Strong Convergence Theorems of Ishikawa Iteration Process With Errors For Fixed points of Lipschitz Continuous Mappings in Banach Spaces

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Dedicated to Professor H. C. Lai

Abstract. Let $q > 1$ and E be a real q-uniformly smooth Banach space, K be a nonempty closed convex subset of E and $T : K \to K$ be a Lipschitz continuous mapping. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in K and $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ satisfying some restrictions. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iteration process with errors: $y_n = (1 - \beta_n)x_n + \beta_n T x_n + v_n$, $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n, n \geq 1$. Sufficient and necessary conditions for the strong convergence $\{x_n\}$ to a fixed point of T is established.

Keywords: Fixed point, Ishikawa iteration process with errors, $q-$uniformly smooth Banach space.

MR(2000) Subject Classfication: 47H09, 47H10, 47H17.

1. Introduction and Preliminaries

Let E be an arbitrary real Banach space and let $J_q (q > 1)$ denotes the generalized duality mapping

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from E into $2^{E^\star}$ given by

$$J_q(x) = \{f \in E^\star : \langle x, f\rangle = \|x\|^q = \|x\|\|f\|\},$$

where $E^\star$ denote the dual space of E and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between E and $E^\star$. In particular, J_2 is called the normalized duality mapping and it is usually denote by J. It is well known (see [11]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$, and that if $E^\star$ is strictly convex then J_q is single-valued. The single-valued generalized duality mapping will be denoted by j_q in the sequel.

Recall that a mapping T with domain $D(T)$ and range $R(T)$ in E is called strictly pseudocontractive [1] if for all $x, y \in D(T)$, there exist $\lambda > 0$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|x - y - (Tx - Ty)\|^2. \tag{1.1}$$

The mapping T is said to be Lipschitz continuous with constant $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in D(T).$$

We remark that a strictly pseudocontractive mapping is Lipschitz continuous with constant $L = (1+\lambda)/\lambda > 1$ (see, e.g., [14]).

The Mann iterative process (with errors) and the Ishikawa iterative process (with errors) have been extensively applied to approximating the solutions of nonlinear operator equations or fixed points of nonlinear mappings in Hilbert spaces or Banach spaces in the literature. See, e.g., [3-10]. In 1974, Rhoades [9] proved the following convergence theorem using the Mann iterative process.

Theorem 1.1. Let H be a real Hilbert space and K a nonempty compact convex subset of H. Let $T : K \to K$ be a strictly pseudocontractive mapping and let $\{\alpha_n\}$ be a real sequence satisfying the conditions: (i) $\alpha_0 = 1$; (ii) $0 < \alpha_n < 1, n \geq 1$; (iii)$\sum_{n=1}^{\infty} \alpha_n = \infty$; (iv) $\lim_{n\to\infty} \alpha_n = \alpha < 1$. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by the Mann iterative process,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0,$$

converges strongly to a fixed point of T.

Let E be a real Banach space. The modulus of smoothness of E is defined as the function $\rho_E : [0,\infty) \to [0,\infty)$:

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau\}.$$

E is said to be uniformly smooth if and only if $\lim_{\tau\to 0_+}(\rho_E(\tau)/\tau) = 0$. Let $q > 1$. The space E is said to be $q-$uniformly smooth (or to have a modulus of smoothness of power type $q > 1$), if there exists a constant $c_q > 0$ such that $\rho_E(\tau) \leq c_q\tau^q$. It is well known that Hilbert spaces are 2-uniformly smooth while if $1 < p \leq 2$, L_p, l_p, and the Sobolev spaces W_m^p are $p-$uniformly smooth. If $p \geq 2$, L_p, l_p and W_m^p are 2-uniformly smooth.

Theorem 1.2 [11]. Let $q > 1$ and E be a real Banach space. Then the following are equivalent:

(1) E is $q-$uniformly smooth.

