# Convergence Analysis of a Hybrid Relaxed-Extragradient Method for Monotone Variational Inequalities and Fixed Point Problems

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**Abstract.** In this paper we introduce a hybrid relaxed-extragradient method for finding a common element of the set of common fixed points of $N$ nonexpansive mappings and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping. The hybrid relaxed-extragradient method is based on two well-known methods: hybrid and extragradient. We derive a strong convergence theorem for three sequences generated by this method. Based on this theorem, we also construct an iterative process for finding a common fixed point of $N+1$ mappings, such that one of these mappings is taken from the more general class of Lipschitz pseudocontractive mappings and the rest $N$ mappings are nonexpansive.

**Key words.** Variational inequality, nonexpansive mapping, extragradient method, hybrid method, monotone mapping, fixed point, strong convergence, demiclosedness principle, Opial's condition.

**AMS subject classifications.** 47H09, 47J20

## 1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection from $H$ onto $C$. When $\{x_n\}$ is a sequence in $H$, then $x_n \to x$ (resp. $x_n \rightharpoonup x$) will denote strong (resp. weak) convergence of the sequence $\{x_n\}$ to $x$. Let $A$ be a mapping of $C$ into $H$. Then $A$ is called monotone if for all $u, v \in C$

$$\langle Au - Av, u - v\rangle \geq 0.$$

$A$ is called $\alpha$-inverse-strongly-monotone (see [6,17]) if there exists a positive constant $\alpha$ such that for all $u, v \in C$

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2.$$

$A$ is called $\beta$-strongly-monotone if there exists a positive constant $\beta$ such that for all $u, v \in C$

$$\langle Au - Av, u - v\rangle \geq \beta\|u - v\|^2.$$

$A$ is called $k$-Lipschitz-continuous if there exists a positive constant $k$ such that for all $u, v \in C$

$$\|Au - Av\| \leq k\|u - v\|.$$

Obviously, it is easy to see that every $\alpha$-inverse-strongly-monotone mapping $A$ is monotone and Lipschitz-continuous. Let $S$ be a mapping of $C$ into itself. Then $S$ is called nonexpansive if for all $u, v \in C$

$$\|Su - Sv\| \leq \|u - v\|.$$

We denote by $F(S)$ the set of fixed points of $S$, i.e., $F(S) = \{u \in C : Su = u\}$.

Let $A$ be a mapping of $C$ into $H$. The variational inequality problem is to find a $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 \ \forall v \in C.$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. The variational inequality problem was first discussed by Lions [16]. Since then, this problem has been being studied widely. It is well known that, if $A$ is a strongly monotone and Lipschitz-continuous mapping on $C$, then the variational inequality problem has a unique solution. How to actually find a solution of the variational inequality problem is one of the best important topics in the study of the variational inequality problem. Indeed, there are a lot of different approaches towards solving this problem in finite-dimensional and infinite-dimensional spaces, and the research is intensively continued. A great deal of effort has gone into this problem; see [1,2,5,7-15,17,19-28].

Recently, Antipin considered a finite-dimensional variant of the variational inequality problem, where the solution should satisfy some related constraint in inequality form [1] or some systems of constraints in inequality and equality form [2]. Yamada [8] considered an infinite-dimensional variant of the solution of the variational inequality problem on the set of fixed points of some mapping. Takahashi and Toyoda [9] also formulated an infinite-dimensional variant of the problem of finding a common point of the set of the variational inequality solutions and the set of fixed points of some mapping.

For finding an element of $F(S) \cap VI(C, A)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping $S$ of $C$ into itself is nonexpansive, and a mapping $A$ of $C$ into $H$ is $\alpha$-inverse-strongly-monotone, Takahashi and Toyoda [9] introduced the following iterative scheme:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n) \end{cases} \tag{1.1}$$

for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap VI(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.1) converges weakly to some $z \in F(S) \cap VI(C, A)$.

For finding an element of $F(S) \cap VI(C, A)$ Iiduka and Takahashi [12] introduced the following iterative scheme by a hybrid method:

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases} \tag{1.2}$$

for all $n \geq 0$, where $0 \leq \alpha_n \leq c < 1$ and $0 < a \leq \lambda_n \leq b < 2\alpha$. They showed that if $F(S) \cap VI(C, A) \neq \emptyset$, then the sequence $\{x_n\}$, generated by this iterative process, converges strongly to $P_{F(S) \cap VI(C,A)} x$.

