Solution existence of variational inequalities with pseudomonotone operators in the sense of Brézis *

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Abstract. This paper is concerned with the study of solution existence of variational inequalities and generalized variational inequalities in reflexive Banach spaces with pseudomonotone operators in the sense of H. Brézis. The obtained results covers some preceding results in [4], [11] and [14].

Keywords. Variational inequality, generalized variational inequality, pseudomonotone operator, solution existence.

1 Introduction

Variational inequality (VI, for brevity), generalized variational inequality (GVI), and quasivariational inequality (QVI) have been recognized as suitable mathematical models for dealing with many problems arising in different fields, such as optimization theory, partial differential equations, economic equilibrium, mechanics, etc. In the last four decades, since the time of the celebrated Hartman-Stampacchia theorem (see [9] and [11]), solution existence of VIs, GVIs, QVIs and other related problems has become a basic research topic which continues to attract attention of researchers in applied mathematics (see for instance [2], [6], [8], [11]-[21] and the references therein). Some difficult questions do exist in this field (see, for example, [16] and [20]).

Let us assume that X is a reflexive Banach space over the reals, $K \subset X$ is a nonempty closed convex set, $\Phi : K \rightrightarrows X^*$ is a multifunction from K into the dual space X^* (which is equipped with the weak* topology).

The *generalized variational inequality* defined by K and Φ, denoted by GVI(K, Φ), is the problem of finding a point $x \in K$ such that

$$\exists x^* \in \Phi(x), \quad \langle x^*, y - x\rangle \geq 0 \ \ \forall y \in K. \tag{1.4}$$

Here $\langle , \rangle$ denotes the canonical pairing between X^* and X. The set of all $x \in K$ satisfying (1.4) is denoted by SOL(K, Φ). If $\Phi(x) = \{F(x)\}$ for all $x \in K$, where $F : K \to X^*$ is a single-valued map, then the problem GVI(K, Φ) is called a *variational inequality* and the abbreviation VI(f, K) is used instead of GVI(K, Φ).

If for any $x, y \in K$ and $x^* \in \Phi(x)$, $y^* \in \Phi(y)$ one has $\langle x^* - y^*, x - y\rangle \geq 0$, then one says that Φ is a *monotone operator*. The first substantial results concerning monotone operators were obtained by G. Minty [15] and F. E. Browder [3]. Then the properties of monotone operators were studied systematically by F. E. Browder in order to obtain existence theorems for quasi-linear elliptic and parabolic partial differential equations. The existence theorems of F. E. Browder were generalized to more general classes of quasi-linear elliptic differential equations by P. Hartman and G. Stampacchia (see [9] and [11]).

In 1968, H. Brézis [5]introduced a vast class of operators of pseudomonotone type as follows. A operator $T : K \to 2^{X^*}$ is called pseudomonotone iff the following holds. Let (u_n, u_n^*) be such that $u_n^* \in T(u_n)$, $u_n \rightharpoonup u$ as $n \to \infty$ and $\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle \leq 0$. Then for each $v \in K$, there exists $v^* \in T(u)$ such that

$$\langle v^*, u - v\rangle \leq \liminf_{n\to\infty}\langle u_n^*, u_n - v\rangle.$$

The theory pseudomonotone operators plays an important role in the study of solvability of operator equations and quasi-linear elliptic equations. This class have been studied systematically. We refer the readers to [4] and Chapter 27 in [22] for more information and properties of the class.

Beside the concept of pseudomonotone operators in the sense of Brézis, another concept of pseudomonotone operators was proposed by Karamardian in 1976 (see [10]). Let us recall the definition of pseudomonotone operators in the sense of Karamardian. A operator $\Phi : K \to 2^{X^*}$ is said to be pseudomonotone if for any $x, y \in K$ and $x^* \in \Phi(x)$, $y^* \in \Phi(y)$ implies

$$\langle y^*, x - y \rangle \geq 0 \implies \langle x^*, x - y \rangle \geq 0.$$

It is clear that monotonicity implies pseudomonotonicity in the sense of Karamardian. The converse implication is not true in general (take, for instance, $K = \mathbb{R}$ and $F(x) = x^2 + 1$ for all $x \in K$).

Existence of solutions of variational inequalities for pseudomonotone operators in the sense of Karamardian has been investigated intensively in recent years (see, for instance [2], [8], [13], [18], [19] and references therein). We cite here some existence results of VIs that have a close connection with the present work.