(2) There exists a constant $c_q > 0$ such that for all $x, y \in E$

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q. \tag{1.1}$$

(3) There exists a constant d_q such that for all $x, y \in E$ and $t \in [0,1]$

$$\|(1-t)x+ty\|^q \geq (1-t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x-y\|^q, \tag{1.2}$$

where $\omega_q(t) = t^q(1-t) + t(1-t)^q$.

Furthermore, it was shown in [12, Remark 5] that if E is $q-$uniformly smooth $(q > 1)$, then for all $x, y \in E$, there exists a constant $L_\star > 0$ such that

$$\|j_q(x) - j_q(y)\| \leq L_\star\|x-y\|^{q-1}.$$

Recently, Osilike and Udomene [13] improved, unified and developed the above Theorem 1.1 and Browder and Petryshyn's corresponding result [1] in two aspects: (i) Hilbert spaces are extended to the setting of

$q-$uniformly smooth Banach spaces ($q > 1$); (ii) Mann iterative process is extended to the case of Ishikawa iterative process.

Theorem 1.3 [13, Theorem 2]. Let E be a real $q-$uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and $T : K \to K$ be a strictly pseudocontractive mapping with a nonempty fixed-point set $F(T)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ satisfying the conditions:

(i) $0 < a \leq \alpha_n^{q-1} \leq b < (q\lambda^{q-1}/c_q)(1 - \beta_n), \forall n \geq 1$ and for some constants $a, b \in (0, 1)$;

(ii) $\sum_{n=1}^{\infty} \beta_n^{\tau} < \infty$, where $\tau = \min\{1, (q - 1)\}$.

If $\{x_n\}$ is the sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iterative process

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, n \geq 1. \end{cases}$$

then $\{x_n\}$ converges weakly to a fixed point of T.

Let E be a real $q-$uniformly smooth Banach space, K be a nonempty closed convex (not necessarily bounded) subset of E, and $T : K \to K$ be a Lipschitz continuous mapping with constant $L > 0$ such that $F(T) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in E and $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0, 1]$ satisfying certain restrictions. Let $\{x_n\}$ be the sequence generated from $x_1 \in K$ by the Ishikawa iterative process with errors:

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n + v_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n, n \geq 1. \end{cases} \tag{1.3}$$

Here we assume that after perturbations by errors all x_n and y_n still belongs to K. We note that the Ishikawa iterative process with errors (1.3) was introduced by Liu [3] for approximating solutions of a nonlinear equation in Banach spaces. In this paper we will establish the sufficient and necessary conditions for the strong convergence of $\{x_n\}$ to a fixed point of T. The case where v_n equal to the zero vectors was studied in [14] under the assumption that T is a strictly pseudocontractive mapping.

The following lemma will be useful in the sequel.

Lemma 1.1 [10]. Let$\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and $a_{n+1} \leq a_n + b_n, \forall n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.

2. Main Results

Lemma 2.1. Let E be a real $q-$uniformly smooth Banach space and K be a nonempty convex subset of E, and $T : K \to K$ be a Lipschitz continuous mapping with constant $L > 0$ such that $F(T) \neq \emptyset$. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ be bounded sequences in E, and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$ satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, (ii) $\sum_{n=1}^{\infty} \|v_n\| < \infty$, and (iii) $\sum_{n=1}^{\infty} \alpha_n < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iterative process (1.3) with errors. Then

(i) $\|x_{n+1} - x^\star\|^q \leq (1 + \delta_n)\|x_n - x^\star\|^q + \theta_n, \forall n \geq 1, \forall x^\star \in F(T)$,

where

$$\delta_n = q\alpha_n(1 + L + L^2) + \alpha_n^q c_q(1 + qL(1 + L) + c_q L^q(2 + L)^{q-1}(1 + L))$$

and

$$\begin{aligned}\theta_n &= (q\alpha_n L + 1)\|x_n - x^\star\|^{q-1}\|v_n\| + \alpha_n^q c_q^2 L^q(2 + L)^{q-1}\|v_n\|^q \\ &\quad +q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q.\end{aligned}$$

