A BANACH-STONE THEOREM FOR RIESZ ISOMORPHISMS OF BANACH LATTICES

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Abstract. Let X and Y be compact Hausdorff spaces, and E, F be Banach lattices. Let $C(X,E)$ denote the Banach lattice of all continuous E-valued functions on X equipped with the pointwise ordering and the sup norm. We prove that if there exists a Riesz isomorphism $\Phi : C(X,E) \to C(Y,F)$ such that Φf is non-vanishing on Y if and only if f is non-vanishing on X, then X is homeomorphic to Y, and E is Riesz isomorphic to F. In this case, Φ can be written as a weighted composition operator: $\Phi f(y) = \Pi(y)(f(\varphi(y)))$, where φ is a homeomorphism from Y onto X, and $\Pi(y)$ is a Riesz isomorphism from E onto F for every y in Y. This generalizes some known results obtained recently.

1. Introduction

Let X and Y be compact Hausdorff spaces, and $C(X)$, $C(Y)$ denote the spaces of real-valued continuous functions defined on X, Y respectively. There are three versions of the Banach-Stone theorem. That is to say, surjective linear isometries, ring isomorphisms and lattice isomorphisms from $C(X)$ onto $C(Y)$ yield homeomorphisms between X and Y, respectively (cf. [1, 6, 14]).

Jerison [13] got the first vector-valued version of the Banach-Stone theorem. He proved that if the Banach space E is strictly convex, then every surjective linear isometry $\Phi : C(X,E) \to C(Y,E)$ can be written as a weighted composition operator

$$\Phi f(y) = \Pi(y)(f(\varphi(y))), \qquad \forall\, f \in C(X,E),\ \forall\, y \in Y.$$

Here φ is a homeomorphism from Y onto X, and Π is a continuous map from Y into the space $(\mathcal{L}(E,E), SOT)$ of bounded linear operators on E equipped with the strong operator topology (SOT). Furthermore, $\Pi(y)$ is a surjective linear isometry on E for every y in Y. After Jerison [13], many vector-valued versions of the Banach-Stone theorem have been obtained in different ways (see, e.g., [3, 4, 5, 7, 9, 10, 12, 16]).

Let E, F be nonzero real Banach lattices, and $C(X,E)$ be the Banach lattice of all continuous E-valued functions on X equipped with the pointwise ordering and the sup norm. Note that, in general, a Riesz isomorphism (i.e., lattice isomorphism) from $C(X,E)$ onto $C(Y,F)$ does not necessarily induce a topological homeomorphism from X onto Y (cf. [16, Example 3.5]). To consider the Banach-Stone theorems for continuous Banach lattice valued functions, we would like to mention the papers [5, 7, 16]. In particular, when E, F are both Banach lattices

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and Riesz algebras, Miao, Cao and Xiong [16] recently proved that if F has no zero-divisor and there exists a Riesz algebraic isomorphism $\Phi : C(X,E) \to C(Y,F)$ such that Φf is non-vanishing on Y if f is non-vanishing on X, then X is homeomorphic to Y, and E is Riesz algebraically isomorphic to F. By saying f in $C(X,E)$ is *non-vanishing*, we mean that $0 \notin f(X)$. Indeed, under these conditions they obtained that $\Phi^{-1}g$ is non-vanishing on X if $g \in C(Y,F)$ is non-vanishing on Y. Note that every Riesz algebraic isomorphism must be a Riesz isomorphism.

Let E and F be Banach lattices. More recently, Ercan and Önal [7] have established that if F is an AM-space with unit, i.e., a $C(K)$-space, and there exists a Riesz isomorphism $\Phi : C(X,E) \to C(Y,F)$ such that Φf is non-vanishing on Y if and only if f is non-vanishing on X, that is, both Φ and Φ^{-1} are non-vanishing preserving, then X is homeomorphic to Y, and E is Riesz isomorphic to F.

Inspired by [5, 7, 16], one can set a natural question:

Question 1. Is X homeomorphic to Y if E, F are Banach lattices and there exists a Riesz isomorphism $\Phi : C(X,E) \to C(Y,F)$ such that both Φ and Φ^{-1} are non-vanishing preserving?