Theorem 1.1 ([[18], Theorem 3.3]) *Let X be a real reflexive Banach space and $K \subset X$ be a closed convex and bounded set. Assume that $F : K \to X^*$ is a pseudomonotone operator (in the sense of Karamardian) which is continuous on finite dimensional subspaces of X. Then $VI(F, K)$ has a solution.*

Particularly, in [12] the authors obtained a necessary and sufficient condition for the solution existence of pseudomonotone VIs in reflexive Banach spaces.

Theorem 1.2 ([[12], Theorem 3.1]) *Let X be a real reflexive Banach space and $K \subset X$ be a closed convex set. Assume that $F : K \to X^*$ is a pseudomonotone operator (in the sense of Karamardian) which is continuous on finite dimensional subspaces of X. Then the following statements are equivalent:*

(a) *There exists a reference point $x^{\text{ref}} \in K$ such that the set*

$$L_<(F, x^{\text{ref}}) := \{x \in K : \langle F(x), x - x^{\text{ref}} \rangle < 0\}$$

is bounded (possibly empty);

(b) *There exist an open ball Ω and a vector $x^{\text{ref}} \in \Omega \cap K$ such that*

$$\langle F(x), x - x^{\text{ref}} \rangle \geq 0 \quad \forall x \in K \cap \partial\Omega;$$

(c) *The problem* $\mathrm{VI}(K, F)$ *has a solution.*

Besides, if there exists a vector $x^{\text{ref}} \in K$ such that the set

$$L_{\leq}(F, x^{\text{ref}}) := \left\{x \in K : \langle F(x), x - x^{\text{ref}} \rangle \leq 0\right\}$$

is bounded, then the solution set $\mathrm{SOL}(K, F)$ *is nonempty and bounded.*

It is natural to ask whether the conclusions of Theorem 1.1 and Theorem 1.2 are still valid for the case of pseudomonotone operators in the sense of Brézis. Our aim in this paper is to obtain such conclusions. It is noted that the proofs of Theorem 1.1 and Theorem 1.2 are based on the Minty lemma. However, when operators are not pseudomonotone in the sense of Karamadian, the lemma is invalid. Hence in this case the scheme of proofs of Theorem 1.1 and Theorem 1.2 fails to apply to our problem. In order to obtain existence results as Theorem 1.1 and Theorem 1.2 for the case of pseudomonotone operators in the sense of Brézis, we derived a new scheme for proofs which based on the Galerkin method. Using this scheme, we can establish some existence theorems for VIs and GVIs in reflexive Banach spaces. The obtained results extend preceding results in [4], [22] and [14]. Besides, the results allow us to retrieve existence results of VIs for the case of finite dimensional spaces as a special case.

It is worth pointing out that although there have been many existence results of VIs for pseudomonotone operators in the sense of Karamardian, there are very few results of VIs for pseudomonotone operators in the sense of Brézis in the literature. Our paper aims at a small contribution for this gap.

2 Existence results for VIs

Throughout of the paper we denote by $\mathcal{B}_K$ the set of operators $T : K \to 2^{X^*}$ which are pseudomonotone in the sense of Brézis and by $\mathcal{K}_K$ the set of operators $\Phi : K \to 2^{X^*}$ which are pseudomonotone in the sense of Karamardian.

Recall that a single-valued operator $A : K \to X^*$ belongs to class $\mathcal{B}_K$ iff the following holds: $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$ imply

$$\langle Au, u - w\rangle \leq \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle \quad \text{for all } w \in K.$$

A is said to be of class $(S)_+$ if for any sequence $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$ imply $u_n \to u$.

We say that A is demicontinuous iff $u_n \to u$ implies $Au_n \rightharpoonup Au$. A is said to be hemicontinuous iff the real function

$$t \mapsto \langle A((1-t)u + tv), w\rangle$$

is continuous on $[0, 1]$ for all $u, v, w \in K$. If the set $A(K)$ is bounded then A is called bounded on K.

We begin with the first existence theorem for class $\mathcal{B}_K$.

