J. Aust. Math. Soc. **89** (2010), 245–254
doi:10.1017/S1446788710001515

LINEAR ORTHOGONALITY PRESERVERS OF HILBERT BUNDLES

CHI-WAI LEUNG ✉, CHI-KEUNG NG and NGAI-CHING WONG

(Received 27 August 2009; accepted 14 September 2010)

Communicated by G. A. Willis

Abstract

A $\mathbb{C}$-linear map θ (not necessarily bounded) between two Hilbert C^*-modules is said to be 'orthogonality preserving' if $\langle\theta(x), \theta(y)\rangle = 0$ whenever $\langle x, y\rangle = 0$. We prove that if θ is an orthogonality preserving map from a full Hilbert $C_0(\Omega)$-module E into another Hilbert $C_0(\Omega)$-module F that satisfies a weaker notion of $C_0(\Omega)$-linearity (called 'localness'), then θ is bounded and there exists $\phi \in C_b(\Omega)_+$ such that $\langle\theta(x), \theta(y)\rangle = \phi \cdot \langle x, y\rangle$ for all $x, y \in E$.

2010 *Mathematics subject classification*: primary 46L08; secondary 46M20, 46H40, 46E40.

Keywords and phrases: automatic continuity, orthogonality preserving maps, module homomorphisms, local maps, Hilbert C^*-modules, Hilbert bundles.

1. Introduction

It is common knowledge that the inner product of a Hilbert space determines both the norm and orthogonality; and conversely, the norm structure determines the inner product structure. It may be slightly less well known that the orthogonality structure of a Hilbert space also determines its norm structure. Indeed, if θ is a linear map between Hilbert spaces preserving orthogonality, then it is easy to see that θ is a scalar multiple of an isometry (see [5, 6]).

We are interested in the corresponding relations for Hilbert C^*-modules. Note that, in the case of a commutative C^*-algebra $C_0(\Omega)$, Hilbert $C_0(\Omega)$-modules are the same as Hilbert bundles, or equivalently, continuous fields of Hilbert spaces over Ω. By modifying the proof of [12, Theorem 6] (see also [9, 13, 16]), one may show that any surjective isometry between two continuous fields of Hilbert spaces with nonzero fibers over each point is given by a homeomorphism and a field of unitary operators. Thus, the norm structure (and linearity) determines the unitary structure in this situation.

The authors were supported by a Hong Kong RGC Research Grant (2160255), the National Natural Science Foundation of China (10771106), and a Taiwan NSC grant (NSC96-2115-M-110-004-MY3).

Our primary concern is the question of whether the orthogonality structure of a Hilbert C^*-module determines its unitary structure. More precisely, let A be a C^*-algebra, and E and F be two Hilbert A-modules. If $\theta : E \to F$ is an A-module homomorphism, not necessarily bounded, which preserves orthogonality, that is, $\langle \theta(x), \theta(y) \rangle_A = 0$ whenever $\langle x, y \rangle_A = 0$, then we ask whether there is a central positive multiplier u in $M(A)$ such that

$$\langle \theta(e), \theta(f) \rangle_A = u \langle e, f \rangle_A \quad \forall e, f \in E.$$

When $A = \mathbb{C}$, this reduces to the case of Hilbert spaces. Recently, Ilišević and Turnšek [10] gave a positive answer in the case where A is a standard C^*-algebra, that is, when $\mathcal{K}(H) \subseteq A \subseteq \mathcal{L}(H)$.

In this paper, we will give a positive answer when A is a commutative C^*-algebra (actually, we prove a slightly stronger result that replaces A-linearity with the 'localness' property; see Definition 2.1). On the other hand, we will also consider bijective biorthogonality preserving maps between Hilbert C^*-modules over different commutative C^*-algebras. We show that if such a map also satisfies a certain local-type property (see Definition 3.12) but is not assumed to be bounded, then it is given by a homeomorphism (between the base spaces) and a 'continuous field of unitary operators'. We remark that in this case of Hilbert C^*-modules over different commutative C^*-algebras, one cannot define 'A-linearity', but has to consider the localness property. This is one of the reasons for considering local maps. We remark also that this case does not cover the case of Hilbert C^*-modules over the same commutative C^*-algebra, because we need to assume that the map is both bijective and biorthogonality preserving.

Note that if Ω is a locally compact Hausdorff space and H is a Hilbert space, then $C_0(\Omega, H)$ is a Hilbert $C_0(\Omega)$-module. As far as we know, even in this case our results are new, and the techniques in the proofs are nonstandard and nontrivial, compared to those in the literature [1, 4, 8, 11] on separating or zero-product preservers (although some statements look similar). In a forthcoming paper, the authors will study the case where the underlying C^*-algebra is not commutative.

2. Terminology and notation

Recall that a (right) *Hilbert C^*-module* E over a C^*-algebra A is a right A-module equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ such that the following conditions hold for all $x, y \in E$ and all $a \in A$:

(i) $\langle x, ya \rangle = \langle x, y \rangle a$;
(ii) $\langle x, y \rangle^* = \langle y, x \rangle$;
(iii) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ exactly when $x = 0$.

