GENERALIZED VECTOR VARIATIONAL INEQUALITIES WITH STAR-PSEUDOMONOTONE AND DISCONTINUOUS OPERATORS[1]

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Abstract. In this paper we deal with the following generalized vector variational inequality problem: let X, Y and Z be topological vector spaces, K be a convex set in X, D be a nonempty set in Y, and C be a closed convex cone in Z. Let $T : K \rightarrow 2^D$ be a multifunction and $f : K \times D \times K \rightarrow Z$ be a single-valued mapping. Find a point $\hat{x} \in K$ and $\hat{y} \in T(\hat{x})$ such that

$$f(\hat{x}, \hat{y}, z) \notin -\mathrm{int}C, \quad \forall z \in K.$$

We prove some existence theorems in which T can be discontinuous, or K can be unbounded, and a existence theorem in which T is pseudomonotone.

1. Introduction

Throughout this paper, let Z be a (Hausdorff) topological vector space, and let C be a closed and convex cone of Z such that $\text{int}C \neq \emptyset$ and $C \neq Z$. Here, $\text{int}C$ denotes the interior of C, while A^c, $\overline{A}$ and $\text{co}A$ denotes the complement, the closure and the convex hull of a subset A of Z, respectively.

A vector ordering in Z is defined by setting

$$z_1 \leq z_2 \text{ if and only if } z_2 - z_1 \in C.$$

Write

$$z_1 < z_2 \text{ if and only if } z_2 - z_1 \in \text{int}C,$$

and

$$z_1 \not< z_2 \text{ if and only if } z_2 - z_1 \notin \text{int}C.$$

Let X, Y be topological vector spaces, and let $K \subseteq X$ and $D \subseteq Y$ be nonempty. Let $T : K \rightarrow 2^D$ be a multifunction and $f : K \times D \times K \rightarrow Z$ be a single-valued mapping. The *generalized vector variational inequality* $\text{GVVI}(K, T, f)$ is the following problem:

(V) Find $\hat{x} \in K$ such that

$$\exists \hat{y} \in T(\hat{x}), \forall z \in K, \text{ we have } f(\hat{x}, \hat{y}, z) \not< 0.$$

Together with model (V) of $\text{GVVI}(K, T, f)$, the following auxiliary models is also of interest:

(W) Find $\hat{x} \in K$ such that

$$\forall z \in K, \ \exists \hat{y} \in T(\hat{x}), \text{ we have } f(\hat{x}, \hat{y}, z) \not< 0,$$

and the so-called *Minty vector variational inequality*:

(M) Find $\hat{x} \in K$ such that

$$\forall z \in K, \forall \hat{y} \in T(z), \text{ we have } f(\hat{x}, \hat{y}, z) \not< 0.$$

We will denote by S_V, S_W and S_M the solution set of problems (V), (W) and (M), respectively. A point $\hat{x} \in S_V$ is called a strong solution of $\text{GVVI}(K, T, f)$, while a point x_0 in S_W is called a weak solution.

It is interesting that problem (V) covers several generalized variational inequalities and equilibrium problems. Here are some examples.

(GVI) Put $Z = \mathbb{R}$, $C = \mathbb{R}_+$, $Y = X^* = D$ and $f(x, y, z) = \langle y, z - x \rangle$. Then (V) reduces to the generalized variational inequality problem of finding:

(1) $\qquad \hat{x} \in K$ and $\hat{y} \in T(\hat{x})$ such that $\langle \hat{y}, z - \hat{x} \rangle \geq 0, \forall z \in K$.

(EP) Put $Z = \mathbb{R}$, $C = \mathbb{R}_+$, $Y = X = D$ and $f(x, y, z) = g(x, z) + h(z) - h(x)$ where g is a real valued function on $K \times K$ such that $g(x, x) = 0$ for all $x \in K$. Then (V) reduces to the equilibrium problem of finding:

(2) $\hat{x} \in K$ such that $g(\hat{x}, z) + h(z) - h(\hat{x}) \geq 0, \forall z \in K$.

(VI) Put $Z = \mathbb{R}, C = \mathbb{R}_+, Y = D$ and $f(x, y, z) = \phi(y, z) - \phi(y, x)$ then (V) reduces to the general variational inequality problem of finding:

(3) $\hat{x} \in K$ and $\hat{y} \in T(\hat{x})$ such that $\phi(\hat{y}, z) - \phi(\hat{y}, \hat{x}) \geq 0, \forall z \in K$.

Problem **(VI)** has been studied recently by Aussel and Luc (2005).

The existence of solutions of **(GVI)** has been studied by many authors. See for instance Cubiotti (1997-2005), Li (2004), Yao (1992-1994), Yen (1994, 1995), and references therein. Also, results about the existence of weak solutions for generalized vector variational inequality problem were investigated by many authors. See for instance Ansari (1999), Chen and Craven (1990), Lee et al. (1997). Very few papers appeared in the literature discuss about the existence of strong solution of GVVI(K, T, f), however.

The aim of this paper is to derive some existence results of strong solutions for GVVI(K, T, f) with discontinuous operators and star-pseudomonotone operators. Namely, we will prove some existence theorems in which T can be discontinuous and K can be unbounded. Our results extends some previous results. Beside, an existence theorem involving star-pseudomonotne operators is also established. In order to obtain these results, we utilize the fixed point theorem due to Ky Fan and the scalarization method.

In Section 2 we recall some auxiliary results. Section 3 is destined for presenting existence results in the case discontinuous operators are involved. Section 4 will be devoted to a existence result for the case where the operators are star-pseudomonotone.