Theorem 2.1 *Let K be a nonempty closed convex set in a real reflexive Banach space X and $A \in \mathcal{B}_K$ a single-valued operator. Assume that A is continuous on finite dimensional subspaces of X and there exists a reference point $x^{\text{ref}} \in K$ such that the* *set*

$$L_<(A, x^{\mathrm{ref}}) := \{x \in K : \langle Ax, x - x^{\mathrm{ref}}\rangle < 0\}$$

is bounded (possibly empty).

Then problem VI(K, A) has a solution.

Proof. Since the set $L_<(A, x^{\mathrm{ref}})$ is bounded, there exist an open ball Ω such that $\{x^{\mathrm{ref}}\} \cup L_<(A, x^{\mathrm{ref}}) \subset \Omega$. Consequently,

$$\langle Ax, x - x^{\mathrm{ref}}\rangle \geq 0 \quad \forall x \in K \cap \partial\Omega. \tag{2.1}$$

Let us denote by $\mathcal{L}$ the set of all finite dimensional subspaces L of X containing x^{ref}. Fixing any $L \in \mathcal{L}$ we put $K_L = K \cap L$, $\Omega_L = \Omega \cap L$, and let $\partial_L \Omega_L$ stand for the boundary of Ω_L in the induced topology of L. Then $\partial_L\Omega_L = (\partial\Omega) \cap L$. Consider the map $A_L : K_L \to L^*$ defined by

$$\langle A_L x, y\rangle = \langle Ax, y\rangle \quad \forall y \in L. \tag{2.2}$$

From (2.1) and (2.2), all the conditions stated in the statement (b) of Proposition 2.2.3 in [8], where $(K_L, A_L, \Omega_L, x^{\mathrm{ref}})$ plays the role of the $(K, A, \Omega, x^{\mathrm{ref}})$, are fulfilled. Hence there exists a vector $x_L \in \Omega_L$ such that

$$\langle Ax_L, y - x_L\rangle \geq 0 \quad \forall y \in K_L. \tag{2.3}$$

For each $Y \in \mathcal{L}$ we denote by S_Y the set of all $\hat{x} \in K$ such that there exists a subspace $L \supseteq Y$ with the property that $\hat{x} \in \Omega_L$ and

$$\langle A\hat{x}, y - \hat{x}\rangle \geq 0 \quad \forall y \in K_L.$$

From (2.3) we see that $S_Y \neq \emptyset$ because $x_Y \in S_Y$. Moreover, the family $\{\overline{S}_Y\}_{Y\in\mathcal{L}}$ has a finite intersection property, where $\overline{S}_Y$ is the weak closure of S_Y in X. Indeed, taking any $L_1, L_2, ..., L_n \in \mathcal{L}$ and putting $M = \mathrm{span}\{L_1, L_2, \ldots, L_n\}$, we have $M \in \mathcal{L}$ and

$$S_M \subseteq \bigcap_{i=1}^{n} S_{L_i}.$$

This implies that

$$\emptyset \neq S_M \subseteq \overline{S}_M \subseteq \overline{\bigcap_{i=1}^{n} S_{L_i}} \subseteq \bigcap_{i=1}^{n} \overline{S}_{L_i}.$$

We now see that $\overline{S}_Y \subset \overline{\Omega}$ and $\overline{\Omega}$ is a weakly compact set. The finite intersection property of $\overline{\Omega}$ implies

$$\bigcap_{Y\in\mathcal{L}} \overline{S}_Y \neq \emptyset.$$

Thus there is a point x_0 such that $x_0 \in \overline{S}_Y$ for all $Y \in \mathcal{L}$.

Fix any $y \in K$. Take $Y \in \mathcal{L}$ such that Y contains y and x_0. Since $x_0 \in \overline{S}_Y$, there exists a sequence $x_n \in S_Y$ such that $x_n \rightharpoonup x_0$. By the definition of S_Y one has

$$\langle Ax_n, v - x_n\rangle \geq 0 \ \forall v \in K_Y.$$

Particularly,

$$\langle Ax_n, x_n - x_0\rangle \leq 0 \text{ and } \langle Ax_n, x_n - y\rangle \leq 0.$$

Hence

$$\limsup_{n\to\infty}\langle Ax_n, x_n - x_0\rangle \leq 0.$$

By the pseudomonotonicity of A, we obtain

$$\langle Ax_0, x_0 - y\rangle \leq \liminf_{n\to\infty}\langle Ax_n, x_n - y\rangle \leq 0.$$