Moreover, E is a Banach space equipped with the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. We also call E a *Hilbert A-module* in this case. A complex linear map $\theta : E \to F$ between two Hilbert A-modules is called an *A-module homomorphism* if $\theta(xa) = \theta(x)a$

for all $a \in A$ and $x \in E$. See, for example, [15] or [20] for a general introduction to the theory of Hilbert C^*-modules. In this paper, we are interested in the case where the underlying C^*-algebra A is abelian, that is, the space $A = C_0(\Omega)$ of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω.

DEFINITION 2.1. Let A be a C^*-algebra. Suppose that E and F are Hilbert A-modules. A $\mathbb{C}$-linear map $\theta : E \to F$ is said to be *local* if $\theta(e)a = 0$ whenever $ea = 0$ for any $e \in E$ and $a \in A$.

The idea of local linear maps is often found in research in analysis. For example, a theorem of Peetre [19] states that local linear maps of the space of smooth functions defined on a manifold modeled on $\mathbb{R}^n$ are exactly the linear differential operators (see [18]). This was extended to the case of vector-valued differentiable functions defined on a finite-dimensional manifold by Kantrowitz and Neumann [14] and Araujo [3], and to the Banach $C^1[0, 1]$-module setting by Alaminos *et al.* [2]. Note that every A-module homomorphism is local. Conversely, every *bounded* local map is an A-module homomorphism (see [17, Proposition A.1]). See Remark 3.4 below for more information.

Throughout this paper, Ω and Δ are two locally compact Hausdorff spaces, and Ω_∞ is the one-point compactification of Ω. Moreover, E and F are a (right) Hilbert $C_0(\Omega)$-module and a (right) Hilbert $C_0(\Delta)$-module respectively, while $\theta : E \to F$ is a $\mathbb{C}$-linear map (not assumed to be bounded). We denote by $\mathcal{B}_{C_0(\Omega)}(E, F)$ the set of all bounded $C_0(\Omega)$-module homomorphisms from E into F. For any $\omega \in \Omega$, we let $\mathcal{N}_\Omega(\omega)$ be the set of all compact neighborhoods of ω in Ω. If $S \subseteq \Omega$, we denote by $\mathrm{Int}_\Omega(S)$ the interior of S in Ω. Moreover, when $U, V \subseteq \Omega$ and the closure of V is a compact subset of $\mathrm{Int}_\Omega(U)$, we denote by $\mathcal{U}_\Omega(V, U)$ the collection of all functions $\lambda \in C_0(\Omega)$ such that $0 \le \lambda \le 1$, $\lambda \equiv 1$ on V and λ vanishes outside U.

Note that any Hilbert $C_0(\Omega)$-module E may be regarded as a Hilbert $C(\Omega_\infty)$-module, and the results in [7] may be applied. In particular, E is the space of C_0-sections (that is, continuous sections that vanish at infinity) of an (F)-Hilbert bundle Ξ^E over Ω_∞ (see [7, p. 49]).

We define the modulus function $|f|(\omega) := \|f(\omega)\|$ for all $f \in E$ and $\omega \in \Omega$. For any closed subset S of Ω_∞ and $\omega \in \Omega_\infty$, we set

$$K_S^E := \{f \in E : f(\omega) = 0 \text{ for some } \omega \in S\} \quad \text{and} \quad I_\omega := \bigcup_{V \in \mathcal{N}_{\Omega_\infty}(\omega)} K_V^E$$

(for simplicity, we also denote $K_{\{\omega\}}^E$ by K_ω^E). Note that $K_\infty^E = E$ and the fiber Ξ_ω^E of Ξ^E at $\omega \in \Omega_\infty$ is E/K_ω^E. Furthermore, K_S^E is a Hilbert $K_S^{C_0(\Omega)}$-module and

$$K_S^E = \overline{E \cdot K_S^{C_0(\Omega)}}.$$

We also define

$$\Delta_\theta := \{\nu \in \Delta : \theta(E) \nsubseteq K_\nu^F\} = \{\nu \in \Delta : \theta(e)(\nu) \neq 0 \text{ for some } e \in E\}.$$

Then Δ_θ is an open subset of Δ and we put

$$\Omega_E := \{\omega \in \Omega : \Xi^E_\omega \neq (0)\}.$$

Let $\Omega_0 \subseteq \Omega$ be an open set. As in [7, p. 10], we denote by $\Xi^E|_{\Omega_0}$ the restriction of Ξ^E to Ω_0 and by E_{Ω_0} the set of C_0-sections on $\Xi^E|_{\Omega_0}$. One may make the following identifications:

$$C_0(\Omega_0) = K^{C_0(\Omega)}_{\Omega\setminus\Omega_0} \quad \text{and} \quad E_{\Omega_0} = K^E_{\Omega\setminus\Omega_0}.$$

3. Orthogonality preserving maps between Hilbert $C_0(\Omega)$-modules

We first recall two technical lemmas from [17, Lemmas 3.1 and 3.3, and Theorem 3.7] (see also [17, Remark 3.4]), which summarize, unify, and generalize techniques used sporadically in the literature [4, 8, 11].