Generally speaking, the algorithm suggested by Takahashi and Toyoda [9] is based on two well-known types of methods, namely, on the projection-type methods for solving variational inequality problems and so-called hybrid or outer-approximation methods for solving fixed point problem. The idea of "hybrid" or "outer-approximation" types of methods was originally introduced by Haugazeau in 1968; see [5] for more details.

In 1976, for finding a solution of the nonconstrained variational inequality problem in the finite-dimensional Euclidean space $\mathcal{R}^n$ under the assumption that a set $C \subset \mathcal{R}^n$ is closed and convex and a mapping $A$ of $C$ into $\mathcal{R}^n$ is monotone and $k$-Lipschitz-continuous, Korpelevich [15] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A \bar{x}_n) \end{cases} \tag{1.3}$$

for all $n \geq 0$, where $\lambda \in (0, 1/k)$. He proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.3), converge to the same point $z \in VI(C, A)$.

Recently, motivated by the idea of Korpelevich's extragradient method [15], Nadezhkina and Takahashi [28] introduced the following iterative scheme for finding an element of $F(S) \cap VI(C, A)$ and proved the following weak convergence result.

**Theorem 1.1** [28, Theorem 3.1]. Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$ and $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ be the sequences generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda A y_n) \end{cases} \tag{1.4}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c,d]$ for some $c, d \in (0,1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ converge weakly to the same point $z \in F(S) \cap VI(C,A)$ where $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

At the same time, the idea of the extragradient method introduced by Korpelevich was successively generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see e.g., the recent papers of He, Yang and Yuan [11], Solodov and Svaiter [26], Solodov [24], and Ceng and Yao [22,23,27].

Very recently, utilizing the combination of hybrid-type method and extragradient-type method Nadezhkina and Takahashi [21] introduced the following iterative method for finding an element of $F(S) \cap VI(C,A)$ and established the following strong convergence theorem.

**Theorem 1.2** [21, Theorem 3.1]. Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$ and let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases} \tag{1.5}$$

for every $n \geq 0$, where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [0,c]$ for some $c \in [0,1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to the same element of $P_{F(S)\cap VI(C,A)} x$.

Let $\{S_i\}_{i=1}^N$ be $N$ nonexpansive mappings of $C$ into itself, and $A$ be a monotone, Lipschitz-continuous mapping of $C$ into $H$. In the present paper, for finding an element of $\bigcap_{i=1}^N F(S_i) \cap VI(C,A)$, by the combination of extragradient and hybrid methods we introduce a hybrid relaxed-extragradient method

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n \mu_n A x_n - \lambda_n(1-\mu_n)Ay_n), \\ t_n = P_C(x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)At_n), \\ z_n = \alpha_n x_n + (1-\alpha_n)S_n t_n, \\ C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases} \tag{1.6}$$

for every $n = 0, 1, ...$, where $S_n = S_{n \bmod N}$, and the following hold:

(i) $\{\mu_n\} \subset (0, 1]$ and $\lim_{n\to\infty} \mu_n = 1$;

(ii) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$;

(iii) $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$.

Moreover, it is shown that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by the hybrid relaxed-extragradient method converge strongly to $q = P_{\bigcap_{i=1}^N F(S_i)\cap VI(C,A)}x$. Utilizing this theorem, we derive some strong convergence results in a real Hilbert space. Based on our main result, we construct an iterative process for finding a common fixed point of $N+1$ mappings, one of which is taken from the more general class of Lipschitz pseudocontractive mappings and the rest $N$ mappings are nonexpansive. We remark that, in the case when $N = 1$ and $\mu_n = 1$ $\forall n \ge 0$, the iterative scheme (1.6) reduces to the one (1.5). Thus, our results are the improvements and extension of many known results in the earlier and recent literature; see e.g., [9,12,13,18,21,28].

## 2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $C$ be a nonempty closed convex subset of $H$. For every point $x \in H$ there exists a unique nearest point in $C$, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. $P_C$ is called the metric projection of $H$ onto $C$. It is known that $P_C$ is a nonexpansive mapping from $H$ onto $C$. It is also known that $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \ge 0 \tag{2.1}$$

for all $x \in H,\ y \in C$; see [7] for more details. It is easy to see that (2.1) is equivalent to

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.2}$$

for all $x \in H,\ y \in C$.