In this paper we show the answer to the above question is affirmative. Moreover, in this case Φ can be written as a weighted composition operator:

$$\Phi f(y) = \Pi(y)(f(\varphi(y))), \qquad \forall\, f \in C(X,E),\, \forall\, y \in Y,$$

where φ is a homeomorphism from Y onto X, and $\Pi(y)$ is a Riesz isomorphism from E onto F for every y in Y. This generalizes the results obtained by Cao, Reilly and Xiong [5], Miao, Cao, and Xiong [16], and Ercan and Önal [7].

Our notions are standard. For the undefined notions and basic facts concerning Banach lattices we refer the readers to the monographs [1, 2, 14].

2. A Banach-Stone theorem for Riesz isomorphisms

In the following we always assume X and Y are compact Hausdorff spaces, E and F are nonzero Banach lattices, and $\mathcal{L}(E,F)$ is the space of bounded linear operators from E into F equipped with SOT. For x in X and y in Y, let M_x and N_y be defined as

$$M_x = \{f \in C(X,E) : \, f(x) = 0\}, \quad N_y = \{g \in C(Y,F) : g(y) = 0\}.$$

Clearly, M_x and N_y are closed (order) ideals in $C(X,E)$ and $C(Y,F)$, respectively.

Lemma 2. *Let $\Phi : C(X,E) \to C(Y,F)$ be a Riesz isomorphism such that $\Phi(f)$ is non-vanishing on Y if and only if f is non-vanishing on X. Then for each x in X there exits a unique y in Y such that*

$$\Phi M_x = N_y.$$

In particular, this defines a bijection φ from Y onto X by $\varphi(y) = x$.

Proof. For each x in X, let

$$\mathcal{Z}(\Phi M_x) = \{y \in Y : \Phi f(y) = 0 \text{ for all } f \in M_x\}.$$

We first claim that $\mathcal{Z}(\Phi M_x)$ is non-empty. Suppose, on the contrary, that $\mathcal{Z}(\Phi M_x)$ is empty. Then for each y in Y there would exist an f_y in M_x such that $\Phi f_y(y) \neq 0$, and thus Φf_y is non-vanishing in an open neighborhood of y. Note that $|f_y| \in M_x$, and $\Phi|f_y| = |\Phi f_y|$ since Φ is a Riesz isomorphism. Therefore, we can assume further that both f_y and Φf_y are positive, by replacing them by their absolute values if necessary. By the compactness of Y, we can choose finitely many $f_1, \ldots, f_n$ from M_x^+ such that the positive functions $\Phi f_1, \ldots, \Phi f_n$ have no common zero in Y. Hence $\Phi(f_1 + \cdots + f_n)$ is strictly positive, that is, $\Phi(f_1 + \cdots + f_n)(y) > 0$ for each y in Y. This contradicts the fact that $f_1 + \cdots + f_n$ vanishes at x. We thus prove that $\mathcal{Z}(\Phi M_x) \neq \phi$.

Next, we claim that $\mathcal{Z}(\Phi M_x)$ is a singleton. Indeed, if $y_1, y_2 \in \mathcal{Z}(\Phi M_x)$ then we would have $\Phi M_x \subseteq N_{y_i}, i = 1, 2$. Applying the above argument to Φ^{-1}, we shall have $\Phi^{-1} N_{y_i} \subseteq M_{x_i}$ for some x_i in X, $i = 1, 2$. It follows that $\Phi M_x \subseteq N_{y_i} \subseteq \Phi M_{x_i}$, $i = 1, 2$. Then $x = x_1 = x_2$ since Φ is bijective and X is Hausdorff. Thus,

$$y_1 = y_2 \quad \text{and} \quad \Phi M_x = N_{y_1} = N_{y_2}.$$

Now, we can define a bijective map $\varphi : Y \to X$ such that

$$\Phi M_{\varphi(y)} = N_y, \quad \forall\, y \in Y.$$

□

The following main result answers affirmatively the question mentioned in the introduction, and solves the conjecture of Ercan and Önal in [7].