LEMMA 3.1. *If $\sigma : \Delta_\theta \to \Omega_\infty$ is a map satisfying $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ for all $\nu \in \Delta_\theta$, then σ is continuous.*

LEMMA 3.2. *Let $\sigma : \Delta \to \Omega$ be a map (not necessarily continuous) with the property that $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ for every $\nu \in \Delta$.*

(a) *If $\mathfrak{U}_\theta := \{\nu \in \Delta : \sup_{\|e\|\leq 1} \|\theta(e)(\nu)\| = \infty\}$, then $\sigma(\mathfrak{U}_\theta)$ is a finite set.*

(b) *If $\mathfrak{N}_{\theta,\sigma} := \{\nu \in \Delta : \theta(K^E_{\sigma(\nu)}) \nsubseteq K^F_\nu\}$, then $\mathfrak{N}_{\theta,\sigma} \subseteq \mathfrak{U}_\theta$ and $\sigma(\mathfrak{N}_{\theta,\sigma})$ consists of nonisolated points in Ω.*

(c) *If σ is injective and sends isolated points in Δ to isolated points in Ω, then $\mathfrak{N}_{\theta,\sigma} = \emptyset$ and there exist a finite set T consisting of isolated points of Δ, a bounded linear map $\theta_0 : K^E_{\sigma(T)} \to K^F_T$ as well as linear maps $\theta_\nu : \Xi^E_{\sigma(\nu)} \to \Xi^F_\nu$ for all $\nu \in T$, such that $E = K^E_{\sigma(T)} \oplus \bigoplus_{\nu\in T} \Xi^E_{\sigma(\nu)}$,*

$$F = K^F_T \oplus \bigoplus_{\nu\in T} \Xi^F_\nu \quad \textit{and} \quad \theta = \theta_0 \oplus \bigoplus_{\nu\in T} \theta_\nu.$$

For any $\nu \in \Delta \setminus \mathfrak{N}_{\theta,\sigma}$, one may define $\theta_\nu : \Xi^E_{\sigma(\nu)} \to \Xi^F_\nu$ by

$$\theta_\nu(e + K^E_{\sigma(\nu)}) = \theta(e) + K^F_\nu \quad \forall e \in E, \tag{3.1}$$

or equivalently, $\theta_\nu(e(\sigma(\nu))) = (\theta(e))(\nu)$ for all $e \in E$.

LEMMA 3.3. *Let σ and $\mathfrak{U}_\theta$ be as in Lemma 3.2. Suppose, in addition, that σ is injective and θ is orthogonality preserving. Then there exists a bounded function $\psi : \Delta \setminus \mathfrak{U}_\theta \to \mathbb{R}_+$ such that*

$$\langle\theta(e), \theta(g)\rangle(\nu) = \psi(\nu)^2\langle e, g\rangle(\sigma(\nu)) \quad \forall e, g \in E, \forall \nu \in \Delta \setminus \mathfrak{U}_\theta. \tag{3.2}$$

Moreover, for each $\nu \in \Delta_\theta$, there is an isometry $\iota_\nu : \Xi^E_{\sigma(\nu)} \to \Xi^F_\nu$ such that

$$\theta(e)(\nu) = \psi(\nu)\iota_\nu(e(\sigma(\nu))) \quad \forall e \in E, \forall \nu \in \Delta_\theta \setminus \mathfrak{U}_\theta.$$

PROOF. Fix any $\nu \in \Delta_\theta \setminus \mathfrak{U}_\theta$. By Lemma 3.2(b), the map θ_ν, as in (3.1), is well defined. Suppose that η_1 and η_2 are orthogonal elements in $\Xi^E_{\sigma(\nu)}$ and $\eta_1 \neq 0$ (this is possible because $\Delta_\theta \setminus \mathfrak{N}_{\theta,\sigma} \subseteq \sigma^{-1}(\Omega_E)$), and that $g_1, g_2 \in E$ and $g_i(\sigma(\nu)) = \eta_i$ when $i = 1, 2$. If $V \in \mathcal{N}_\Omega(\sigma(\nu))$ and g_1 does not vanish on V, then by replacing g_2 with

$$\left(g_2 - \frac{\langle g_2, g_1\rangle}{|g_1|^2} g_1\right)\lambda,$$

where $\lambda \in \mathcal{U}_\Omega(\{\sigma(\nu)\}, V)$, we see that there are orthogonal elements $e_1, e_2 \in E$ such that $e_i(\sigma(\nu)) = \eta_i$ when $i = 1, 2$. Hence θ_ν is nonzero, because $\nu \in \Delta_\theta$, and is an orthogonality preserving $\mathbb{C}$-linear map between Hilbert spaces. Consequently, there exist an isometry $\iota_\nu : \Xi^E_{\sigma(\nu)} \to \Xi^F_\nu$ and a unique scalar $\psi(\nu) > 0$ such that $\theta_\nu = \psi(\nu)\iota_\nu$. For any $\nu \in \Delta \setminus \Delta_\theta$, we set $\psi(\nu) = 0$. Then clearly (3.2) holds. Next, we show that ψ is a bounded function on $\Delta \setminus \mathfrak{U}_\theta$. Suppose that this is not the case. Then there exist distinct points $\nu_n \in \Delta_\theta \setminus \mathfrak{U}_\theta$ such that $\psi(\nu_n) > n^3$. If $e_n \in E$ such that $\|e_n\| = 1$ and its modulus function satisfies