2. Preliminaries

Here are the basic setting in this paper. Let X and Y be topological vector spaces, and let Z be a locally convex space. Let C be a closed and convex proper cone in Z with nonempty interior. Let K be a nonempty convex subset of X, and D be a nonempty convex subset of Y. Let $T : K \to 2^D$ be a multifunction. We denote by Γ_T the graph of T, that is, the set $\{(x, y) \in X \times Y \mid y \in T(x)\}$.

Definition 2.1. (a) T is said to be *lower semicontinuous* (l.s.c., shortly) at $\overline{x} \in K$ if for any open set V in E such that $V \cap T(\overline{x}) \neq \emptyset$,

there exists a neighborhood U of $\overline{x}$ such that

$$V \cap T(x) \neq \emptyset, \quad \forall x \in U \cap K.$$

(b) T is said to be *upper semicontinuous* (u.s.c., shortly) at $\overline{x} \in K$ if for any open set V in E such that $T(\overline{x}) \subset V$, there exists a neighborhood W of $\overline{x}$ with the property that

$$T(x) \subset V, \quad \forall x \in W \cap K.$$

(c) T is said to be lower semicontinuous (resp., upper semicontinuous) on K if it is l.s.c. (resp., u.s.c.) at every point $x \in K$.

(d) A single-valued mapping $g : X \to Z$ is said to be $C-$*upper semicontinuous* on X if for every $z \in Z$ the set $g^{-1}(z - \mathrm{int}C)$ is open in X.

Tanaka (1997) proved that g is $C - upper\ semicontinuous$ on X if and only if for each fixed $x \in X$ and for any $y \in \mathrm{int}C$, there exists a neighborhood U of x such that

$$g(u) \in g(x) + y - \mathrm{int}C, \quad \forall u \in U.$$

(e) g is C-*lower semicontinuous* on X if for each fixed $x \in X$ and for any $y \in \mathrm{int}C$, there exists a neighborhood U of x such that

$$g(x) \in g(u) + y - \mathrm{int}C, \quad \forall u \in U.$$

(f) If g is C-lower semicontinuous and C-upper semicontinuous on X then g is called $C - continuous$ on X.

(g) A single-valued mapping $h : K \to Z$ is called C-*convex* if for every $x, x' \in K$ and $t \in [0, 1]$ one has

$$th(x) + (1 - t)h(x') - h(tx + (1 - t)x') \in C.$$

It is called C-*strongly convex* if for every $x, x' \in K$ and $t \in [0, 1]$ one has

$$th(x) + (1 - t)h(x') - h(tx + (1 - t)x') \in \mathrm{int}C \cup \{0\}.$$

If $-h$ is C-convex (resp., C-strongly convex) then h is said to be C-*concave* (resp., C-*strongly concave*) on K.

We continue by recalling the scalarization method. Let Z^* be the topological dual of Z. The set

$$C^* := \{z^* \in Z^* : \langle z^*, z \rangle \geq 0 \text{ for all } z \in C\}$$

is the polar cone of C. Note that C^* has a weakly-star compact base B, that is, $C^* = \bigcup_{t>0} tB$, and B is convex and weakly-star compact with $0 \notin B$ (see Luc, 1989). When $\mathrm{int}C \neq \emptyset$ and $\overline{z} \in \mathrm{int}C$, $\overline{z} \neq 0$, the set

$$B = \{z^* \in C^* : \langle z^*, \overline{z} \rangle = 1\}$$

is a weakly- star compact convex base for C^*. We put $C^*_+ = C^* \backslash \{0\}$.

Note that from the bipolar theorem we have

(4) $z \in C \Longleftrightarrow [\langle z^*, z \rangle \geq 0, \forall z^* \in C^*] \Longleftrightarrow [\langle z^*, z \rangle \geq 0, \forall z^* \in B]$

(5) $z \in \mathrm{int}C \Longleftrightarrow [\langle z^*, z \rangle > 0, \forall z^* \in C_+^*] \Longleftrightarrow [\langle z^*, z \rangle > 0, \forall z^* \in B].$

(see Jeyakumar et al., (1993) for developments).

The following proposition is a useful tool for the scalarization method.

Proposition 2.2. *Let g be a single-valued mapping from K into Z and $u^* \in C_+^*$. Let $\phi : K \to \mathbb{R}$ be a mapping defined by $\phi(x) = \langle u^*, g(x) \rangle$ for all $x \in K$. Then the following assertions are valid:*

(a) If g is $C-$convex (resp., $C-$concave) then ϕ is convex (resp., concave);

(b) If g is $C-$upper semicontinuous (resp., $C-$lower semicontinuous) then ϕ u.s.c. (resp., l.s.c.)

Proof. (a) Since g is $C-$convex, then for all $x, x' \in K$ and $t \in [0, 1]$ one has
$$tg(x) + (1 - t)g(x') - g(tx + (1 - t)x') \in C.$$
By (5) we have $\langle u^*, tg(x) + (1 - t)g(x') - g(tx + (1 - t)x') \rangle \geq 0$. Hence
$$t\langle u^*, g(x) \rangle + (1 - t)\langle u^*, g(x') \rangle \geq \langle u^* g(tx + (1 - t)x') \rangle.$$
This implies that
$$t\phi(x) + (1 - t)\phi(x') \geq \phi(tx + (1 - t)x')$$

and so ϕ is convex. By a similar argument we show that if g is $C-$concave then ϕ is concave.