Let $A$ be a monotone mapping of $C$ into $H$. In the context of the variational inequality problem the characterization of projection (2.1) implies

$$u \in VI(C, A) \ \Leftrightarrow\ u = P_C(u - \lambda A u)\ \forall \lambda > 0.$$

It is also known that $H$ satisfies Opial's condition [7], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

The following results will be used in the rest of this paper.

**Lemma 2.1.** Let $H$ be a real Hilbert space. If $\{x_n\}$ is a sequence in $H$ such that $x_n \rightharpoonup \hat{x} \in H$ and $\|x_n\| \to \|\hat{x}\|$, then $x_n \to \hat{x}$.

**Proof.** Observe that

$$\|x_n - \hat{x}\|^2 = \|x_n\|^2 - 2\langle x_n, \hat{x}\rangle + \|\hat{x}\|^2.$$

Since $x_n \rightharpoonup \hat{x} \in H$ and $\|x_n\| \to \|\hat{x}\|$, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \hat{x}\|^2 &= \lim_{n\to\infty} (\|x_n\|^2 - 2\langle x_n, \hat{x}\rangle + \|\hat{x}\|^2) \\ &= \|\hat{x}\|^2 - 2\langle \hat{x}, \hat{x}\rangle + \|\hat{x}\|^2 = 0.\end{aligned}$$

□

**Lemma 2.2** Demiclosedness Principle [7]. Assume that $S$ is a nonexpansive self-mapping of a closed convex subset $C$ of a Hilbert space $H$. If $S$ has a fixed point, then $I - S$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in $C$ converging weakly to some $x \in C$ and the sequence $\{(I - S)x_n\}$ converges strongly to some $y \in H$, it follows that $(I - S)x = y$. Here $I$ is the identity operator of $H$.

A mapping $T : C \to C$ is called pseudocontractive if for all $x, y \in C$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2.$$

We remark that, if a mapping $T : C \to C$ is pseudocontractive and $k$-Lipschitz-continuous, then the mapping $A = I - T$ is monotone and $k + 1$-Lipschitz-continuous; moreover, $F(T) = VI(C, A)$ (see e.g., [21, proof of Theorem 4.5]).

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $T$ is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. Let $A$ be a monotone, $k$-Lipschitz-continuous mapping of $C$ into $H$ and let $N_C v$ be the normal cone to $C$ at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0 \text{ for all } u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

It is known that in this case $T$ is maximal monotone, and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [3].

Throughout the rest of the paper, we shall use the following notation: for a given sequence $\{x_n\} \subset H$, $\omega_w(x_n)$ denotes the weak $\omega$-limit set of $\{x_n\}$; that is,

$$\omega_w(x_n) := \{x \in H : \{x_{n_j}\} \text{ converges weakly to } x \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\}.$$

## 3. Strong Convergence Theorem

We are now in a position to prove our main result in this paper. Given $N$ nonexpansive mappings $\{S_i\}_{i=1}^N$ of $C$ into itself, for each integer $n \geq 1$ we write

$$S_n = S_{n \bmod N}$$

with the mod function taking values in the set $\{1, 2, ..., N\}$; i.e., if $n = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $S_n = S_N$ if $q = 0$ and $S_n = S_q$ if $1 < q < N$.

**Theorem 3.1.** Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$ and let $\{S_i\}_{i=1}^N$ be $N$ nonexpansive mappings of $C$ into itself such that $\bigcap_{i=1}^N F(S_i) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n \mu_n A x_n - \lambda_n(1-\mu_n) A y_n), \\ t_n = P_C(x_n - \lambda_n A y_n - \lambda_n (1-\mu_n) A t_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) S_n t_n, \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases} \tag{3.1}$$

for every $n = 0, 1, ...$, where $S_n = S_{n \bmod N}$, and the following hold:
(i) $\{\mu_n\} \subset (0, 1]$ and $\lim_{n\to\infty} \mu_n = 1$;
(ii) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$;
(iii) $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$.
Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_{\bigcap_{i=1}^N F(S_i) \cap VI(C,A)} x$.