$$|e_n|(\sigma(\nu_n)) = \sqrt{\langle e_n, e_n\rangle}(\sigma(\nu_n)) \geq (n-1)/n$$

(note that $\nu_n \in \sigma^{-1}(\Omega_E)$), then in light of (3.2),

$$|\theta(e_n)|(\nu_n) = \psi(\nu_n)|e_n|(\sigma(\nu_n)) > n^2(n-1).$$

As $\{\sigma(\nu_n)\}$ is a set of distinct points (note that σ is injective), by passing to a subsequence if necessary, we may assume that there are $U_n \in \mathcal{N}_\Omega(\sigma(\nu_n))$ such that $U_n \cap U_m = \emptyset$ when $m \neq n$. Now pick any $V_n \in \mathcal{N}_\Omega(\sigma(\nu_n))$ such that $V_n \subseteq \mathrm{Int}_\Omega(U_n)$ and choose a function $\lambda_n \in \mathcal{U}_\Omega(V_n, U_n)$ for all $n \in \mathbb{N}$. Define $e := \sum_{k=1}^\infty e_k\lambda_k^2/k^2 \in E$. As $n^2 e - e_n\lambda_n^2 \in K^E_{U_n}$ and $e_n - e_n\lambda_n^2 = e_n(1-\lambda_n^2) \in K^E_{V_n}$ for all $n \in \mathbb{N}$,

$$\|\theta(e)\| \geq \|\theta(e)(\nu_n)\| = \frac{\|\theta(e_n\lambda_n^2)(\nu_n)\|}{n^2} = \frac{\|\theta(e_n)(\nu_n)\|}{n^2} > n-1,$$

by the relation between θ and σ, which is a contradiction. □

3.1. Hilbert bundles over the same base space.

REMARK 3.4. For any $e \in E$, we define

$$\mathrm{supp}_\Omega\, e := \overline{\{\omega \in \Omega : e(\omega) \neq 0\}}.$$

It is not hard to check that the following statements are equivalent (and this tells us that local maps are the same as *support shrinking maps* [8]):

(i) θ is local (see Definition 2.1);
(ii) $\theta(K^E_V) \subseteq K^F_V$ for all nonempty open set V;
(iii) $\mathrm{supp}_\Omega\, \theta(e) \subseteq \mathrm{supp}_\Omega\, e$ for all $e \in E$;
(iv) $\mathrm{supp}_\Omega\, \theta(e)\lambda \subseteq \mathrm{supp}_\Omega\, e$ for all $e \in E$ and $\lambda \in C_0(\Omega)$.

THEOREM 3.5. *Let Ω be a locally compact Hausdorff space, and let E and F be two Hilbert $C_0(\Omega)$-modules. Suppose that $\theta : E \to F$ is an orthogonality preserving local $\mathbb{C}$-linear map. The following assertions hold.*

(a) $\theta \in \mathcal{B}_{C_0(\Omega)}(E, F)$.

(b) *There is a bounded nonnegative function φ on Ω, continuous on Ω_E, such that*

$$\langle \theta(e), \theta(g) \rangle = \varphi \cdot \langle e, g \rangle \quad \forall e, g \in E.$$

(c) *There exist a strictly positive element $\psi_0 \in C_b(\Omega_\theta)_+$ and $J \in \mathcal{B}_{C_0(\Omega_\theta)}(E_{\Omega_\theta}, F_{\Omega_\theta})$ such that the fiber map J_ω is an isometry for all $\omega \in \Omega_\theta$ and*

$$\theta(e)(\omega) = \psi_0(\omega) J(e)(\omega) \quad \forall e \in E, \forall \omega \in \Omega_\theta.$$

PROOF. Note that the conclusions of Lemmas 3.2 and 3.3 hold when $\Omega = \Delta$ and $\sigma = \mathrm{Int}_\Omega$.

We prove (a). By Remark 3.4 and Lemma 3.2(c), θ is a $C_0(\Omega)$-module homomorphism. Further, as θ_ν (as in Lemma 3.2(c)) is an orthogonality preserving, hence bounded, linear map between Hilbert spaces for all $\nu \in T$ (where T is as in Lemma 3.2(c) and $\sigma = \mathrm{Int}_\Omega$), we know from Lemma 3.2(c) that θ is bounded (note that T is finite).