(ii) There exists a constant $M > 0$ (e.g., $M = e^{\sum_{n=1}^{\infty} \delta_n}$) such that

$$\|x_{n+m} - x^\star\|^q \leq M\|x_n - x^\star\|^q + M \sum_{k=n}^{n+m-1} \theta_k, \forall n, m \geq 1, \forall x^\star \in F(T).$$

Proof. (i) Let $x^\star$ be an arbitrary element in $F(T)$. Then it follows from (1.1) and (1.3) that

$$\begin{aligned}\|x_{n+1} - x^\star\|^q &= \|(1 - \alpha_n)x_n + \alpha_n T y_n + u_n - x^\star\|^q \\ &\leq \|(1 - \alpha_n)x_n + \alpha_n T y_n - x^\star\|^q + q\langle u_n, j_q(x_{n+1} - u_n - x^\star)\rangle + c_q\|u_n\|^q \\ &\leq \|(1 - \alpha_n)x_n + \alpha_n T y_n - x^\star\|^q + q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q. \quad (2.1)\end{aligned}$$

Observe that

$$
\begin{aligned}
\|(1-\alpha_n)x_n+\alpha_n Ty_n-x^\star\|^q &= \|x_n-x^\star-\alpha_n(x_n-Ty_n)\|^q \\
&\leq \|x_n-x^\star\|^q-q\alpha_n\langle x_n-Ty_n, j_q(x_n-x^\star)\rangle \\
&\quad +\alpha_n^q c_q\|x_n-Ty_n\|^q \\
&\leq \|x_n-x^\star\|^q+q\alpha_n|\langle x_n-Ty_n, j_q(x_n-x^\star)\rangle| \\
&\quad +\alpha_n^q c_q\|x_n-Ty_n\|^q. \qquad (2.2)
\end{aligned}
$$

Since

$$
\begin{aligned}
|\langle x_n-Ty_n, j_q(x_n-x^\star)\rangle| &\leq \|x_n-Ty_n\|\|j_q(x_n-x^\star)\| \\
&= \|(x_n-x^\star)-(Ty_n-Tx^\star)\|\|x_n-x^\star\|^{q-1} \\
&\leq (\|x_n-x^\star\|+L\|y_n-x^\star\|)\|x_n-x^\star\|^{q-1} \\
&= \|x_n-x^\star\|^q+L\|y_n-x^\star\|\|x_n-x^\star\|^{q-1},
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_n-x^\star\| &\leq \|(1-\beta_n)x_n+\beta_n Tx_n+v_n-x^\star\| \\
&= \|(1-\beta_n)(x_n-x^\star)+\beta_n(Tx_n-Tx^\star)+v_n\| \\
&\leq (1-\beta_n)\|x_n-x^\star\|+\beta_n\|Tx_n-Tx^\star\|+\|v_n\| \\
&\leq (1-\beta_n+L\beta_n)\|x_n-x^\star\|+\|v_n\| \\
&\leq (1+L)\|x_n-x^\star\|+\|v_n\|,
\end{aligned}
$$

we have

$$
\begin{aligned}
|\langle x_n-Ty_n, j_q(x_n-x^\star)\rangle| &\leq \|x_n-x^\star\|^q+L\|x_n-x^\star\|^{q-1}((1+L)\|x_n-x^\star\|+\|v_n\|) \\
&= (1+L(1+L))\|x_n-x^\star\|^q+L\|x_n-x^\star\|^{q-1}\|v_n\|. \qquad (2.3)
\end{aligned}
$$