(b) Let $x_n \to x$. We want to show that $\limsup_{n \to \infty} \phi(x_n) \leq \phi(x)$. Choose $y_j \in \mathrm{int}C$ such that $y_j \to 0$. Then for each $j > 0$ there exists a neighborhood U_j of x such that
$$g(u) \in g(x) + y_j - \mathrm{int}C, \quad \forall u \in U_j.$$
Therefore for each j there exists $n_j > 0$ such that
$$g(x_n) \in g(x) + y_j - \mathrm{int}C, \quad \forall n > n_j.$$
By (5) it follows that $\langle u^*, g(x_n) - g(x) - y_j \rangle < 0$. Hence
$$\begin{aligned}
\phi(x_n) &= \langle u^*, g(x_n) - g(x) - y_j + g(x) + y_j \rangle \\
&= \langle u^*, g(x_n) - g(x) - y_j \rangle + \langle u^*, g(x) + y_j \rangle \\
&< \langle u^*, g(x) \rangle + \langle u^*, y_j \rangle
\end{aligned}$$
for all $n > n_j$. This implies that
$$\limsup_{n \to \infty} \phi(x_n) \leq \langle u^*, g(x) \rangle + \langle u^* y_j \rangle.$$

Letting $j \to \infty$ and noting that $\langle u^*, y_j \rangle \to 0$, we obtain

$$\limsup_{n \to \infty} \phi(x_n) \leq \langle u^*, g(x) \rangle = \phi(x).$$

By the same way, we obtain the second assertion of (b). $\qquad \square$

One of the tools in deriving our results is Ky Fan theorem. We state it below for the convenience.

Theorem 2.3 (Ky Fan). *Let K be a nonempty subset of a topological vector space X and $G : K \to 2^X$ be a multifunction with closed values such that the convex hull of every finite subset $\{x_1, x_2, ..., x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n G(x_i)$. If there exists $x_0 \in K$ such that $G(x_0)$ is compact, then $\bigcap_{x \in K} G(x) \neq \emptyset$.*

We recall that the multifunction G satisfying the property in Theorem 2.3 is called a KKM mapping.

We also need the following Sion's minimax theorem (see Jeyakumar et al., 1993).

Theorem 2.4 (Sion). *Let A and B be convex subset of some real topological vector spaces with B compact, and let $p : A \times B \to \mathbb{R}$. If $p(., b)$ is lower semicontinuous and quasiconvex on A for all $b \in B$, and if $p(a, .)$ is upper semicontinuous and quasiconcave on B for all $a \in A$, then*

$$\inf_{a \in A} \max_{b \in B} p(a, b) = \max_{b \in B} \inf_{a \in A} p(a, b).$$

3. Existence of solutions in case of discontinuous operators

In addition to the basic assumptions stated in the beginning of Section 2, we further assume in this section that $f : K \times D \times K \to Z$ is a single-valued mapping satisfying some of the following conditions

   (i) the set $T(x)$ is nonempty, weakly compact, and convex for all $x \in K$;
  (ii) there exists $u^* \in C_+^*$ such that for each $z \in K$, the set

$$F(z) = \{x \in K \mid \sup_{y \in T(x)} \langle u^*, f(x, y, z) \rangle \geq 0\}$$

      is closed;
 (iii) for each $x \in K$ and $y \in T(x)$ one has $f(x, y, x) = 0$;
 (iv) for each $x \in K$ and $y \in T(x)$, the function $f(x, y, \cdot)$ is C−convex and C−l.s.c. on K;
  (v) for each $(x, z) \in K \times K$, the function $f(x, \cdot, z)$ is C−concave and C−u.s.c. on D.

Theorem 3.1. *Let X be a dual Banach space and Y be a locally convex space. Assume conditions (i), (ii), (iii), (vi), and (v) hold. Assume further that $F(z)$ is weakly* closed for all z in K, and*

> *(vi) There is a bounded subset K_0 of K such that for each $x \in K \backslash K_0$ there exists $z \in K_0$ with $f(x, y, z) < 0$ for all $y \in T(x)$.*

Then there exists a point $\hat{x} \in K_0$ such that

$$\max_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, z) \rangle \geq 0, \quad \forall z \in K.$$

Moreover, there exists $\hat{y} \in T(\hat{x})$ such that

(6)
$$f(\hat{x}, \hat{y}, z) \not< 0, \quad \forall z \in K.$$

Proof. We choose $r > 0$ such that int B_r contains K_0, where $B_r = \{x \in X : \|x\| \leq r\}$ is the closed ball in X with radius r. Set $\Omega = K \cap B_r$. We define a multifunction $G : \Omega \to 2^X$ by the formula

$$G(z) = \{x \in \Omega \mid \sup_{y \in T(x)} \langle u^*, f(x, y, z) \rangle \geq 0\} \text{ for } z \in \Omega.$$

By Proposition 2.2 , the function $y \mapsto \langle u^*, f(x, y, z) \rangle$ is concave and u.s.c. In particular, for each real number δ, the convex set $\{y \in Y : \langle u^*, f(x, y, z) \rangle \geq \delta\}$ is closed, and thus weakly closed, in the locally convex space Y. Hence, this function is also weakly u.s.c. Since $T(x)$ is a weakly compact set, we have

$$G(z) = \{x \in \Omega \mid \max_{y \in T(x)} \langle u^*, f(x, y, z) \rangle \geq 0\}.$$

We claim that G is a KKM mapping. Indeed, by (iii) we have $z \in G(z)$ for all $z \in \Omega$. Hence $G(z) \neq \emptyset$ for all $z \in \Omega$. Since

$$G(z) = F(z) \cap B_r,$$

$G(z)$ is a weak* compact subset of Ω, by Banach-Alaoglu's Theorem. Taking any $z_1, z_2 \in \Omega$ we show that co$\{z_1, z_2\} \subset G(z_1) \cup G(z_2)$. Let $z \in \text{co}\{z_1, z_2\}$. Then $z = tz_1 + (1-t)z_2$ for some $t \in [0, 1]$. When $t = 0$ or $t = 1$, the claim is trivial. Consider $t \in (0, 1)$, with $z \notin G(z_i); i = 1, 2$. Since $z \in \Omega$, we obtain