Theorem 3. *Let $\Phi : C(X, E) \to C(Y, F)$ be a Riesz isomorphism such that Φf is non-vanishing on Y if and only if f is non-vanishing on X. Then Y is homeomorphic to X, and Φ can be written as a weighted composition operator*

$$\Phi f(y) = \Pi(y)(f(\varphi(y))), \qquad \forall\, f \in C(X, E), \forall\, y \in Y.$$

Here φ is a homeomorphism from Y onto X, and $\Pi(y)$ is a Riesz isomorphism from E onto F for every y in Y. Moreover, $\Pi : Y \to (\mathcal{L}(E, F), SOT)$ is continuous, and $\|\Phi\| = \sup_{y \in Y} \|\Pi(y)\|$.

Proof. First, we show that the bijection φ given in Lemma 2 is a homeomorphism from Y onto X. It suffices to verify the continuity of φ since Y is compact and X is Hausdorff. To this end, suppose, to the contrary, that there would exist a net $\{y_\lambda\}$ in Y converging to y_0 in Y, but $\varphi(y_\lambda)$ converges to $x_0 \neq \varphi(y_0)$ in X.

Let U_{x_0} and $U_{\varphi(y_0)}$ be disjoint open neighborhoods of x_0 and $\varphi(y_0)$, respectively. First, for any f in $C(X, E)$ vanishing outside $U_{\varphi(y_0)}$ we claim that $\Phi f(y_0) = 0$. Indeed, since $\varphi(y_\lambda)$ belongs to U_{x_0} for λ large enough and $f(x) = 0$ for any x in U_{x_0}, we have that $f \in M_{\varphi(y_\lambda)}$. It follows from Lemma 2 that $\Phi f \in N_{y_\lambda}$, that is, $\Phi f(y_\lambda) = 0$ when λ is large enough. Thus, $\Phi f(y_0) = 0$ since $y_\lambda \to y_0$ and Φf is continuous.

Let $\chi \in C(X)$ such that χ vanishes outside $U_{\varphi(y_0)}$ and $\chi(\varphi(y_0)) = 1$. Then, for any h in $C(X, E)$ we have $h = \chi h + (1 - \chi)h$. Since χh vanishes outside $U_{\varphi(y_0)}$, by the above argument, we can see that $\Phi(\chi h)(y_0) = 0$. Clearly, $\Phi((1 - \chi)h)$ vanishes at y_0 since $(1 - \chi)h \in M_{\varphi(y_0)}$. Thus, $\Phi h(y_0) = \Phi(\chi h)(y_0) + \Phi((1 - \chi)h)(y_0) = 0$ for any h in $C(X, E)$. This leads to a contradiction since Φ is surjective. So φ is continuous, and thus a homeomorphism from Y onto X satisfying $\Phi M_{\varphi(y)} = N_y$ for each y in Y.

Next, note that $\ker \delta_{\varphi(y)} = \ker \delta_y \circ \Phi$, where δ_y is the Dirac functional. Hence, there is a linear operator $\Pi(y) : E \to F$ such that $\delta_y \circ \Phi = \Pi(y) \circ \delta_{\varphi(y)}$. In other words,

$$\Phi f(y) = \Pi(y)(f(\varphi(y))), \qquad \forall\, f \in C(X,E), \forall\, y \in Y.$$

See, e.g., [8, p. 67].