**Remark 3.1.** First, observe that for all $x, y \in C$ and all $n \geq 0$

$$\begin{aligned} &\|P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1-\mu_n)Ax) - P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1-\mu_n)Ay)\| \\ &\leq \|(x_n - \lambda_n \mu_n A x_n - \lambda_n(1-\mu_n)Ax) - (x_n - \lambda_n \mu_n A x_n - \lambda_n(1-\mu_n)Ay)\| \\ &= \lambda_n(1-\mu_n)\|Ax - Ay\| \\ &\leq \lambda_n k \|x - y\|. \end{aligned}$$

Thus, by Banach Contraction Principle, we know that for each $n \geq 0$ there exists a unique $y_n \in C$ such that

$$y_n = P_C(x_n - \lambda_n \mu_n A x_n - \lambda_n(1-\mu_n)Ay_n). \tag{3.2}$$

Also, observe that for all $x, y \in C$ and all $n \geq 0$

$$\begin{aligned}
&\|P_C(x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)Ax) - P_C(x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)Ay)\| \\
&\leq \|(x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)Ax) - (x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)Ay)\| \\
&= \lambda_n(1-\mu_n)\|Ax - Ay\| \\
&\leq \lambda_n k \|x - y\|.
\end{aligned}$$

Utilizing Banach Contraction Principle, we know that for each $n \geq 0$ there exists a unique $t_n \in C$ such that

$$t_n = P_C(x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)At_n). \tag{3.3}$$

**Proof of Theorem 3.1.** We divide the proof into several steps.

**Step 1.** We claim that every $C_n$ is closed and convex, and that $\bigcap_{i=1}^N F(S_i) \cap VI(C,A) \subset C_n \ \forall n \geq 0$.

Indeed, it is obvious that $C_n$ is closed for all $n \geq 0$. Since

$$C_n = \{z \in C : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle \leq 0\},$$

we deduce that $C_n$ is convex for all $n \geq 0$. Note that $t_n = P_C(x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)At_n)$ for all $n \geq 0$. Let $u \in \bigcap_{i=1}^N F(S_i) \cap VI(C,A)$ be an arbitrary element. From (2.2), monotonicity of $A$, and $u \in VI(C,A)$, we have

$$\begin{aligned}
\|t_n - u\|^2 &\leq \|(x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)At_n) - u\|^2 \\
&\quad - \|(x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)At_n) - t_n\|^2 \\
&= \|x_n - \lambda_n(1-\mu_n)At_n - u\|^2 \\
&\quad - \|x_n - \lambda_n(1-\mu_n)At_n - t_n\|^2 + 2\lambda_n\langle Ay_n, u - t_n\rangle \\
&= \|x_n - \lambda_n(1-\mu_n)At_n - u\|^2 - \|x_n - \lambda_n(1-\mu_n)At_n - t_n\|^2 \\
&\quad + 2\lambda_n(\langle Ay_n, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\
&= \|x_n - \lambda_n(1-\mu_n)At_n - u\|^2 - \|x_n - \lambda_n(1-\mu_n)At_n - t_n\|^2 \\
&\quad + 2\lambda_n(\langle Ay_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\
&\leq \|x_n - \lambda_n(1-\mu_n)At_n - u\|^2 - \|x_n - \lambda_n(1-\mu_n)At_n - t_n\|^2 \\
&\quad + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - t_n\|^2 - 2\lambda_n(1-\mu_n)\langle At_n, t_n - u\rangle \\
&\quad + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\
&\quad + 2\lambda_n\langle Ay_n, y_n - t_n\rangle - 2\lambda_n(1-\mu_n)(\langle At_n - Au, t_n - u\rangle \\
&\quad + \langle Au, t_n - u\rangle) \\
&\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle.
\end{aligned}$$

Further, since $y_n = P_C(x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n)$ and $A$ is $k$-Lipschitz-continuous, we have

$$
\begin{aligned}
&\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle \\
&= \langle x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n - y_n, t_n - y_n\rangle + \lambda_n\mu_n\langle Ax_n - Ay_n, t_n - y_n\rangle \\
&\le \lambda_n\mu_n\langle Ax_n - Ay_n, t_n - y_n\rangle \\
&\le \lambda_n k\|x_n - y_n\|\|t_n - y_n\|.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\|t_n - u\|^2 &\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|x_n - y_n\|\|t_n - y_n\| \\
&\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2k^2\|x_n - y_n\|^2 + \|y_n - t_n\|^2 \\
&= \|x_n - u\|^2 + (\lambda_n^2k^2 - 1)\|x_n - y_n\|^2 \\
&\le \|x_n - u\|^2.
\end{aligned} \tag{3.4}
$$