$$\max_{y \in T(z)} \langle u^*, f(z, y, z_i) \rangle < 0; \ i = 1, 2.$$

By (iv) and Proposition 2.2, the function $\langle u^*, f(x, y, \cdot) \rangle$ is convex. By (iii) we have

$$0 = \max_{y \in T(z)} \langle u^*, f(z, y, z) \rangle$$
$$\leq t \max_{y \in T(z)} \langle u^*, f(z, y, z_1) \rangle + (1-t) \max_{y \in T(z)} \langle u^*, f(z, y, z_2) \rangle,$$
$$< 0,$$

which is an absurd. Thus we must have $z \in G(z_1) \cup G(z_2)$ and so $\text{co}\{z_1, z_2\} \subset G(z_1) \cup G(z_2)$. By a similar argument we can show that

$$\text{co}\{z_1, z_2, ..., z_n\} \subset \bigcup_{i=1}^{n} G(z_i)$$

for any finite subset $\{x_1, x_2, ..., x_n\}$ in Ω.

By Theorem 2.3,

$$\exists \hat{x} \in \Omega \text{ such that } \hat{x} \in \bigcap_{z \in \Omega} G(z).$$

In other words,

(7) $$\exists \hat{x} \in \Omega, \quad \max_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, z) \rangle \geq 0, \quad \forall z \in \Omega.$$

By (vi), we see that $\hat{x} \in K_0$.

We need to show that

(8) $$\max_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, z) \rangle \geq 0, \quad \forall z \in K.$$

Take any $z \in K$. Since $\hat{x} \in \text{int } B_r$ we have $\hat{x} + t(z - \hat{x}) \in K \cap \text{int } B_r$ for a sufficiently small $t \in (0, 1)$. Then (7) implies that

$$\max_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, \hat{x} + t(z - \hat{x})) \rangle$$

(9) $$= \max_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, (1 - t)\hat{x} + tz) \rangle \geq 0.$$

By the convexity of the function $\langle u^*, f(x, y, \cdot) \rangle$, we obtain from (9) the following:

$$0 \leq \max_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, (1 - t)\hat{x} + tz) \rangle$$

$$\leq t \max_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, z) \rangle + (1 - t) \max_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, \hat{x}) \rangle$$

Since $f(\hat{x}, y, \hat{x}) = 0$, it follows (8).

To complete the proof we will use the Sion's theorem (Theorem 2.4). Put $p(z, y) = \langle u^*, f(\hat{x}, y, z) \rangle$. By the assumptions of the theorem and Proposition 2.2, the function $p(\cdot, y)$ is l.s.c. and convex. Also, the function $p(z, \cdot)$ is u.s.c. and concave, and thus weakly u.s.c. By Theorem 2.4 and (8) we get

$$0 \leq \inf_{z \in K} \max_{y \in T(\hat{x})} p(z, y) = \max_{y \in T(\hat{x})} \inf_{z \in K} p(z, y).$$

Since $y \mapsto p(z, y)$ is weakly u.s.c., the function $y \mapsto \inf_{z \in K} p(z, y)$ is also weakly u.s.c. Hence, there exists $\hat{y} \in T(\hat{x})$ such that

$$\inf_{z \in K} p(z, \hat{y}) = \max_{y \in T(\hat{x})} \inf_{z \in K} p(z, y) \geq 0.$$

It follows that $\langle u^*, f(\hat{x}, \hat{y}, z) \rangle \geq 0, \forall z \in K$. This is equivalent to $\langle u^*, -f(\hat{x}, \hat{y}, z) \rangle \leq 0, \forall z \in K$. By (5) we have $-f(\hat{x}, \hat{y}, z) \notin \text{int} C$ for all $z \in K$, and (6) is obtained. The proof of the theorem is complete. $\square$

In Theorem 3.1, condition (ii) is checked for a particular element $u^* \in C_+^*$. It seems that we do not know when such u^* exists. Fortunately, under certain conditions, for an example, T is upper semicontinuous and $f(\cdot, \cdot, z)$ is $C-$continuous, we can choose any $u^* \in C_+^*$.

The following example illustrates our result.

Example 3.2. Let $X = \mathbb{R}, K = [0, 1] \subset X, Y = \mathbb{R}, D = [1, 4], Z = \mathbb{R}^2$ and
$$C = \mathbb{R}_+^2 = \{(x, y) \mid x \geq 0, y \geq 0\}.$$
Let T and f be defined by
$$T(x) = \begin{cases} [2, 4], & \text{if } x = 0; \\ \{1\}, & \text{if } 0 < x \leq 1, \end{cases}$$
and
$$f(x, y, z) = (f_1(x, y, z), f_2(x, y, z)),$$
where
$$f_1(x, y, z) = y(z^2 - x^2) \text{ and } f_2(x, y, z) = y(z - x).$$
Then the set $\{0\} \times [2, 4]$ is a solution set of GVVI(K, T, f). Moreover, T is not upper semicontinuous on $[0, 1]$.

Proof. First we note that
$$C^* = \{(u, v) \mid u \geq 0, v \geq 0\}; \quad -\text{int} C = \{(x, y) \mid x < 0, y < 0\}.$$
Taking $u^* = (1, 0)$ we want to check all conditions of Theorem 3.1.

(i) It is obvious that $T(x)$ is a nonempty compact convex set for each $x \in [0, 1]$.

(ii) For each $z \in [0, 1]$ we have
$$\{x \in K \mid \sup_{y \in T(x)} \langle u^*, f(x, y, z) \rangle \geq 0\}$$
$$= \{x \in [0, 1] \mid \sup_{y \in T(x)} y(z^2 - x^2) \geq 0\}$$
$$= [0, z].$$
This is a closed set.

(iii) For each $x \in M$ and $y \in T(x)$ we have $f(x, y, x) = (0, 0)$.