Thus we have shown that

$$\langle Ax_0, y - x_0\rangle \geq 0 \text{ for all } y \in K.$$

This implies that x_0 is a solution of VI(K, A). The proof is complete. □

When K is a bounded set we obtain

Theorem 2.2 *Let K be a weakly compact convex set in a reflexive Banach space X and $A \in \mathcal{B}_K$. Assume that A is continuous on finite dimensional subspaces of X. Then problem VI(K, A) has a solution.*

Proof. The proof of the theorem is similar to the proof of Theorem 2.1, where instead of using Proposition 2.2.3 in [8], we use Theorem 3.1 in [[11], p. 12]. □

Let us present some corollaries of Theorem 2.2.

Corollary 2.3 ([[11], Theorem 1.4, p. 84]) *Let K be a closed convex and bounded set in a reflexive Banach space X and $A : K \to X^*$ be a operator which is monotone and continuous on finite dimensional subspaces of X. Then the problem VI(K, A) has a solution.*

Proof. Since A is continuous on finite dimensional subspaces of X, A is hemicontinuous. According to Proposition 27.6 (a) in [22], we have $A \in \mathcal{B}_K$. The conclusion follows from Theorem 2.2. □

Corollary 2.4 ([[14], Theorem 2.1]) *Suppose that K is a nonempty weakly compact convex subset of the reflexive Banach space X and $A : K \to X^*$. Assume conditions:*
(i) A is of class $(S)_+$;
(ii) A is continuous on finite dimensional subspaces;
(iii) if $x_n \to x$ then $\{Ax_n\}$ has a weakly convergent subsequence with limit Ax.
Then VI(K, A) has a solution.

Proof. We shall show that A is of class $\mathcal{B}_K$. In fact, assume that $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$. Fix any $v \in K$ and put

$$\alpha = \liminf_{n\to\infty}\langle Au_n, u_n - v\rangle.$$

Then there exists a subsequence $\{u_{n_k}\}$ such that

$$\alpha = \lim_{k\to\infty}\langle Au_{n_k}, u_{n_k} - v\rangle.$$

By (i) we have $u_{n_k} \to u$ as $k \to \infty$. By (iii) we can assume that $Au_{n_k} \rightharpoonup Au$ as $k \to \infty$. Hence we obtain

$$\langle Au, u - v\rangle = \lim_{k\to\infty}\langle Au_{n_k}, u_{n_k} - v\rangle = \alpha.$$

This implies that A is pseudomontone. The conclusion now follows from Theorem 2.2. □

Let us give an example which shows that there exists an operator $A \in \mathcal{B}_K \setminus \mathcal{K}_K$.

Example 2.1 Consider VI(K, A), where $K \subset R^2$ defined by

$$K = \{(x, y) \in R^2 : x + y \leq 1, x \geq -1, y \geq -1\}$$

and A defined by $A(x, y) = (-x^3, y^2)$. Since A is continuous and X is a finite dimensional space, $A \in \mathcal{B}_K$ (see [[22], Proposition 27.6]). Thus VI(K, A) satisfies all conditions of Theorem 2.2. Moreover $A \notin \mathcal{K}_K$. Indeed, taking $u = (1, 0)$ and $v = (-1, 0)$, we have $\langle Av, u - v\rangle = 2$ and $\langle Au, u - v\rangle = -2$. Hence A is not pseudomonotone in the sense of Karamardian.

The class of pseudomonotone operators in the sense of Brezis is rather large. For examples, maximal monotone operators are of class $\mathcal{B}_K$ (see [[4], Proposition 8]), monotone and hemicontinuous operators are of $\mathcal{B}_\mathcal{K}$ (see [[22], Proposition 27.6]). The following example shows that there exists a pseudomonotone operator acting on a Sobolev space which is not monotone.