Now we consider (b). By part (a), $\mathfrak{U}_\theta = \emptyset$. Thus, Lemma 3.3 tells us that there exists a bounded nonnegative function ψ on Ω such that $\langle \theta(e), \theta(f) \rangle = |\psi|^2 \cdot \langle e, f \rangle$. Let $\omega \in \Omega_E$ and pick any $e \in E$ for which there exists $U_\omega \in \mathcal{N}_\Omega(\omega)$ such that $e(\nu) \neq 0$ for all $\nu \in U_\omega$. Then $\psi(\omega) = |\theta(e)|(\omega)/|e|(\omega)$ for all $\omega \in U_\omega$. Hence ψ is continuous on Ω_E, and $\varphi = \psi^2$ is the required function.

It remains to prove (c). Note that $\Omega_\theta \subseteq \Omega_E$, by part (a). Since $\varphi(\omega) > 0$ for all $\omega \in \Omega_\theta$, we know from part (b) that $\psi = \varphi^{1/2}$ is a strictly positive element ψ_0 in $C_b(\Omega_\theta)_+$. The equivalence in [7, (2.2)] (consider E and F as Hilbert $C(\Omega_\infty)$-bundles) tells us that the restriction of θ induces a bounded Banach bundle map, again denoted by θ, from $\Xi^E|_{\Omega_\theta}$ into $\Xi^F|_{\Omega_\theta}$. For each $\eta \in \Xi^E|_{\Omega_\theta}$, we define $J(\eta) := \psi_0(\pi(\eta))^{-1}\theta(\eta)$, where $\pi : \Xi^E \to \Omega$ is the canonical projection. Then $J : \Xi^E|_{\Omega_\theta} \to \Xi^F|_{\Omega_\theta}$ is a Banach bundle map, as $\eta \mapsto \psi_0(\pi(\eta))^{-1}$ is continuous, which is an isometry on each fiber (hence J is bounded) such that $\theta(\eta) = \psi(\pi(\eta))J(\eta)$. This map J induces a map, again denoted by J, in $\mathcal{B}_{C_0(\Omega_\theta)}(E_{\Omega_\theta}, F_{\Omega_\theta})$ that satisfies the requirement of part (c). □

It is natural to ask if one can find $\varphi \in C_b(\Omega)$ such that the conclusion of Theorem 3.5(b) holds. Unfortunately, the following example tells us that this is not the case in general.

EXAMPLE 3.6. Let $\Omega = \mathbb{R}_\infty$, the one-point compactification of the real line $\mathbb{R}$. Let E and F be the Hilbert $C(\Omega)$-module $C_0(\mathbb{R})$, and define $\theta(f)(t) = f(t)\cos t$ for all $f \in E$ and $t \in \mathbb{R}$. Then $\Omega \setminus \Omega_E = \{\infty\}$ and $\varphi(t) = \cos t$ for all $t \in \mathbb{R} = \Omega_E$. Thus φ does not extend to a continuous function on Ω.

We can now obtain the following commutative analog of [10, Proposition 2.3]. This, together with Corollary 3.11, asserts that the orthogonality structure of a Hilbert bundle essentially determines its unitary structure, as we claimed in the introduction. Note also that a large portion of Lemma 3.2 was used to deal with the possibility of $\theta(K^E_{\sigma(\nu)}) \nsubseteq K^F_\nu$ (this situation does not arise for $C_0(\Omega)$-module homomorphism), and this corollary actually has a much easier proof.

COROLLARY 3.7. *Let Ω be a locally compact Hausdorff space, and E and F be Hilbert $C_0(\Omega)$-modules. Suppose that $\theta : E \to F$ is a $C_0(\Omega)$-module homomorphism that preserves orthogonality. Then θ is bounded and there exists a bounded nonnegative function φ on Ω that is continuous on Ω_E and satisfies $\langle\theta(e), \theta(f)\rangle = \varphi \cdot \langle e, f\rangle$ for all $e, f \in E$.*

Recall that a Hilbert $C_0(\Omega)$-module E is *full* if the $\mathbb{C}$-linear span $\langle E, E\rangle$ of the set

$$\{\langle e, f\rangle : e, f \in E\}$$

is dense in $C_0(\Omega)$.

REMARK 3.8. A Hilbert $C_0(\Omega)$-module E is full if and only if $E \nsubseteq K^E_\omega$ for all $\omega \in \Omega$ (or equivalently, $\Omega_E = \Omega$). In fact, if $E \subseteq K^E_\omega$, then $f(\omega) = 0$ for all $f \in \langle E, E\rangle$ and E is not full. Conversely, if E is not full, then there exists $\omega \in \Omega$ such that $f(\omega) = 0$ for all $f \in \langle E, E\rangle$, because the closure of $\langle E, E\rangle$ is an ideal of $C_0(\Omega)$, and $E \subseteq K^E_\omega$.

REMARK 3.9. If E is full, then by the previous remark, the function φ in Theorem 3.5(b) (and Corollary 3.7) is an element of $C_b(\Omega)$. However, there is no guarantee that this function is strictly positive.