Therefore, from (3.4), $z_n = \alpha_n x_n + (1-\alpha_n)S_n t_n$, and $u = S_n u$, we have

$$
\begin{aligned}
\|z_n - u\|^2 &= \|\alpha_n x_n + (1-\alpha_n)S_n t_n - u\|^2 \\
&= \|\alpha_n(x_n - u) + (1-\alpha_n)(S_n t_n - u)\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|S_n t_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|x_n - u\|^2 + (\lambda_n^2k^2 - 1)\|x_n - y_n\|^2] \\
&= \|x_n - u\|^2 + (1-\alpha_n)(\lambda_n^2k^2 - 1)\|x_n - y_n\|^2 \\
&\le \|x_n - u\|^2
\end{aligned} \tag{3.5}
$$

for all $n \ge 0$ and hence $u \in C_n$. So, $\bigcap_{i=1}^N F(S_i) \cap VI(C,A) \subset C_n$ for all $n \ge 0$.

**Step 2.** We claim that $\{x_n\}$ is well defined and $\bigcap_{i=1}^N F(S_i) \cap VI(C,A) \subset C_n \cap Q_n$ for all $n \ge 0$.

Indeed, let us show by mathematical induction that $\{x_n\}$ is well defined and $\bigcap_{i=1}^N F(S_i) \cap VI(C,A) \subset C_n \cap Q_n$ for all $n \ge 0$. First, it is obvious that $Q_n$ is closed and convex for all $n \ge 0$. As $Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}$, we have $\langle x_n - z, x - x_n\rangle \ge 0$ for all $z \in Q_n$ and, by (2.1), $x_n = P_{Q_n}x$. Second, according to Remark 3.1 we know that for each $n \ge 0$ there exist a unique $y_n \in C$ and a unique $t_n \in C$ such that (3.2) and (3.3) hold, respectively. For $n = 0$ we have $Q_0 = C$. Hence we obtain $\bigcap_{i=1}^N F(S_i) \cap VI(C,A) \subset C_0 \cap Q_0$. Suppose that $x_k$ is given and $\bigcap_{i=1}^N F(S_i) \cap VI(C,A) \subset C_k \cap Q_k$ for some $k \ge 0$. Since $\bigcap_{i=1}^N F(S_i) \cap VI(C,A)$ is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of $C$. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k\cap Q_k}x$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for all $z \in C_k \cap Q_k$. Since $\bigcap_{i=1}^N F(S_i) \cap VI(C,A) \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for $z \in \bigcap_{i=1}^N F(S_i) \cap VI(C,A)$ and hence $\bigcap_{i=1}^N F(S_i) \cap VI(C,A) \subset Q_{k+1}$. Therefore, we obtain $\bigcap_{i=1}^N F(S_i) \cap VI(C,A) \subset C_{k+1} \cap Q_{k+1}$.

**Step 3.** We claim that the following statements hold:
(1) $\{x_n\}$ is bounded, and $\lim_{n\to\infty}\|x_{n+i} - x_n\| = 0$ for each $i = 1, 2, ..., N$;

(2) $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Indeed, let $q = P_{\bigcap_{i=1}^N F(S_i)\cap VI(C,A)}x$. From $x_{n+1} = P_{C_n\cap Q_n}x$ and $q \in \bigcap_{i=1}^N F(S_i)\cap VI(C,A) \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \le \|q - x\| \quad \forall n \ge 0. \tag{3.6}$$

Therefore, $\{x_n\}$ is bounded and so are $\{z_n\}$ and $\{t_n\}$ due to (3.4) and (3.5). Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n}x$, we have

$$\|x_n - x\| \le \|x_{n+1} - x\| \quad \forall n \ge 0.$$

Therefore, there exists $\lim_{n\to\infty}\|x_n - x\|$. Since $x_n = P_{Q_n}x$ and $x_{n+1} \in Q_n$, using (2.2) we have

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x\|^2 - \|x_n - x\|^2 \quad \forall n \ge 0.$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0,$$

and hence $\lim_{n\to\infty}\|x_{n+i} - x_n\| = 0$ for each $i = 1, 2, ..., N$. Since $x_{n+1} \in C_n$, we have $\|z_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$ and hence

$$\|z_n - x_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_n - x_{n+1}\| \quad \forall n \ge 0.$$

Consequently, from $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

**Step 4.** We claim that the following statements hold:

(1) $\lim_{n\to\infty}\|x_n - y_n\| = 0$;

(2) $\lim_{n\to\infty}\|S_l x_n - x_n\| = 0$ for each $l = 1, 2, ..., N$.