Example 2.2 Let G be a bounded open set in R^n with $n = 1, 2, 3$ and let $X = \overset{\circ}{W}{}^1_2(G)$ be a closed subspace of the Sobolev space $W^1_2(G)$. Let $A_n : X \to X^*$ be an operator defined by

$$\langle A_n u, v\rangle = \alpha \int_G \sum_{i=1}^{n}(D_i u)(\sin u)v dx,$$

where $\alpha < 0$ and $D_i u(x) := \frac{\partial u}{\partial x_i}(x)$. By the Proposition 27.11 in [22], A_n is pseudomonotone. However, A_n is not monotone in general. In fact, for n=1 and $G = (0,1)$ we have

$$\langle A_1 u, v\rangle = \alpha \int_0^1 \dot{u}(\sin u) v dx.$$

Take two functions $u_1(x) \equiv 0$ and $u_2(x) = -x^3 + x^2$. We see that $u_1, u_2 \in C^\infty(0,1)$ and $u_i(0) = u_i(1) = 0$, i=1, 2. Hence $u_1, u_2 \in X$. We now have

$$\langle A_1 u_1 - A_1 u_2, u_1 - u_2\rangle = \langle A_1 u_2, u_2\rangle = \alpha \int_0^1 (-3x^2 + 2x)(-x^3 + x^2) \sin(-x^3 + x^2) dx.$$

Since $0 \le -x^3 + x^2 \le 4/27$ for all $x \in (0,1)$, we have $\sin(-x^3 + x^2) \ge 0$. Also $(-3x^2 + 2x)(-x^3 + x^2) \ge 0$ for all $x \in (0,1)$. Hence we obtain

$$\int_0^1 (-3x^2 + 2x)(-x^3 + x^2) \sin(-x^3 + x^2) dx > 0.$$

Consequently, $\langle A_1 u_1 - A_1 u_2, u_1 - u_2\rangle < 0$ and so A_1 is not monotone.

3 Existence results for GVIs

This section is devoted to deriving some solution existence theorems for GVIs.

Let us recall the definition of upper semicontinuous multifunctions. A multifunction $F : X \to 2^Z$ from a topological space X to a topological space Z is said to be upper semicontinuous (u.s.c. for brevity) on X if for any $x_0 \in X$ and open set V in Z such that $F(x_0) \subset V$, there exists a neighborhood U of x_0 satisfying $F(x) \subset V$ for all $x \in U$.

As an auxiliary result, we will use the following well-known existence theorem for GVIs in finite dimensional spaces. For the convenience of the reader we provide below another proof which is based on approximate selections.

Theorem 3.1 *Let K be a convex and compact set in R^n and $F : K \to R^n$ be a multifunction which is u.s.c with compact convex values. Then problem $GVI(F, K)$ has a solution.*

Proof. For the proof, we will use a lemma which is due to Cellina.

Lemma 3.2 ([[1], Theorem 1, p. 84]) *Let X and Y be Banach space, $M \subset X$ and $T : M \to 2^Y$ be an u.s.c multifunction with closed and convex values. Then for each $\epsilon > 0$ there exists a continuous map $f_\epsilon : M \to Y$ such that for all $x \in M$ one has*

$$f_\epsilon(x) \in T((x + \epsilon B_X) \cap M) + \epsilon B_Y, \tag{3.1}$$

where B_X and B_Y are open unit balls of X and Y respectively.

We choose a sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0^+$. By Lemma 2.1, for each n there exists a continuous function f_{ϵ_n} such that condition (3.1) is satisfied. We now consider VI(f_{ϵ_n}, K). By Theorem 3.1 in [[11], p. 12], there exists $x_n \in K$ such that

$$\langle f_{\epsilon_n}(x_n), y - x_n \rangle \geq 0 \quad \forall\, y \in K. \tag{3.2}$$

From (3.1), there exists y_n, $\|y_n\| = 1$ and $z_n \in T(x_n + \epsilon_n y_n)$ such that $\|f_{\epsilon_n}(x_n) - z_n\| \leq \epsilon_n$. Since K is a compact set, we can assume that $x_n \to x_0 \in K$. Consequently, $x_n + \epsilon_n y_n \to x_0$. Since $F(K)$ is a compact set and $F(x_n + \epsilon_n y_n) \subset F(K)$, we can assume further that $z_n \to z_0$ and so $f_{\epsilon_n}(x_n) \to z_0$. By the upper semicontinuity of F we have $z_0 \in F(x_0)$. Fixing any $y \in K$ and letting $n \to \infty$, we obtain from (3.2) that

$$\langle z_0, y - x_0 \rangle \geq 0.$$

This implies that x_0 is a solution of GVI(F, K). The proof is complete. $\square$

We are ready to state and prove existence theorems for GVIs in reflexive Banach spaces.