Also since

$$
\begin{aligned}
\|x_n-Ty_n\|^q &= \|(x_n-x^\star)-(Ty_n-Tx^\star)\|^q \\
&\leq \|x_n-x^\star\|^q-q\langle Ty_n-Tx^\star, j_q(x_n-x^\star)\rangle+c_q\|Ty_n-Tx^\star\|^q \\
&\leq \|x_n-x^\star\|^q+q\|Ty_n-Tx^\star\|\|j_q(x_n-x^\star)\|+c_qL^q\|y_n-x^\star\|^q \\
&\leq \|x_n-x^\star\|^q+qL\|y_n-x^\star\|\|x_n-x^\star\|^{q-1}+c_qL^q\|y_n-x^\star\|^q,
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_n-x^\star\|^q &\leq ((1+L)\|x_n-x^\star\|+\|v_n\|)^q \\
&= (2+L)^q(\tfrac{1+L}{2+L}\|x_n-x^\star\|+\tfrac{1}{2+L}\|v_n\|)^q \\
&\leq (2+L)^q(\tfrac{1+L}{2+L}\|x_n-x^\star\|^q+\tfrac{1}{2+L}\|v_n\|^q) \text{ (by Jensen's Inequality)} \\
&= (2+L)^{q-1}(1+L)\|x_n-x^\star\|^q+(2+L)^{q-1}\|v_n\|^q,
\end{aligned}
$$

we get

$$
\begin{aligned}
\|x_n-Ty_n\|^q &\leq \|x_n-x^\star\|^q+qL((1+L)\|x_n-x^\star\|+\|v_n\|)\|x_n-x^\star\|^{q-1} \\
&\quad +c_qL^q[(2+L)^{q-1}(1+L)\|x_n-x^\star\|^q+(2+L)^{q-1}\|v_n\|^q] \\
&= (1+qL(1+L)+c_qL^q(2+L)^{q-1}(1+L))\|x_n-x^\star\|^q \\
&\quad +c_qL^q(2+L)^{q-1}\|v_n\|^q+\|v_n\|\|x_n-x^\star\|^{q-1}. \qquad (2.4)
\end{aligned}
$$

Consequently from (2.1) -(2.4), we have

$$\begin{array}{rcl}\|x_{n+1}-x^\star\|^q & \le & \|x_n-x^\star\|^q+q\alpha_n[(1+L(1+L))\|x_n-x^\star\|^q+L\|x_n-x^\star\|^{q-1}\|v_n\|]\\ & & +\alpha_n^q c_q[(1+qL(1+L)+c_qL^q(2+L)^{q-1}(1+L))\|x_n-x^\star\|^q\\ & & +c_qL^q(2+L)^{q-1}\|v_n\|^q+\|v_n\|\|x_n-x^\star\|^{q-1}]+q\|u_n\|\|x_{n+1}-u_n-x^\star\|^{q-1}+c_q\|u_n\|^q\\ & = & (1+\delta_n)\|x_n-x^\star\|^q+\theta_n.\end{array}$$

Therefore (i) is valid.

(ii) It follows from conclusion (i) that for all $n, m \geq 1$ and $x^\star \in F(T)$

$$\begin{array}{rcl}\|x_{n+m}-x^\star\|^q & \le & (1+\delta_{n+m-1})\|x_{n+m-1}-x^\star\|^q+\theta_{n+m-1}\\ & \le & (1+\delta_{n+m-1})(1+\delta_{n+m-2})\|x_{n+m-3}-x^\star\|^q\\ & & +(1+\delta_{n+m-1})\theta_{n+m-2}+\theta_{n+m-1}\\ & \le & (1+\delta_{n+m-1})(1+\delta_{n+m-2})(1+\delta_{n+m-3})\|x_{n+m-2}-x^\star\|^q\\ & & +(1+\delta_{n+m-1})(1+\delta_{n+m-2})\theta_{n+m-3}+(1+\delta_{n+m-1})\theta_{n+m-2}+\theta_{n+m-1}\\ & \le & \ldots\\ & \le & e^{\sum_{k=n}^{n+m-1}\delta_k}\|x_n-x^\star\|^q+e^{\sum_{k=n}^{n+m-1}\delta_k}\sum_{k=n}^{n+m+1}\theta_k\\ & \le & M\|x_n-x^\star\|^q+M\sum_{k=n}^{n+m+1}\theta_k,\end{array}$$

where $M = e^{\sum_{k=1}^{\infty}\delta_k}$. This shows that conclusion (ii) is also valid.