Indeed, for $u \in \bigcap_{i=1}^N F(S_i)\cap VI(C,A)$, from (3.5) we derive

$$\|z_n - u\|^2 \le \|x_n - u\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2.$$

Therefore, we have

$$\begin{aligned}\|x_n - y_n\|^2 &\le \tfrac{1}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n - u\|^2 - \|z_n - u\|^2)\\ &= \tfrac{1}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n - u\| - \|z_n - u\|)(\|x_n - u\| + \|z_n - u\|)\\ &\le \tfrac{1}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.\end{aligned} \tag{3.7}$$

Since $\|z_n - x_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|x_n - y_n\| \to 0$. By the same process as in (3.4), we also have

$$\begin{aligned}\|t_n - u\|^2 &\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|x_n - y_n\|\|t_n - y_n\|\\ &\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \|x_n - y_n\|^2 + \lambda_n^2k^2\|y_n - t_n\|^2\\ &= \|x_n - u\|^2 + (\lambda_n^2k^2 - 1)\|y_n - t_n\|^2\\ &\le \|x_n - u\|^2,\end{aligned}$$

and hence $\{t_n\}, \{At_n\}$ are bounded. Then, in contrast with (3.5),

$$\begin{aligned}\|z_n - u\|^2 &= \|\alpha_n x_n + (1-\alpha_n)S_n t_n - u\|^2 \\ &= \|\alpha_n(x_n - u) + (1-\alpha_n)(S_n t_n - u)\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|S_n t_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)(\|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - t_n\|^2) \\ &= \|x_n - u\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|y_n - t_n\|^2 \\ &\le \|x_n - u\|^2\end{aligned}$$

and, rearranging as in (3.7),

$$\begin{aligned}\|t_n - y_n\|^2 &\le \tfrac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - u\|^2 - \|z_n - u\|^2) \\ &= \tfrac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - u\| - \|z_n - u\|)(\|x_n - u\| + \|z_n - u\|) \\ &\le \tfrac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.\end{aligned}$$

Since $\|z_n - x_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|t_n - y_n\| \to 0$. As $A$ is $k$-Lipschitz-continuous, we have $\|Ay_n - At_n\| \to 0$. From $\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|$ we also have $\|x_n - t_n\| \to 0$. Since $z_n = \alpha_n x_n + (1-\alpha_n)S_n t_n$, we have $(1-\alpha_n)(S_n t_n - t_n) = \alpha_n(t_n - x_n) + (z_n - t_n)$. Then

$$\begin{aligned}(1-c)\|S_n t_n - t_n\| &\le (1-\alpha_n)\|S_n t_n - t_n\| \\ &\le \alpha_n\|t_n - x_n\| + \|z_n - t_n\| \\ &\le (1+\alpha_n)\|t_n - x_n\| + \|z_n - x_n\|\end{aligned}$$

and hence $\|S_n t_n - t_n\| \to 0$. Also, observe that

$$\begin{aligned}\|S_n x_n - x_n\| &\le \|S_n x_n - S_n t_n\| + \|S_n t_n - t_n\| + \|t_n - x_n\| \\ &\le 2\|x_n - t_n\| + \|S_n t_n - t_n\|.\end{aligned}$$

Since $\|x_n - t_n\| \to 0$ and $\|S_n t_n - t_n\| \to 0$, we have $\|S_n x_n - x_n\| \to 0$. Consequently, we have for each $i = 1, 2, ..., N$

$$\begin{aligned}\|x_n - S_{n+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\| + \|S_{n+i}x_{n+i} - S_{n+i}x_n\| \\ &\le 2\|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\|\end{aligned}$$

and so $\lim_{n\to\infty}\|x_n - S_{n+i}x_n\| = 0$ for each $i = 1, 2, ..., N$. This implies that for each $l = 1, 2, ..., N$

$$\lim_{n\to\infty}\|x_n - S_l x_n\| = 0.$$

**Step 5.** We claim that $\omega_w(x_n) \subset \bigcap_{i=1}^N F(S_i) \cap VI(C, A)$, where $\omega_w(x_n)$ denotes the weak $\omega$-limit set of $\{x_n\}$, i.e.,

$$\omega_w(x_n) = \{u \in H : \{x_{n_j}\} \text{ converges weakly to } u \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\}.$$