(iv) For each $x \in K$ and $y \in T(x) \subset [1, 4]$, the functions $f_1(x, y, \cdot)$ and $f_2(x, y, \cdot)$ are convex and continuous. Hence, f is $C-$convex and $C-$lower semicontinuous.

(v) For each $x, z \in K \times K$ the functions $f_1(x, ., z)$ and $f_2(x, \cdot, z)$ are linear and continuous.

(vi) is plain, as K is compact.

Thus all conditions of Theorem 3.1 are fulfilled. Taking $\hat{x} = 0$ and $\hat{y} \in T(0) = [2, 4]$, we have

$$f(0, \hat{y}, z) = (yz^2, yz^2) \not< 0, \ \forall z \in [0, 1].$$

Hence the set $\{0\} \times [2, 4]$ is a solution set of the problem.

It is observed that T is not upper semicontinuous on K. Indeed, $x_n = 1/n \to 0$, $y_n = 1 \in T(x_n)$ but $1 \notin T(0) = [2, 4]$. $\square$

Theorem 3.1 requires that X is a dual Banach space. This requirement guarantees the weak* compactness of weakly* closed bounded sets. The following results relax this assumption by strengthening the coercive condition (vi).

Theorem 3.3. *Let Y be a locally convex space (resp. topological vector space). In addition to conditions (i), (ii), (iii), (iv) and (v), suppose further that $T(x)$ is weakly compact (resp. compact) each x in K, $F(z)$ is weakly closed (resp. closed) for each z in K, and*

(vi)' there exists a weakly compact (resp. compact) K_0 of K and $z_0 \in K$ such that for each $x \in K \backslash K_0$ one has $f(x, y, z_0) < 0$ for all $y \in T(x)$.

Then there exists a point $\hat{x} \in K$ and $\hat{y} \in T(\hat{x})$ such that

$$f(\hat{x}, \hat{y}, z) \not< 0, \quad \forall z \in K.$$

Proof. Assume that Y is locally convex. By condition (v),

$$F(z) = \{x \in K \mid \max_{y \in T(x)} \langle u^*, f(x, y, z) \rangle \geq 0\}.$$

We claim that $F(z_0) \subset K_0$. Indeed, suppose that the claim is false. Then there is an $x \in F(z_0) \setminus K_0$ and so $x \in K \setminus K_0$. By (vi), $-f(x, y, z_0) \in \text{int} C$ for all $y \in T(x)$. By (5), it follows that $\langle u^*, -f(x, y, z_0) \rangle > 0$ for all $y \in T(x)$. Hence

$$\max_{y \in T(x)} \langle u^*, f(x, y, z_0) \rangle < 0.$$

This implies that $x \notin F(z_0)$, which is absurd. Thus $F(z_0)$ is a weakly compact set. As in the proof of Theorem 3.1, we can show that $F(z)$ is a KKM mapping, and the desired conclusion follows accordingly. The case Y is not locally convex is similar. $\square$

Theorem 3.4. *Let X and Y be a topological vector spaces. Assume conditions (i), (ii), (iii), (iv) and (v) hold. Suppose further that*

(vi)" *there exist compact sets $K_1 \subseteq K_2$ in K such that K_1 is finite-dimensional, and for each $x \in K \setminus K_2$ there exists $z \in K_1$ with $f(x, y, z) < 0$ for all $y \in T(x)$.*

Then there exists a point $\hat{x} \in K_2$ and $\hat{y} \in T(\hat{x})$ such that

$$f(\hat{x}, \hat{y}, z) \not< 0, \quad \forall z \in K.$$

Proof. We denote by $\mathcal{F}$ the set of all finite-dimensional subspaces of X containing K_1. Fix any $S \in \mathcal{F}$ and put

$$\Omega = K \cap S, \quad T^S = T \mid_\Omega, \quad f^S = f \mid_{\Omega \times D \times \Omega}.$$

The task is to check that Theorem 3.1 can be applied to the problem $\text{GVVI}(\Omega, T^S, f^S)$, where Ω plays a role as K in Theorem 3.1.

(a_1) For each $x \in \Omega$ we have $T(x) = T^S(x) \neq \emptyset$. Hence condition (i) of Theorem 3.1 is valid.

(a_2) For each $z \in \Omega$, we have

$$\{x \in \Omega \mid \sup_{y \in T(x)} \langle u^*, f(x, y, z) \rangle \geq 0\}$$
$$= \{x \in K \mid \sup_{y \in T(x)} \langle u^*, f(x, y, z) \rangle \geq 0\} \cap S.$$

Since S is finite-dimensional, the set

$$\{x \in \Omega \mid \sup_{y \in T(x)} \langle u^*, f(x, y, z) \rangle \geq 0\}$$

is closed. This implies that condition (ii) of Theorem 3.1 is valid.

(a_3) Conditions (iii)-(vi) are automatically fulfilled.

By Theorem 3.1, there exists $x_S \in \Omega$ such that

$$\max_{y \in T^S(x_S)} \langle u^*, \ f^S(x_S, \hat{y}, z) \rangle \geq 0, \quad \forall z \in \Omega.$$

Since $T^S(x_S) = T(x_S)$, $f^S(x_S, \hat{y}, z) = f(x_S, \hat{y}, z)$ we have

$$(10) \qquad \max_{y \in T(x_S)} \langle u^*, \ f(x_S, \hat{y}, z) \rangle \geq 0, \quad \forall z \in K \cap S.$$

Put $Q_S = \{x \in K \mid x \text{ satisfies (10)}\}$. Then $Q_S \neq \emptyset$ because $x_S \in Q_S$. Moreover, the family $\{\overline{Q}_S : S \in \mathcal{F}\}$ has the finite intersection property. Indeed, taking any $V_1, V_2 \in \mathcal{F}$ and putting $M = \text{span}\{V_1, V_2\}$, we have $Q_M \subseteq Q_{V_1} \cap Q_{V_2}$. This implies that $\overline{Q}_M \subseteq \overline{Q_{V_1} \cap Q_{V_2}} \subseteq \overline{Q}_{V_1} \cap \overline{Q}_{V_2}$.