Theorem 2.1. Let $q > 1$ and E be a real $q-$uniformly smooth Banach space, K be a nonempty closed convex subset of E, and $T : K \to K$ be a Lipschitz continuous mapping with constant $L > 0$ such that $F(T) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in E. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ satisfying $\sum_{n=1}^{\infty}\|u_n\| < \infty$, $\sum_{n=1}^{\infty}\|v_n\| < \infty$ and $\sum_{n=1}^{\infty}\alpha_n < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iterative process (1.3) with errors. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\{x_n\}$ is bounded and

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x_n, F(T))$ is the distance of x_n to set $F(T)$, i.e., $d(x_n, F(T)) = \inf_{u^\star\in F(T)}\|x_n - u^\star\|$.

Proof. The necessity is rather straightforword. We verify the sufficiency. Suppose that $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. First, from Lemma 2.1(i), we obtain

$$\|x_{n+1}-x^\star\|^q \le (1+\delta_n)\|x_n-x^\star\|^q+\theta_n, \quad \forall n \geq 1, x^\star \in F(T),$$

where

$$\delta_n = q\alpha_n(1+L+L^2) + \alpha_n^q c_q(1+qL(1+L)+c_qL^q(2+L)^{q-1}(1+L)$$

and

$$\begin{aligned}\theta_n &= (q\alpha_n L+1)\|x_n - x^\star\|^{q-1}\|v_n\| + \alpha_n^q c_q^2 L^q(2+L)^{q-1}\|v_n\|^q \\ &\quad + q\|u_n\|\|x_{n+1}-u_n-x^\star\|^{q-1} + c_q\|u_n\|^q.\end{aligned}$$

Since $\sum_{n=1}^{\infty}\|u_n\| < \infty$ and $\sum_{n=1}^{\infty}\|v_n\| < \infty$, we have $\sum_{n=1}^{\infty}\|u_n\|^q < \infty$ and $\sum_{n=1}^{\infty}\|v_n\|^q < \infty$. Note that $\{x_n\}$ and $\{u_n\}$ are both bounded. Thus, there is a number $\tilde{M} > 0$ such that $\|x_{n+1} - u_n - x^\star\| \leq \tilde{M}$ and $\|x_n - x^\star\| \leq \tilde{M}, \forall n \geq 1$. Hence

$$\sum_{n=1}^{\infty}\theta_n \leq \sum_{n=1}^{\infty}((qL+1)\tilde{M}^{q-1}\|v_n\| + c_q^2L^q(2+L)^{q-1}\|v_n\|^q + q\|u_n\|\tilde{M}^{q-1} + c_q\|u_n\|^q)$$

$$\leq (qL+1)\tilde{M}^{q-1}\sum_{n=1}^{\infty}\|v_n\| + c_q^2L^q(2+L)^{q-1}\sum_{n=1}^{\infty}\|v_n\|^q + q\tilde{M}^{q-1}\sum_{n=1}^{\infty}\|u_n\| + c_q\sum_{n=1}^{\infty}\|u_n\|^q < \infty.$$

On the other hand, we have

$$\sum_{n=1}^{\infty}\delta_n = q(1+L+L^2)\sum_{n=1}^{\infty}\alpha_n + c_q(1+qL(1+L)+c_qL^q(2+L)^{q-1}(1+L))\sum_{n=1}^{\infty}\alpha_n^q < \infty.$$

Also, observe that

$$\|x_{n+1} - x^\star\|^q \leq (1+\delta_n)\|x_n - x^\star\|^q + \theta_n \leq \|x_n - x^\star\|^q + \delta_n\tilde{M}^q + \theta_n. \tag{2.5}$$