Indeed, since $\{x_n\}$ is bounded, it has a subsequence which converges weakly to some point in $C$ and hence $\omega_w(x_n) \neq \emptyset$. Let $u \in \omega_w(x_n)$ be an arbitrary point. Then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges weakly to $u$ and hence we have $\lim_{j\to\infty} \|x_{n_j} - S_l x_{n_j}\| = 0$ for each $l = 1, 2, ..., N$. Note that from Lemma 2.2 it follows that $I - S$ is demiclosed at zero. Thus $u \in F(S_l)$ for each $l = 1, 2, ..., N$, i.e., $u \in \bigcap_{i=1}^N F(S_i)$. Now, we show $u \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then $T$ is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [17]. Let $(v, w) \in G(T)$. Then we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So, we have $\langle v - t, w - Av\rangle \geq 0$ for all $t \in C$. On the other hand, from $t_n = P_C(x_n - \lambda_n Ay_n - \lambda_n(1 - \mu_n)At_n)$ and $v \in C$ we have

$$\langle x_n - \lambda_n Ay_n - \lambda_n(1 - \mu_n)At_n - t_n, t_n - v\rangle \geq 0$$

and hence

$$\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + Ay_n + (1 - \mu_n)At_n\rangle \geq 0.$$

From $\langle v - t, w - Av\rangle \geq 0$ for all $t \in C$ and $t_{n_j} \in C$, we have

$$\begin{aligned} \langle v - t_{n_j}, w\rangle &\geq \langle v - t_{n_j}, Av\rangle \\ &\geq \langle v - t_{n_j}, Av\rangle - \langle v - t_{n_j}, \tfrac{t_{n_j} - x_{n_j}}{\lambda_{n_j}} + Ay_{n_j} + (1 - \mu_{n_j})At_{n_j}\rangle \\ &= \langle v - t_{n_j}, Av - At_{n_j}\rangle + \langle v - t_{n_j}, At_{n_j} - Ay_{n_j}\rangle \\ &\quad - \langle v - t_{n_j}, \tfrac{t_{n_j} - x_{n_j}}{\lambda_{n_j}}\rangle - (1 - \mu_{n_j})\langle v - t_{n_j}, At_{n_j}\rangle \\ &\geq \langle v - t_{n_j}, At_{n_j} - Ay_{n_j}\rangle - \langle v - t_{n_j}, \tfrac{t_{n_j} - x_{n_j}}{\lambda_{n_j}}\rangle - (1 - \mu_{n_j})\langle v - t_{n_j}, At_{n_j}\rangle. \end{aligned}$$

So, we obtain $\langle v - u, w\rangle \geq 0$ as $j \to \infty$. Since $T$ is maximal monotone, we have $u \in T^{-1}0$ and hence $u \in VI(C, A)$. Therefore, $u \in \bigcap_{i=1}^N F(S_i) \cap VI(C, A)$.

**Step 6.** We claim that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_{\bigcap_{i=1}^N F(S_i) \cap VI(C,A)} x$.

Indeed, let $u \in \omega_w(x_n)$ be an arbitrary point. Then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges weakly to $u$. By Step 5, we know that $u \in \bigcap_{i=1}^N F(S_i) \cap VI(C, A)$. Hence from $q = P_{\bigcap_{i=1}^N F(S_i) \cap VI(C,A)} x$ and (3.6) we derive

$$\|q - x\| \leq \|u - x\| \leq \liminf_{j\to\infty} \|x_{n_j} - x\| \leq \limsup_{j\to\infty} \|x_{n_j} - x\| \leq \|q - x\|.$$

So, we obtain

$$\lim_{j\to\infty} \|x_{n_j} - x\| = \|q - x\|.$$

From $x_{n_j} - x \rightharpoonup u - x$ we have $x_{n_j} - x \to u - x$ and hence $x_{n_j} \to u$. Since $x_n = P_{Q_n} x$ and $q \in \bigcap_{i=1}^N F(S_i) \cap VI(C, A) \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|q - x_{n_j}\|^2 = \langle q - x_{n_j}, x_{n_j} - x\rangle + \langle q - x_{n_j}, x - q\rangle \geq \langle q - x_{n_j}, x - q\rangle.$$

As $j \to \infty$, we get $-\|q-u\|^2 \geq \langle q-u, x-q\rangle \geq 0$ due to $q = P_{\bigcap_{i=1}^N F(S_i)\cap VI(C,A)}x$ and $u \in \bigcap_{i=1}^N F(S_i) \cap VI(C,A)$. Thus we have $u = q$. This implies that $x_n \to q$. Consequently, from $\|x_n - y_n\| \to 0$ and $\|x_n - z_n\| \to 0$ we infer that both $\{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_{\bigcap_{i=1}^N F(S_i)\cap VI(C,A)}x$. This completes the proof. $\square$

## 4. Applications

Utilizing Theorem 3.1 in the above section, we prove some strong convergence theorems in a real Hilbert space.