Theorem 3.3 *Let X be a reflexive Banach space, K be a nonempty closed convex and bounded set in X and $T : K \to 2^{X^*}$ be an operator. Assume that $T \in \mathcal{B}_K$ and the following conditions are satisfied:*
(i) T is upper semicontinuous from each finite dimensional subspace of X to the weak topology on X^*;
(ii) for each $x \in K$, $T(x)$ is a closed convex and bounded set.
Then problem GVI(K, T) has a solution.

Proof. We shall use the scheme as in the proof of Theorem 2.1.

Let us denote by $\mathcal{F}$ the set of all subspaces L of X such that $L \cap K \neq \emptyset$. Fix a subspace $L \in \mathcal{F}$ and consider a mapping $\alpha_L : X^* \to L^*$ defined by $\langle \alpha_L x^*, y \rangle = \langle x^*, y \rangle$ for all $y \in L$. Put $K_L = K \cap L$ and define the mapping $T_L : K_L \to 2^{L^*}$ by the formula

$$T_L(x) = \{\alpha_L x^* : x^* \in T(x)\}.$$

It is easy to see that T_L is upper semicontinuous on K_L. We now consider problem GVI(T_L, K_L). By (i) and (ii), conditions of Theorem 3.1 are fulfilled for GVI(T_L, K_L). According to Theorem 3.1, there exists $x_L \in K_L$ and $z_L^* \in T_L(x_L)$ such that

$$\langle z_L^*, y - x_L \rangle \geq 0 \quad \forall\, y \in K_L.$$

Since $z_L^* = \alpha_L x^*$ for some $x^* \in T(x_L)$, we get

$$\langle x^*, y - x_L \rangle \geq 0 \quad \forall\, y \in K_L.$$

Hence we obtain

$$\sup_{x^* \in T(x_L)} \langle x^*, y - x_L \rangle \geq 0 \quad \forall\, y \in K_L. \tag{3.3}$$

For each $Y \in \mathcal{L}$ we denote by S_Y the set of all $\hat{x} \in K$ such that there exists a subspace $L \supseteq Y$ with the property that $\hat{x} \in K_L$ and

$$\sup_{x^* \in T(\hat{x})} \langle x^*, y - \hat{x} \rangle \geq 0 \quad \forall\, y \in K_L.$$

We claim that the family $\{S_Y\}$ has the finite intersection property. Indeed, for each $Y \in \mathcal{F}$, by putting $L = Y$, we have from (3.3) that $x_Y \in S_Y$. Hence S_Y is nonempty. Take subspaces $L_1, L_2, ..., L_n \in \mathcal{L}$ and put $M = \operatorname{span}\{L_1, L_2, \ldots, L_n\}$. Then we have $M \in \mathcal{L}$ and

$$S_M \subset \bigcap_{i=1}^{n} S_{L_i}.$$

This implies that

$$\emptyset \neq S_M \subseteq \overline{S}_M \subseteq \overline{\bigcap_{i=1}^{n} S_{L_i}} \subseteq \bigcap_{i=1}^{n} \overline{S}_{L_i}.$$

The claim is proved.

Since $\overline{S}_Y \subset K$ and K is weakly compact, we obtain

$$\bigcap_{Y \in \mathcal{F}} \overline{S}_Y \neq \emptyset.$$

This means that there exists a point $x_0 \in K$ such that $x_0 \in \overline{S}_Y$ for all $Y \in \mathcal{F}$. Fix any $y \in K$ and choose $Y \in \mathcal{F}$ such that Y contains y and x_0. Since $x_0 \in \overline{S}_Y$, there exists a sequence $x_n \in S_Y$ such that $x_n \rightharpoonup x_0$. By the definition of S_Y we have

$$\sup_{x^* \in T(x_n)} \langle x^*, v - x_n \rangle \geq 0 \quad \forall\, v \in K_Y.$$

Hence

$$\inf_{v \in K_Y} \sup_{x^* \in T(x_n)} \langle x^*, v - x_n \rangle \geq 0.$$

By the Sion minimax theorem (see [17]), we obtain

$$\sup_{x^* \in T(x_n)} \inf_{v \in K_Y} \langle x^*, v - x_n \rangle = \inf_{v \in K_Y} \sup_{x^* \in T(x_n)} \langle x^*, v - x_n \rangle \geq 0.$$