REMARK 3.10. Suppose that F is full and θ is a surjective orthogonality preserving local $\mathbb{C}$-linear map. If there exists $\omega \in \Omega \setminus \Omega_\theta$, then $F = \theta(E) \subseteq K^F_\omega$, which contradicts the fullness of F (see Remark 3.8). Consequently, $\Omega_\theta = \Omega$. As $\theta \in \mathcal{B}_{C_0(\Omega)}(E, F)$ by Theorem 3.5(a), we see that $\Omega = \Omega_\theta \subseteq \Omega_E$ and E is full.

COROLLARY 3.11. *Let Ω be a locally compact Hausdorff space, and let E and F be two Hilbert $C_0(\Omega)$-modules. Suppose that F is full and $\theta : E \to F$ is an orthogonality preserving surjective local $\mathbb{C}$-linear map. Then $\theta \in \mathcal{B}_{C_0(\Omega)}(E, F)$. Moreover, there exist a strictly positive element $\psi \in C_b(\Omega)_+$ and a unitary map $U \in \mathcal{B}_{C_0(\Omega)}(E, F)$ such that $\theta = \psi \cdot U$.*

PROOF. Remark 3.10 tells us that $\Omega_\theta = \Omega$. By the surjectivity of θ, the bounded Banach bundle map J in Theorem 3.5 is unitary on each fiber. Therefore, the element $U \in \mathcal{B}_{C_0(\Omega)}(E, F)$ corresponding to J, as in [7, (2.2)], is unitary. □

3.2. Hilbert bundles over different base spaces.

DEFINITION 3.12. The map θ is said to be *quasilocal* if it is bijective and, for all $e \in E$ and $\lambda \in C_0(\Delta)$,

$$\operatorname{supp}_\Omega \theta^{-1}(\theta(e)\lambda) \subseteq \operatorname{supp}_\Omega e. \tag{3.3}$$

Note that if $\Delta = \Omega$ and θ is both local and bijective (hence θ^{-1} is also local), then θ is quasilocal by Remark 3.4.

LEMMA 3.13. *Suppose that θ is bijective and quasilocal and that θ and θ^{-1} both preserve orthogonality. Then $|\theta(e)||\theta(g)| = 0$ if $e, g \in E$ and $\operatorname{supp}_\Omega e \cap \operatorname{supp}_\Omega g = \emptyset$.*

PROOF. Suppose, on the contrary, that there exist $e_1, e_2 \in E$ and $\nu \in \Delta$ such that $\operatorname{supp}_\Omega e_1 \cap \operatorname{supp}_\Omega e_2 = \emptyset$ but $\|\theta(e_1)(\nu)\|\|\theta(e_2)(\nu)\| \neq 0$. As θ preserves orthogonality, we may assume that $\theta(e_1)(\nu)$ and $\theta(e_2)(\nu)$ are orthogonal unit vectors in Ξ^F_ν. Take $U, W \in \mathcal{N}_\Delta(\nu)$ such that $W \subseteq \operatorname{Int}_\Delta(U)$ and $\|\theta(e_i)(\mu)\| > 1/2$ for all $\mu \in U$. Pick any $\lambda \in \mathcal{U}_\Delta(W; U)$, and define $h_i \in F \setminus \{0\}$ (when $i = 1, 2$) by

$$h_i(\mu) := \begin{cases} \theta(e_i)(\mu)\dfrac{\lambda(\mu)}{|\theta(e_i)|(\mu)} & \text{if } \mu \in \operatorname{Int}_\Delta(U) \\ 0 & \text{if } \mu \notin \operatorname{Int}_\Delta(U) \end{cases}$$

and set $e'_i := \theta^{-1}(h_i)$. The orthogonality of h_1 and h_2 (recall that e_1 and e_2 are orthogonal), together with that of $h_1 + h_2$ and $h_1 - h_2$ (as $|h_1| = \lambda = |h_2|$), ensures the orthogonality of e'_1 and e'_2, as well as that of $e'_1 + e'_2$ and $e'_1 - e'_2$. It follows that $|e'_1| = |e'_2| \neq 0$, which contradicts the fact that $|e'_1||e'_2| = 0$, as θ is quasilocal. □

THEOREM 3.14. *Let Ω and Δ be locally compact Hausdorff spaces. Suppose that E is a full Hilbert $C_0(\Omega)$-module and F is a full Hilbert $C_0(\Delta)$-module. If $\theta : E \to F$ is a bijective $\mathbb{C}$-linear map such that both θ and θ^{-1} are quasilocal and orthogonality preserving, then θ is bounded and*

$$\theta(e)(\nu) = \psi(\nu)J_\nu(e(\sigma(\nu))) \quad \forall e \in E, \forall \nu \in \Delta, \tag{3.4}$$

where $\sigma : \Delta \to \Omega$ is a homeomorphism, ψ is a strictly positive element of $C_b(\Delta)_+$, and J_ν is a unitary operator from $\Xi^E_{\sigma(\nu)}$ onto Ξ^F_ν such that the map $\nu \mapsto J_\nu(f(\sigma(\nu)))$ is continuous for all fixed $f \in E$.