By condition (vi), $\overline{Q}_S \subset K_2$, which is compact. Then one has $\bigcap_{S \in \mathcal{F}} \overline{Q}_S \neq \emptyset$. Thus there exists $\hat{x} \in K_2$ such that $\hat{x} \in \overline{Q}_S$ for all

$S \in \mathcal{F}$. Fix any $S \in \mathcal{F}$. Then there exists a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in Q_S such that $x_\alpha \to \hat{x}$. By the definition of Q_S one has

$$(11) \qquad \max_{y \in T(x_\alpha)} \langle u^*, \ f(x_\alpha, y, z) \rangle \geq 0, \quad \forall z \in K \cap S.$$

Fixing $z \in K \cap S$ and using assumption (ii), we obtain from (11) that

$$\max_{\hat{y} \in T(\hat{x})} \langle u^*, \ f(\hat{x}, \hat{y}, z) \rangle \geq 0.$$

In summary, we have proved that

$$(12) \qquad \exists \hat{x} \in K_2, \forall z \in K \cap S, \quad \max_{y \in T(\hat{x})} \langle u^*, \ f(\hat{x}, y, z) \rangle \geq 0.$$

We now take any $v \in K$ and put $S' = \mathrm{span}\{S, v\}$. Since $\hat{x}$ also satisfies (12) for S' we have

$$\exists \hat{x} \in K_2, \quad \max_{y \in T(\hat{x})} \langle u^*, \ f(\hat{x}, \hat{y}, v) \rangle \geq 0.$$

Thus we have

$$\exists \hat{x} \in K_2, \forall v \in K, \quad \max_{y \in T(\hat{x})} \langle u^*, \ f(\hat{x}, \hat{y}, v) \rangle \geq 0.$$

Using a similar argument as in the proof of Theorem 3.1, we show that there exists $\hat{y} \in T(\hat{x})$ such that $f(\hat{x}, \hat{y}, z) \not< 0$ for all $z \in K$. The proof is complete. $\qquad \square$

Theorem 3.5. *Let X, Y be locally convex spaces. In addition to conditions (i), (ii), (iii), (iv), (v) and (vi), suppose further that K_0 is weakly compact and $F(z)$ is weakly closed for all z in K. Then there exists a point $\hat{x} \in K_0$ and $\hat{y} \in T(\hat{x})$ such that*

$$f(\hat{x}, \hat{y}, z) \not< 0, \quad \forall z \in K.$$

Proof. For each absolutely convex weakly compact subset S of X, let X_S be the subspace of X generated by S. Namely,

$$X_S = \bigcup_{t>0} tS.$$

Equip X_S with the norm defined by the gauge of S; namely,

$$\|x\| = \inf\{t > 0 : x \in tS\}.$$

Then $(X_S, \|\cdot\|)$ is a dual Banach space by the Dixmier-Ng Theorem, see (Dixmier, 1948) and (Ng, 1972). Indeed, the predual of $(X_S, \|\cdot\|)$ consists of those linear functionals on X continuous on S. Hence, the weak* topology of $(X_S, \|\cdot\|)$ is finer than the weak topology $\sigma(X_S, X')$ of X_S inherited from X. As a result, $F(z)$ is weakly* closed in $(X_S, \|\cdot\|)$ for all z in $K \cap X_S$.

Let $\mathcal{F}$ be the set of all dual Banach spaces $(X_S, \|\cdot\|)$, where S is an absolutely convex weakly compact subset of X containing K_0. We can then proceed as in the proof of Theorem 3.4. $\square$

We remark that in Theorem 3.5 we can assume instead that X is a topological vector space such that X' separates points in X. This condition is enough to ensure the conclusion of the Dixmier-Ng Theorem, as one can check by going through the proof of Ng (1972) carefully.

4. Existence of solution in case of star-pseudomonotone operators

We first introduce some notations on the star-pseudomonotone operator.

Definition 4.1. (a) The operator $T : K \rightarrow 2^D$ is called *star-pseudomonotone with respect to f and $u^* \in C_+^*$* if for any $(x_1, y_1), (x_2, y_2) \in \Gamma_T$,

$$\langle u^*, f(x_1, y_1, x_2) \rangle \geq 0 \quad \text{implies} \quad \langle u^*, f(x_1, y_2, x_2) \rangle \geq 0.$$

(b) T is called *star-quasimonotone with respect to f and $u^* \in C_+^*$* if for any $(x_1, y_1), (x_2, y_2) \in \Gamma_T$,

$$\langle u^*, f(x_1, y_1, x_2) \rangle > 0 \quad \text{implies} \quad \langle u^*, f(x_1, y_2, x_2) \rangle \geq 0.$$

(c) T is called *star-properly quasimonotone with respect to f and $u^* \in C_+^*$* if for any finite set $\{x_1, x_2, ..., x_2\}$ in K and $x \in \mathrm{co}\{x_1, x_2, ..., x_2\}$, there exists some $i \in \{1, 2, ..., n\}$ such that

$$\langle u^*, f(x, y, x_i) \rangle \geq 0, \quad \forall y \in T(x_i).$$

The notion of quasimonotone operators and pseudomonotone operators were first introduced by Karamardian and Schible (1990). For this case we put $Y = D = X^*$, $Z = \mathbb{R}$, $C = \mathbb{R}+$, $f(x, y, z) = \langle y, z - x \rangle$ and $u^* = 1$. Then T is a pseudomonotone operator (resp., quasimonotone) in the sense of theirs. While the notion of proper quasimonotonicity was introduced by Daniilidis and Hadjisavvas (1999). Also, putting $f(x, y, z) = \phi(y, z) - \phi(y, x)$, $Z = \mathbb{R}$, $C = \mathbb{R}+$ and $u^* = 1$ we obtain the notions of ϕ-quasimonotone operators and ϕ-pseudomonotone operators which have introduced recently by Aussel and Luc (2005).