Since the function $x^* \mapsto \inf_{v \in K_Y} \langle x^*, v - x_n \rangle$ is u.s.c., there exists $x_n^* \in T(x_n)$ such that

$$\inf_{v \in K_Y} \langle x_n^*, v - x_n \rangle \geq 0.$$

Hence

$$\langle x_n^*, v - x_n \rangle \geq 0 \ \forall \ v \in K_Y.$$

In particular

$$\langle x_n^*, y - x_n \rangle \geq 0 \quad \text{and } \langle x_n^*, x_0 - x_n \rangle \geq 0. \tag{3.4}$$

From this we get $\limsup_{n\to\infty} \langle x_n^*, x_n - x_0 \rangle \leq 0$. By the pseudomonotonicity of T and (3.4), there exists $x^* \in T(x_0)$ such that

$$\langle x^*, x_0 - y \rangle \leq \liminf_{n\to\infty} \langle x_n^*, x_n - y \rangle \leq 0.$$

This implies that $\langle x^*, y - x_0 \rangle \geq 0$. Thus we have shown that

$$\inf_{y\in K} \sup_{x^*\in T(x_0)} \langle x^*, y - x_0 \rangle \geq 0.$$

Using the Sion minimax theorem again, we can prove that there exists $x_0^* \in T(x_0)$ such that

$$\langle x_0^*, y - x_0 \rangle \geq 0 \quad \forall \ y \in K.$$

The proof is complete. □

Let us present some corollaries of Theorem 3.1.

Corollary 3.4 ([[4], Theorem 15]) *Let X be a reflexive Banach space and K be a nonempty closed convex set of X with $0 \in K$. Let $T : K \to 2^{X^*}$ be an operator of class $\mathcal{B}_K$ such that conditions (i) and (ii) of Theorem 2.6 are fulfilled. Assume further that T is coercive on K, i.e. that there exists a function $c(r)$ from R^+ into R with $c(r) \to +\infty$ as $r \to \infty$, such that for each $u \in K$, $w \in Tu$, we have $\langle w, u \rangle \geq c(\|u\|)\|u\|$.*

Then for each $f_0 \in X^*$, there exists $u_0 \in K$ and $w_0 \in Tu_0$ such that

$$\langle w_0 - f_0, u - u_0 \rangle \geq 0 \text{ for all } u \in K. \tag{3.5}$$

Proof. Put $T_{f_0} = T - f_0$. From the coercive condition, for each $u \in K$, $w \in Tu$ we have

$$\langle w - f_0, u \rangle \geq (c(\|u\|) - \|f_0\|)\|u\|.$$

Hence there exists a number $\gamma > 0$ such that for all $u \in K$ satisfying $\|u\| \geq \gamma$ and $w \in Tu$, one has $\langle w - f_0, u \rangle > 0$. Since $0 \in K$, it follows that if (u_0, w_0) satisfies (3.5) then $\|u_0\| < \gamma$. Put $K_\gamma = K \cap \overline{B}(0, \gamma)$ and consider the problem GVI(T_{f_0}, K_γ). According to Theorem 2.6, there exists $u_0 \in K_\gamma$ such that

$$\inf_{w\in T_{f_0}(u_0)} \langle w, u - u_0 \rangle \geq 0 \text{ for all } u \in K_\gamma. \tag{3.6}$$

Fix any $u \in K$. Since $\|u_0\| < \gamma$, we have $u_0 + \lambda(u - u_0) \in K_\gamma$ for $\lambda > 0$ and sufficiently small. Hence (3.6) implies

$$\inf_{w \in T_{f_0}(u_0)} \langle w, \lambda(u - u_0) \rangle \geq 0.$$

Thus we have proved that

$$\inf_{w \in T_{f_0}(u_0)} \langle w, u - u_0 \rangle \geq 0 \text{ for all } u \in K.$$

Using the Sion minimax theorem we can show that there exists a point $w_0 \in Tu_0$ such that

$$\langle w_0 - f_0, u - u_0 \rangle \geq 0 \text{ for all } u \in K.$$

□

Corollary 3.5 ([14], Theorem 3.1) *Let K be a nonempty weakly compact convex subset of a reflexive Banach space X and $T : K \to 2^{X^*}$. Assume conditions:*
(i) Tx is nonempty closed and convex for each $x \in K$;
(ii) T is bounded and upper semicontinuous on finite-dimensional subspaces;
(iii) if x_n converges weakly to x, $w_n \in Tx_n$, and $\limsup\langle w_n, x_n - x\rangle \leq 0$ then $x_n \to x$ and $\{w_n\}$ has a subsequence converging weakly to some $w \in Tx$.

