_n < \infty$ *and* $F(S)\cap A^{-1}(0)$ *is nonempty and bounded. Let* μ *be a constant in* $(0,1]$*, let* $\{\lambda_n\}$ *be a sequence in* $[a, 3b/4]$ *with* $0 < 4a/3 < b < \frac{3}{8k\mu}$*, and let* $\{\alpha_n\}$*,* $\{\beta_n\}$*, and* $\{\delta_n\}$ *be three sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n+\beta_n\leq 1, \forall n\geq 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$;

(iii) $\liminf_{n\to\infty}\beta_n > 0$;

(iv) $\lim_{n\to\infty}\delta_n = 1$ *and* $\delta_n > \frac{3}{4}$ *for all* $n\geq 0$*.*

Set

$$\begin{aligned} \Delta_n &= \sup\{\|x_n - u\| : u\in F(S)\cap A^{-1}(0)\},\\ d_n &= 2b(1-\mu)\alpha_n\Delta_n,\\ w_n &= b^2\mu\alpha_n + 4b^2\mu^2\beta_n(1-\delta_n)(1+\gamma_n),\\ v_n &= b^2(1-\mu)\alpha_n + 4b^2(1-\mu)^2\beta_n(1-\delta_n)(1+\gamma_n), \text{ and}\\ \vartheta_n &= \beta_n\gamma_n\Delta_n^2 + \beta_n c_n, \end{aligned}$$

for all $n\geq 0$*.*

Let $\{x_n\}$*,* $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_0\in C \text{ chosen arbitrarily,}\\ y_n = x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu)Ay_n,\\ z_n = (1-\beta_n)x_n - \alpha_n\mu Ax_n - \alpha_n\lambda_n(1-\mu)Ay_n + \beta_n S^n(x_n - \frac{\lambda_n}{\delta_n}Ay_n),\\ C_n = \{z\in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + d_n\|Ay_n\| + w_n\|Ax_n\|^2 + v_n\|Ay_n\|^2 + \vartheta_n\},\\ Q_n = \{z\in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \quad \forall n\geq 0. \end{cases} \tag{18}$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in (18) are well-defined and converge strongly to the same point $q = P_{F(S)\cap A^{-1}(0)}(x_0)$.

Proof. Replace λ_n by $\lambda'_n = \frac{\lambda_n}{\delta_n}$. Then $a \leq \lambda'_n < \frac{4}{3}\lambda_n < b < \frac{3}{8k\mu}$. For $C = H$, we have $P_C = I$ and $VI(C, A) = A^{-1}(0)$. In view of Theorem 1, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are well-defined and converge strongly to the same point $q = P_{F(S)\cap A^{-1}(0)}(x_0)$. $\square$

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