This implies that

$$(d(x_{n+1}, F(T)))^q \leq [d(x_n, F(T))]^q + \delta_n\tilde{M}^q + \theta_n.$$

From Lemma 1.1 we know that the sequence $\{\|x_n - x^\star\|^q\}$ converges, so does the sequence $\{\|x_n - x^\star\|\}$. By Lemma 1.1 again, we infer that $\lim_{n\to\infty}(d(x_n, F(T))^q$ exists, so does $\lim_{n\to\infty} d(x_n, F(T))$. Since $\lim_{n\to\infty}\inf d(x_n, F(T)) = 0$, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Now, we claim that $\{x_n\}$ is Cauchy sequence. Indeed, according to Lemma 2.1(ii), we deduce that there exists a constant $M > 0$ such that

$$\|x_{n+m} - x^\star\| \leq M\|x_n - x^\star\|^q + M\sum_{k=n}^{n+m+1}\theta_k, \forall n, m \geq 1, x^\star \in F(T).$$

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} \theta_n < \infty$, for an arbitrary $\varepsilon > 0$, there exists an integer $N_1 \geq 1$ such that for all $n \geq N_1$

$$d(x_n, F(T)) < (\frac{\varepsilon}{3M})^{1/q} \cdot \frac{1}{2^{(q-1)/q}}, \text{ and } \sum_{k=n}^{\infty} \theta_k < \frac{\varepsilon}{6M} \cdot \frac{1}{2^{q-1}}.$$

Hence, $d(x_{N_1}, F(T)) < (\frac{\varepsilon}{3M})^{1/q} \cdot \frac{1}{2^{(q-1)/q}}$. This implies that there exists an $x_1^\star \in F(T)$ such that

$$d(x_{N_1}, x_1^\star) < (\frac{\varepsilon}{3M})^{1/q} \cdot \frac{1}{2^{(q-1)/q}}.$$

In view of Jensen's Inequality, we conclude that

$$\|x_{n+m} - x_n\|^q \leq 2^{q-1}(\|x_n - x_1^\star\|^q + \|x_{n+m} - x_1^\star\|^q). \tag{2.6}$$

Since for all $n \geq N_1$, we have

$$\begin{aligned} \|x_n - x_1^\star\|^q &\leq M\|x_{N_1} - x_1^\star\|^q + M \textstyle\sum_{k=N_1}^{n} \theta_k \\ &\leq M\|x_{N_1} - x_1^\star\|^q + M \textstyle\sum_{k=N_1}^{\infty} \theta_k \\ &\leq M\frac{\varepsilon}{3M} \cdot \frac{1}{2^{(q-1)}} + M\frac{\varepsilon}{6M} \cdot \frac{1}{2^{q-1}} \\ &= \frac{\varepsilon}{2} \cdot \frac{1}{2^{q-1}}, \end{aligned}$$

and

$$\begin{aligned} \|x_{n+m} - x_1^\star\|^q &\leq M\|x_{N_1} - x_1^\star\|^q + M \textstyle\sum_{k=N_1}^{n+m-1} \theta_k \\ &\leq M\|x_{N_1} - x_1^\star\|^q + M \textstyle\sum_{k=N_1}^{\infty} \theta_k \\ &\leq M\frac{\varepsilon}{3M} \cdot \frac{1}{2q-1} + M\frac{\varepsilon}{6M} \cdot \frac{1}{2^{q-1}} \\ &= \frac{\varepsilon}{2} \cdot \frac{1}{2^{q-1}}, \end{aligned}$$

so, from (2.6), we get

$$\|x_{n+m} - x_n\|^q \leq 2^{q-1}(\frac{\varepsilon}{2} \cdot \frac{1}{2^{q-1}} + \frac{\varepsilon}{2} \cdot \frac{1}{2^{q-1}}) = \varepsilon, \quad \forall n \geq N_1, m \geq 1.$$

This shows that $\{x_n\}$ is Cauchy sequence. Since the space E is complete, $\lim_{n\to\infty} x_n$ exists. Thus, we may assume that $\lim_{n\to\infty} x_n = u^\star$ and it is easy to show that $u^\star$ is a fixed point of T. This completes the proof.

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