The following are some relations among star-monotone operators.

Proposition 4.2. *(a) If T is star-pseudomonotone with respect to f and u^* then T is also star-quasimonotone with respect to f and u^*.*
(b) Suppose T is star-pseudomonotone with respect to f and u^*, and the following conditions are fulfilled:

(i) *for each $x \in K$ and each $y \in T(x)$, $f(x, y, x) = 0$;*
(ii) *for each $x \in K$ and for each $y \in T(x)$, the function $f(x, y, \cdot)$
    is $C-$convex.*

*Then T is also star-properly quasimonotone with respect to f and
u^*.*

Proof. (*a*) is obvious. For (*b*), let $\{x_1, x_2, ..., x_n\} \subset K$ and $x \in$
$\mathrm{co}\{x_1, x_2, ..., x_n\}$. Then $x = \sum_{i=1}^n t_i x_i$ with $t_i \geq 0$ and $\sum_{i=1}^n t_i = 1$.
By (i) we have $\langle u^*, f(x, y, x) \rangle = 0$ for all $y \in T(x)$. Since $f(x, y, \cdot)$
is $C-$convex, by Proposition 2.2 the function $\langle u^*, f(x, y, \cdot) \rangle$ is convex.
Hence

$$0 = \langle u^*, f(x, y, x) \rangle \leq \sum_{i=1}^n t_i \langle u^*, f(x, y, x_i) \rangle.$$

It follows that there exists $i \in \{1, 2, ..., n\}$ satisfying $\langle u^*, f(x, y, x_i) \rangle \geq 0$
for some $y \in T(x)$. Since T is star-pseudomonotone, $\langle u^*, f(x, y, x_i) \rangle \geq$
0 for all $y \in T(x_i)$. The proof is complete. $\square$

Definition 4.3. *T is called upper sign-continuous on K with respect
to f and $u^* \in C_+^*$ if for all x, z in K the following is satisfied:*

$$(\forall t \in (0, 1), \ \inf_{y \in T(x_t)} \langle u^*, f(x, y, z) \rangle \geq 0) \implies \sup_{y \in T(x)} \langle u^*, f(x, y, z) \rangle \geq 0,$$

where $x_t = (1 - t)x + tz$.

The notion of upper sign-continuous operators was first introduced
by Hadjisavvas (2003). In the above definition by putting $Y = D = X^*$,
$Z = \mathbb{R}$, $C = \mathbb{R}_+$, $u^* = 1$ and $f(x, y, z) = \langle y, z - x \rangle$ we also obtain the
upper sign-continuous concept in the sense of Hadjisavvas.

The following lemma is a type of Minty's lemma.

Lemma 4.4. *Suppose that Y is locally convex. Let $u^* \in C_+^*$, $\hat{x} \in K$
and the following conditions hold:*

(i) *for each $x \in K$ and $y \in D$ one has $f(x, y, x) = 0$;*
(ii) *for each $x \in K$ and $y \in T(x)$ the function $f(x, y, \cdot)$ is $C-$convex
    and $C-$lower semicontinuous;*
(iii) *for each $(x, z) \in K \times K$, the function $f(x, \cdot, z)$ is $C-$concave and
    $C-$upper semicontinuous;*
(iv) *T is upper sign-continuous with respect to f and u^*, with weakly
    compact and convex values;*
(v) *$\langle u^*, f(\hat{x}, y, z) \rangle \geq 0$ for all $z \in K$ and $y \in T(z)$.*

Then $\hat{x} \in S_V$.

Proof. Rewrite condition (v) as

(13) $$\forall z \in K, \forall y \in T(z), \quad \langle u^*, f(\hat{x}, y, z) \rangle \geq 0.$$

Taking any $z \in K$, we have $z_t = (1-t)\hat{x} + tz \in K$ for $t \in (0, 1)$. Hence (13) implies that

(14) $$\langle u^*, f(\hat{x}, y, z_t) \rangle \geq 0, \quad \forall y \in T(z_t).$$

We claim that

(15) $$\langle u^*, f(\hat{x}, y, z) \rangle \geq 0, \quad \forall y \in T(z_t).$$

Suppose, on contrary, that the assertion is false. Then there exists $t \in (0, 1)$ and $y_t \in T(z_t)$ such that

(16) $$\langle u^*, f(\hat{x}, y_t, z) \rangle < 0.$$

By (ii) and Proposition 2.2 we have

$$t\langle u^*, f(\hat{x}, y_t, z) \rangle + (1-t)\langle u^*, f(\hat{x}, y_t, \hat{x}) \rangle \geq \langle u^*, f(\hat{x}, y_t, z_t) \rangle.$$

Combining this and (i) we obtain

$$t\langle u^*, f(\hat{x}, y_t, z) \rangle \geq \langle u^*, f(\hat{x}, y_t, z_t) \rangle.$$

From here and (16), it follows that

$$\langle u^*, f(\hat{x}, y_t, z_t) \rangle < 0,$$

which contradicts (14). Thus our claim is verified.