Then $GVI(K, T)$ has a solution.

Proof. It is sufficient to show that T is of class $\mathcal{B}_K$, i.e., if $\{(u_n, u_n^*)\}$ is a sequence such that $u_n^* \in T(u_n)$, $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle \leq 0$, then for each $v \in K$, there exists $v^* \in T(u)$ satisfying

$$\langle v^*, u - v \rangle \leq \liminf_{n\to\infty} \langle u_n^*, u_n - v \rangle.$$

Suppose that T is not of class $\mathcal{B}_K$. Then there exists $v \in K$ such that

$$\inf_{v^* \in T(u)} \langle v^*, u - v \rangle > \liminf_{n\to\infty} \langle w_n, u_n - v \rangle$$

By passing to an subsequence, we may assume that

$$\inf_{v^* \in T(u)} \langle v^*, u - v \rangle > \lim_{n\to\infty} \langle w_n, u_n - v \rangle.$$

By (iii), without loss of generality we can assume that $w_n \rightharpoonup w \in Tu$ and $u_n \to u$. Hence we obtain

$$\inf_{v^* \in T(u)} \langle v^*, u - v \rangle > \lim_{n\to\infty} \langle w_n, u_n - v \rangle = \langle w, u - v \rangle,$$

a contradiction. □

The following theorem gives another version on the solution existence of GVIs to the case of unbounded sets.

Theorem 3.6 *Let X be a reflexive Banach space, K be a nonempty closed convex set of X and $T : K \to 2^{X^*}$ be an operator. Assume that $T \in \mathcal{B}_K$ and the following conditions are satisfied:*
(i) T is upper semicontinuous on finite dimensional subspaces of X;
(ii) for each $x \in K$, $T(x)$ is a closed convex and bounded set;
(iii) there exists $x^{\text{ref}} \in K$ such that the set

$$L_{\leq}(T, x^{\text{ref}}) := \left\{ x \in K : \inf_{x^* \in T(x)} \langle x^*, x - x^{\text{ref}} \rangle \leq 0 \right\}$$

is bounded (possibly empty).

Then problem $GVI(K, T)$ has a solution.

Proof. The proof of the theorem proceeds analogously to the proof of Theorem 3.3, where instead of using Theorem 3.1 we use

Lemma 3.7 ([[12], Theorem 2.3]) *Let $K \subset R^n$ be a closed convex set and $\Phi : K \rightrightarrows \mathbb{R}^n$ be a upper semicontinuous multifunction with nonempty compact convex values. Assume that there exists $x^{\text{ref}} \in K$ such that the set*

$$L_{\leq}(\Phi, x^{\text{ref}}) := \left\{ x \in K : \inf_{x^* \in \Phi(x)} \langle x^*, x - x^{\text{ref}} \rangle \leq 0 \right\}$$

is bounded (possibly empty).

Then problem $\mathrm{GVI}(K, \Phi)$ *has a solution.*

□

In summary, we showed that under certain conditions of operators $T \in \mathcal{B}_K$, $\mathrm{VI}(K,T)$ and $\mathrm{GVI}(K,T)$ have a solution. It is interesting that the obtained results not only cover preceding results but also contain existence result for the case of finite dimensional spaces as a special case. Namely, by Proposition 27. 6 (d) in [22], every continuous operators in finite dimensional spaces are pseudomonotone in the sense of Brézis. Hence we obtain Theorem 3.1 in [[11], p.12] from Theorem 2.2 when $X = R^n$. Meanwhile we can not obtain such result from Theorem 1.1 and Theorem 1.4 in [[11], p. 84] because the continuous operators in finite dimensional spaces, are not necessary to be pseudomonotone in the sense of Karamardian. In other word, the existence theorems of VIs and GVIs in infinite dimensional spaces with monotone operators or operators of class $\mathcal{K}_K$, extend existence theorems in finite dimensional spaces but these results can not be derived from existence results in the case of infinite dimensional spaces. This says that, the conditions imposed on class $\mathcal{B}_K$ for the solution existence of VIs and GVIs, are more natural and intrinsic than conditions imposed on class $\mathcal{K}_K$.

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