PROOF. We consider E as a Hilbert $C(\Omega_\infty)$-module. For each $\nu \in \Delta$, let

$$S_\nu := \{\omega \in \Omega_\infty : \theta(K^E_{\Omega_\infty \setminus W}) \nsubseteq K^F_\nu \ \forall W \in \mathcal{N}_{\Omega_\infty}(\omega)\}.$$

We first show that S_ν is a singleton. Indeed, assume that $S_\nu = \emptyset$. Then for all $\omega \in \Omega_\infty$, there is $W_\omega \in \mathcal{N}_{\Omega_\infty}(\omega)$ such that $\theta(K^E_{\Omega_\infty \setminus W_\omega}) \subseteq K^F_\nu$. Consider $\omega_1, \ldots, \omega_n \in \Omega_\infty$ such that

$$\bigcup_{k=1}^n \operatorname{Int}_{\Omega_\infty}(W_{\omega_k}) = \Omega_\infty,$$

and take a partition of unity $\{\varphi_k\}_{k=1}^n$ that is subordinate to $\{\operatorname{Int}_{\Omega_\infty}(W_{\omega_k})\}_{k=1}^n$. Then $e\varphi_k \in K^E_{\Omega_\infty \setminus W_{\omega_k}}$ for all $e \in E$, and so $\theta(e) \in K^F_\nu$. As θ is surjective, this shows that $F = K^F_\nu$, and contradicts the fullness of F (see Remark 3.8). Now, assume that there are distinct elements $\omega_1, \omega_2 \in S_\nu$. Take $V_1 \in \mathcal{N}_{\Omega_\infty}(\omega_1)$ and $V_2 \in \mathcal{N}_{\Omega_\infty}(\omega_2)$ such that $V_1 \cap V_2 = \emptyset$. By the definition of S_ν, there exist $e_1, e_2 \in E$ such that $\operatorname{supp}_\Omega e_i \subseteq V_i \setminus \{\infty\}$ and $\theta(e_i)(\nu) \neq 0$ when $i = 1, 2$, which contradicts Lemma 3.13.

Thus, there is a unique element $\sigma(\nu) \in \Omega_\infty$ such that $S_\nu = \{\sigma(\nu)\}$. Next, we claim that

$$\theta(I^E_{\sigma(\nu)}) \subseteq I^F_\nu \quad \forall \nu \in \Delta. \tag{3.5}$$

Consider any $V \in \mathcal{N}_{\Omega_\infty}(\sigma(\nu))$ and $e \in K^E_V$. Pick $U \in \mathcal{N}_{\Omega_\infty}(\sigma(\nu))$ such that $U \subseteq \mathrm{Int}_{\Omega_\infty}(V)$. By the definition of σ, there exists $g \in K^E_{\Omega_\infty \setminus U}$ such that $\theta(g)(\nu) \neq 0$. Hence, there is $W \in \mathcal{N}_\Delta(\nu)$ such that $\theta(g)(\mu) \neq 0$ for all $\mu \in W$, and Lemma 3.13 implies that $\theta(e) \in K^F_W$, as claimed. If there exists $\nu \in \Delta \setminus \Delta_\theta$, then $f(\nu) = 0$ for all $f \in F$, because θ is surjective, which contradicts the fullness of F. Thus, $\Delta_\theta = \Delta$ and $\sigma : \Delta \to \Omega_\infty$ is continuous, by Lemma 3.1. As θ^{-1} is also quasilocal and orthogonality preserving, a similar argument to the above gives a continuous map $\tau : \Omega \to \Delta_\infty$ satisfying $\theta^{-1}(I^F_{\tau(\omega)}) \subseteq I^E_\omega$ for all $\omega \in \Omega$. Now, the argument of [17, Theorem 5.3] tells us that σ is a homeomorphism from Δ to Ω such that

$$\theta(e \cdot \varphi) = \theta(e) \cdot \varphi \circ \sigma \quad \forall e \in E, \forall \varphi \in C_0(\Omega),$$

and by Lemma 3.2(c), there exists a finite set T consisting of isolated points of Δ such that θ restricts to a bounded map from $K^E_{\sigma(T)}$ to K^F_T. Since any $\nu \in T$ is an isolated point, θ induces an orthogonality preserving, hence bounded, map θ_ν from the Hilbert space $\Xi^E_{\sigma(\nu)}$ onto the Hilbert space Ξ^F_ν. This shows that θ is bounded, by Lemma 3.2(c) and the fact that T is finite. By Lemma 3.3, there is a surjective isometry $J_\nu : \Xi^E_{\sigma(\nu)} \to \Xi^F_\nu$ such that

$$\theta(e)(\nu) = \psi(\nu) J_\nu(e(\sigma(\nu))) \quad \forall e \in E, \forall \nu \in \Delta.$$

Now the fullness of E implies that $\psi(\nu) > 0$ for all $\nu \in \Delta$, and the map $\nu \mapsto \theta(e)(\nu)/\psi(\nu)$ is evidently continuous. $\square$

The following example shows the necessity of the assumption in Theorem 3.14 that θ^{-1} preserves orthogonality.