By (15) we obtain

$$\inf_{y \in T(z_t)} \langle u^*, f(\hat{x}, y, z) \rangle \geq 0.$$

By the upper sign-continuity of T, we have

$$\sup_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, z) \rangle \geq 0.$$

This implies that

(17) $$\inf_{z \in K} \sup_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, z) \rangle \geq 0.$$

Since the function $y \mapsto \langle u^*, f(\hat{x}, y, z) \rangle$ is upper semicontinuous and concave, it is also upper semicontinuous in the weak topology of Y. According to the minimax theorem (Theorem 2.4) we obtain from (21) that

$$\max_{y \in T(\hat{x})} \inf_{z \in K} \langle u^*, f(\hat{x}, y, z) \rangle = \inf_{z \in K} \max_{y \in T(\hat{x})} \langle u^*, f(\hat{x}, y, z) \rangle \geq 0.$$

Hence there exists $\hat{y} \in T(\hat{x})$ such that

$$\inf_{z \in K} \langle u^*, f(\hat{x}, \hat{y}, z) \rangle \geq 0.$$

This means that

$$\langle u^*, f(\hat{x}, \hat{y}, z) \rangle \geq 0, \quad \forall z \in K.$$

By (5) we obtain

$$f(\hat{x}, \hat{y}, z) \not< 0, \quad \forall z \in K.$$

Hence $\hat{x} \in S_V$. $\square$

Based on Lemma 4.4, we have the following existence result of strong solutions for GVVI(K, T, f).

Theorem 4.5. *Let X be a reflexive Banach space. Let D be a nonempty set in the locally convex space Y, K be a nonempty convex set in X, and K_0 be a weakly compact subset of K. Let $T : K \to 2^D$ be a multifunction, $f : K \times D \times K \to Z$ be a single-valued mapping and $u^* \in C_+^*$ be given. Assume that:*

(a) for each $x \in K$ and $y \in D$, $f(x, y, x) = 0$;
(b) for each $x \in K$ and for each $y \in T(x)$, the function $f(x, y, \cdot)$ is $C-$strongly convex and lower semicontinuous;
(c) for each $z \in K$ the function $f(\cdot, \cdot, z)$ is $C-$concave and upper semicontinuous;
(d) T is star-pseudomonotone and upper sign-continuous with respect to f and u^*;
(e) for each $x \in K$, $T(x)$ is a weakly compact and convex set;
(f) for each $x \in K \backslash K_0$ there exists $z \in K_0$ such that $f(x, y, z) < 0$ for all $y \in T(x)$.

Then there exist $\hat{x} \in K_0$ and $\hat{y} \in T(\hat{x})$ such that $f(\hat{x}, \hat{y}, z) \not< 0$ for all $z \in K$.

Proof. We choose $r > 0$ such that $K_0 \subset \mathrm{int} B_r$ where $B_r \subset X$ is the closed ball with radius r. Put $\Omega = K \cap B_r$. We next define the multifunction $G : \Omega \to 2^X$ by setting

$$G(z) = \{x \in \Omega \mid \langle u^*, f(x, y, z) \rangle \geq 0, \forall y \in T(z)\}.$$

By Proposition 4.2, T is also star-properly quasimonotone with respect to f and u^*. Hence for any finite subset $\{z_1, z_2, ..., z_n\}$ in Ω and for every $z \in \mathrm{co}\{z_1, z_2, ..., z_n\}$ there exists $i \in \{1, 2, .., n\}$ such that

$$\langle u^*, f(z, y, z_i) \rangle \geq 0, \quad \forall y \in T(z_i).$$

This implies that $z \in G(z_i)$ and so

$$\mathrm{co}\{z_1, z_2, ..., z_n\} \subseteq \bigcup_{i=1}^n G(z_i).$$

By (c) and Proposition 2.2, the mapping $\langle u^*, f(\cdot, y, z) \rangle$ is concave and u.s.c. on Ω. Hence $G(z)$ is a closed convex set. Consequently, $G(z)$ is a weakly compact subset of Ω. Thus G is a KKM mapping. By Theorem

2.3, there exists $\hat{x} \in \Omega$ such that $\hat{x} \in G(z)$ for all $z \in \Omega$. This implies that

$$\langle u^*, f(\hat{x}, y, z) \rangle \geq 0, \quad \forall z \in \Omega, \quad \forall y \in T(z).$$

By Lemma 4.4, $\hat{x}$ is a strong solution of $\mathrm{GVVI}(\Omega, T, f)$. This means that

$$(18) \qquad\qquad \exists \hat{y} \in T(\hat{x}), \ \forall z \in \Omega, \quad f(\hat{x}, y, z) \not< 0.$$

Combining this and (f) we get $\hat{x} \in K_0 \subset \mathrm{int} B_r$.

It remains to show that

$$(19) \qquad\qquad \exists \hat{y} \in T(\hat{x}), \ \forall z \in K \quad f(\hat{x}, y, z) \not< 0.$$

Taking any $z \in K$ we have $z_t = \hat{x} + t(z - \hat{x}) \in \Omega$ for a sufficiently small $t \in (0,1)$. Hence (18) implies that

$$(20) \qquad\qquad f(\hat{x}, \hat{y}, z_t) \not< 0.$$

If $f(\hat{x}, \hat{y}, z) < 0$ then by (b) we have

$$t f(\hat{x}, \hat{y}, z) + (1-t) f(\hat{x}, \hat{y}, \hat{x}) - f(\hat{x}, \hat{y}, z_t) \in \mathrm{int} C \cup \{0\}.$$

Hence

$$\begin{aligned} -f(\hat{x}, \hat{y}, z_t) &\in -t f(\hat{x}, \hat{y}, z) + \mathrm{int} C \cup \{0\}. \\ &\subset t\,\mathrm{int} C + \mathrm{int} C \cup \{0\}. \\ &\subset \mathrm{int} C + \mathrm{int} C \cup \{0\}. \\ &= \mathrm{int} C. \end{aligned}$$

This implies that $f(\hat{x}, \hat{y}, z_t) < 0$, which contradicts (20). Thus we must have $f(\hat{x}, \hat{y}, z) \not< 0$ and so (19) is fulfilled. The proof is complete. $\square$

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