It is a routine work to verify the other assertions in the statement of this theorem. For the convenience of the readers, we give a sketch of the rest of the proof. For e in E, let $\mathbf{1}_X \otimes e \in C(X,E)$ be defined by $(\mathbf{1}_X \otimes e)(x) = e$ for each x in X. Let y in Y be fixed. If $e \neq 0$, then $\Pi(y)e = \Pi(y)((\mathbf{1}_X \otimes e)(\varphi(y))) = \Phi(\mathbf{1}_X \otimes e)(y) \neq 0$ since $\mathbf{1}_X \otimes e$ is non-vanishing. Hence, $\Pi(y)$ is one-to-one. On the other hand, for u in F we can find a function f in $C(X,E)$ such that $\Phi f = \mathbf{1}_Y \otimes u$ by the surjectivity of Φ. Let $e = f(\varphi(y))$. Then $\Pi(y)e = \Pi(y)(f(\varphi(y))) = \Phi f(y) = u$. That is, $\Pi(y)$ is surjective. To see $\Pi(y)$ is a Riesz isomorphism, let $e_1, e_2 \in E$. Then $\Pi(y)(e_1 \vee e_2) = \Phi(\mathbf{1}_X \otimes (e_1 \vee e_2))(y) = \Phi(\mathbf{1}_X \otimes e_1)(y) \vee \Phi(\mathbf{1}_X \otimes e_2)(y) = \Pi(y)e_1 \vee \Pi(y)e_2$, since Φ is a Riesz isomorphism.

Recall that every positive operator between Banach lattices is continuous. Let $e \in E$. Since $\|\Pi(y)e\| = \|\Phi(\mathbf{1}_X \otimes e)(y)\| \leq \|\Phi(\mathbf{1}_X \otimes e)\| \leq \|\Phi\|\|e\|$, we have $\|\Pi(y)\| \leq \|\Phi\|$ for all y in Y. On the other hand, for any f in $C(X,E)$ and any y in Y, we can see $\|\Phi f(y)\| = \|\Pi(y)(f(\varphi(y)))\| \leq \|\Pi(y)\|\|f\|$. Consequently, $\|\Phi\| \leq \sup_{y \in Y} \|\Pi(y)\|$.

Finally, we prove that $\Pi : Y \to (\mathcal{L}(E,F), SOT)$ is continuous. To this end, let $\{y_\lambda\}$ be a net such that $y_\lambda \to y$ in Y. Then, for any e in E, $\|\Pi(y_\lambda)e - \Pi(y)e\| = \|\Phi(\mathbf{1}_X \otimes e)(y_\lambda) - \Phi(\mathbf{1}_X \otimes e)(y)\| \to 0$, since $\Phi(\mathbf{1}_X \otimes e)$ is continuous on Y. □

In the above results, we have to assume that both Φ and Φ^{-1} are non-vanishing preserving. In the following example, we can see that the inverse of a non-vanishing preserving Riesz isomorphism is not necessarily non-vanishing preserving.

Example 4. Let $X = \{1,2\}$ equipped with the discrete topology and $E = \mathbb{R}$ with its usual ordering and norm, and let $Y = \{0\}$ and $F = \mathbb{R}^2$ with the pointwise ordering and the sup norm. Define $\Phi : C(X,E) \to C(Y,F)$ by $\Phi f(0) = (f(1),\, f(2))$. Clearly, the Riesz isometric isomorphism Φ is non-vanishing preserving, but its inverse Φ^{-1} is not.

Let E, F be both Banach lattices and Riesz algebras, Miao, Cao and Xiong [16] recently proved that if F has no zero-divisor and there exists a Riesz algebraic isomorphism $\Phi : C(X,E) \to C(Y,F)$ such that Φf is non-vanishing on Y if f is non-vanishing on X, then X is homeomorphic to Y, and E is Riesz algebraically isomorphic to F. In fact, from their proof we can see that Φf is non-vanishing on Y if and only if f is non-vanishing on X, that is, both Φ and Φ^{-1} are non-vanishing preserving. Therefore, the result of Miao, Cao and Xiong can be restated as follows.

Corollary 5 ([16])**.** *Let E, F be both Banach lattices and Riesz algebras. If F has no zero-divisor and $\Phi : C(X,E) \to C(Y,F)$ is a Riesz algebraic isomorphism such that Φf is non-vanishing on Y if f is non-vanishing on X, then Φ is a weighted composition operator*

$$\Phi f(y) = \Pi(y)(f(\varphi(y))), \qquad \forall\, f \in C(X,E), \forall\, y \in Y.$$

Here φ is a homeomorphism from Y onto X, and $\Pi(y)$ is a Riesz algebraic isomorphism from E onto F for every y in Y.