EXAMPLE 3.15. Let Ω be a nonempty locally compact Hausdorff space, Ω_2 be the topological disjoint sum of two copies of Ω, and $j_1, j_2 : \Omega \to \Omega_2$ be the embeddings into the first and the second copies of Ω in Ω_2, respectively. Let H be a nonzero Hilbert space, and let H_2 be the Hilbert space direct sum of two copies of H. Then the map $\theta : C_0(\Omega_2, H) \to C_0(\Omega, H_2)$, defined by

$$\theta(f)(\omega) = (f(j_1(\omega)), f(j_2(\omega))),$$

is a bijective $\mathbb{C}$-linear map preserving orthogonality satisfying condition (3.3). However, θ is not of the expected form. Note that θ^{-1} does not preserve orthogonality.

References

[1] Y. A. Abramovich, E. L. Arenson and A. K. Kitover, *Banach $C(K)$-Modules and Operators Preserving Disjointness*, Pitman Research Notes in Mathematics, 277 (John Wiley & Sons, Inc., New York, 1992).

[2] J. Alaminos, M. Brešar, M. Černe, J. Extremera and A. R. Villena, 'Zero product preserving maps on $C^1[0, 1]$', *J. Math. Anal. Appl.* **347** (2008), 472–481.

[3] J. Araujo, 'Linear biseparating maps between spaces of vector-valued differentiable functions and automatic continuity', *Adv. Math.* **187** (2004), 488–520.

[4] J. Araujo and K. Jarosz, 'Automatic continuity of biseparating maps', *Studia Math.* **155** (2003), 231–239.

[5] A. Blanco and A. Turnšek, 'On maps that preserve orthogonality in normed spaces', *Proc. R. Soc. Edinburgh* A **136** (2006), 709–716.

[6] J. Chmieliński, 'Linear mappings approximately preserving orthogonality', *J. Math. Anal. Appl.* **304** (2005), 158–169.

[7] M. J. Dupré and R. M. Gillette, *Banach Bundles, Banach Modules and Automorphisms of C^*-Algebras*, Pitman Research Notes in Mathematics, 92 (John Wiley & Sons, Inc., New York, 1983).

[8] H. L. Gau, J. S. Jeang and N. C. Wong, 'Biseparating linear maps between continuous vector-valued function spaces', *J. Aust. Math. Soc. (Series A)* **74** (2003), 101–111.

[9] M. H. Hsu and N. C. Wong, 'Isometric embeddings of Banach bundles', *Taiwanese J. Math.*, to appear.

[10] D. Ilišević and A. Turnšek, 'Approximately orthogonality preserving mappings on C^*-modules', *J. Math. Anal. Appl.* **341** (2008), 298–308.

[11] K. Jarosz, 'Automatic continuity of separating linear isomorphisms', *Canad. Math. Bull.* **33** (1990), 139–144.

[12] J. S. Jeang and N. C. Wong, 'On the Banach–Stone problem', *Studia Math.* **155** (2003), 95–105.

[13] M. Jerison, 'The space of bounded maps into a Banach space', *Ann. of Math.* (2) **52** (1950), 309–327.

[14] R. Kantrowitz and M. M. Neumann, 'Disjointness preserving and local operators on algebras of differentiable functions', *Glasg. Math. J.* **43** (2001), 295–309.

[15] E. C. Lance, *Hilbert C^*-Modules. A Toolkit for Operator Algebraists*, London Mathematical Society Lecture Note Series, 210 (Cambridge University Press, Cambridge, 1995).

[16] K. S. Lau, 'A representation theorem for isometries of $C(X, E)$', *Pacific J. Math.* **60** (1975), 229–233.

[17] C. W. Leung, C. K. Ng and N. C. Wong, 'Automatic continuity and $C_0(\Omega)$-linearity of linear maps between $C_0(\Omega)$-modules', *J. Operator Theory*, to appear.

[18] R. Narasimhan, *Analysis on Real and Complex Manifolds*, Advanced Studies in Pure Mathematics, 1 (North-Holland, Amsterdam, 1968).

[19] J. Peetre, 'Réctification à l'article "Une caractérisation abstraite des opérateurs différentiels"', *Math. Scand.* **8** (1960), 116–120.

[20] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace C^*-Algebras*, Mathematical Surveys and Monographs, 60 (American Mathematical Society, Providence, RI, 1998).

CHI-WAI LEUNG, Department of Mathematics,
The Chinese University of Hong Kong, Hong Kong
e-mail: cwleung@math.cuhk.edu.hk

CHI-KEUNG NG, Chern Institute of Mathematics and LPMC,
Nankai University, Tianjin 300071, PR China
e-mail: ckng@nankai.edu.cn

NGAI-CHING WONG, Department of Applied Mathematics,
National Sun Yat-sen University, Kaohsiung 80424, Taiwan
e-mail: wong@math.nsysu.edu.tw