In Theorem 3, when X, Y are compact Hausdorff spaces and $E = F = \mathbb{R}$, the lattice hypothesis about Φ can be dropped.

Example 6. Let X, Y be compact Hausdorff spaces, and $C(X), C(Y)$ be the Banach spaces of continuous real-valued functions defined on X, Y, respectively. Assume $\Phi : C(X) \to C(Y)$ is a linear map such that Φf is non-vanishing on Y if and only if f is non-vanishing on X.

Note that $(\Phi \mathbf{1}_X)^{-1}\Phi$ is a unital linear map preserving non-vanishing. Let λ be in the range of f. Then $f - \lambda \mathbf{1}_X$ is not invertible, and thus neither is $(\Phi \mathbf{1}_X)^{-1}\Phi f - \lambda \mathbf{1}_Y$. It follows that λ is in the range of $(\Phi \mathbf{1}_X)^{-1}\Phi f$. The converse also holds. Therefore, the range of $(\Phi \mathbf{1}_X)^{-1}\Phi f$ coincides with the range of f for each f in $C(X)$. In particular, $(\Phi \mathbf{1}_X)^{-1}\Phi$ is a unital linear isometry from $C(X)$ into $C(Y)$. By the Holsztyński Theorem [11], there is a compact subset Y_0 of Y and a quotient map $\varphi : Y_0 \to X$ such that

$$(\Phi \mathbf{1}_X)^{-1}\Phi f \,|_{Y_0} = f \circ \varphi, \qquad \forall f \in C(X).$$

In case Φ is surjective, the classical Banach-Stone Theorem ensures that φ is a homeomorphism from $Y = Y_0$ onto X. Moreover, if $\Phi \mathbf{1}_X$ is strictly positive on Y, then Φ is a Riesz isomorphism. However, when Φ is not surjective the situation is a bit uncontrollable. For example, consider $\Phi : C[0,1] \to C([0,\frac{1}{2}] \cup [1, \frac{3}{2}])$ defined by

$$\Phi f(y) = \begin{cases} f(2y), & \text{if } 0 \le y \le 1/2; \\ (2y-2)f(0) + (3-2y)f(1), & \text{if } 1 \le y \le \frac{3}{2}. \end{cases}$$

Clearly, the thus defined Φ is a non-surjective linear isometry preserving non-vanishing in two ways, but $[0,1]$ is not homeomorphic to $[0,\frac{1}{2}] \cup [1, \frac{3}{2}]$.

Finally, we borrow an example from [15] which shows that the surjectivity cannot be guaranteed by many other properties we usually consider.

Example 7. Let ω and ω_1 be the first infinite and the first uncountable ordinal number, respectively. Let $[0, \omega_1]$ be the compact Hausdorff space consisting of all ordinal numbers x not greater than ω_1 and equipped with the topology generated by order intervals. Note that every continuous function f in $C[0, \omega_1]$ is eventually constant. More precisely, there is a non-limit ordinal x_f such that $\omega < x_f < \omega_1$ and $f(x) = f(\omega_1)$ for all $x \ge x_f$.

Define $\phi : [0, \omega_1] \to [0, \omega_1]$ by setting

$$\phi(0) = \omega_1, \quad \phi(n) = n-1 \text{ for all } n = 1, 2, \ldots, \quad \text{and } \phi(x) = x \text{ for all } x \ge \omega.$$

Let $\Phi : C[0, \omega_1] \to C[0, \omega_1]$ be the *non-surjective* composition operator defined by $\Phi f = f \circ \phi$. It is plain that Φ is an isometric unital algebraic and lattice isomorphism from $C[0, \omega_1]$ onto its range. In fact, one can see in [15, Example 3.3] that the map Φ is a non-surjective linear n-local automorphism of $C[0, \omega_1]$, where $n = 1, 2, \ldots, \omega$, i.e., the action of Φ on any set S of cardinality not greater than n agrees with an automorphism Φ_S.

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