**Theorem 4.1.** Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$ such that $VI(C,A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n\mu_n A x_n - \lambda_n(1-\mu_n)Ay_n), \\ t_n = P_C(x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)At_n), \\ z_n = \alpha_n x_n + (1-\alpha_n)t_n, \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x \end{cases}$$

for every $n = 0, 1, ...$, where the following hold:

(i) $\{\mu_n\} \subset (0,1]$ and $\lim_{n\to\infty}\mu_n = 1$;

(ii) $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, 1/k)$;

(iii) $\{\alpha_n\} \subset [0,c]$ for some $c \in [0,1)$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_{VI(C,A)}x$.

**Proof.** Putting $S_i = I$ $(1 \leq i \leq N)$, $\alpha_n = 0$ for all $n \geq 0$, by Theorem 3.1 we obtain the desired result. $\square$

**Remark 4.1.** See Iiduka, Takahashi and Toyoda [13] for the case when the mapping $A$ is $\alpha$-inverse-strongly-monotone; see Nadezhkina and Takahashi [21, Theorem 4.1] for the case when the mapping $A$ is monotone, Lipschitz-continuous.

**Theorem 4.2.** Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $\{S_i\}_{i=1}^N$ be $N$ nonexpansive mappings of $C$ into itself such that $\bigcap_{i=1}^N F(S_i) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by

$$
\begin{cases}
x_0 = x \in C, \\
y_n = \alpha_n x_n + (1 - \alpha_n) S_n P_C x_n, \\
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x
\end{cases}
$$

for every $n = 0, 1, ...$, where $S_n = S_{n \bmod N}$, and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $q = P_{\bigcap_{i=1}^N F(S_i)} x$.

**Proof.** Putting $A = 0$, by Theorem 3.1 we obtain the desired result. □

**Remark 4.2.** See Nadezhkina and Takahashi [21, Theorem 4.2] for the case when $N = 1$, and see also Nakajo and Takahashi [18].

**Theorem 4.3.** Let $H$ be a real Hilbert space. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $H$ into itself and let $\{S_i\}_{i=1}^N$ be $N$ nonexpansive mappings of $H$ into itself such that $\bigcap_{i=1}^N F(S_i) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$
\begin{cases}
x_0 = x \in H, \\
y_n = x_n - \lambda_n \mu_n A x_n - \lambda_n (1 - \mu_n) A y_n, \\
t_n = x_n - \lambda_n A y_n - \lambda_n (1 - \mu_n) A t_n, \\
z_n = \alpha_n x_n + (1 - \alpha_n) S_n t_n, \\
C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x
\end{cases}
$$

for every $n = 0, 1, ...$, where $S_n = S_{n \bmod N}$, and the following hold:
(i) $\{\mu_n\} \subset (0, 1]$ and $\lim_{n \to \infty} \mu_n = 1$;
(ii) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$;
(iii) $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$.
Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_{\bigcap_{i=1}^N F(S_i) \cap A^{-1}0} x$.

**Proof.** We have $A^{-1}0 = VI(H, A)$ and $P_H = I$. By Theorem 3.1 we obtain the desired result. □

Let $B : H \to 2^H$ be a maximal monotone mapping. Then, for any $x \in H$ and $r > 0$, consider $J_r^B x = \{z \in H : z + rBz \ni x\}$. Such $J_r^B x$ is called the resolvent of $B$ and is denoted by $J_r^B = (I + rB)^{-1}$.

**Theorem 4.4.** Let $H$ be a real Hilbert space. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $H$ into itself and let $B_i : H \to 2^H$, $i = 1, 2, ..., N$ be $N$ maximal monotone mappings such that $\bigcap_{i=1}^{N} B_i^{-1}0 \cap A^{-1}0 \neq \emptyset$. Let $J_r^{B_i}$ be the resolvent of $B_i$ for each $r > 0$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_0 = x \in H, \\ y_n = x_n - \lambda_n \mu_n A x_n - \lambda_n (1-\mu_n) A y_n, \\ t_n = x_n - \lambda_n A y_n - \lambda_n (1-\mu_n) A t_n, \\ z_n = \alpha_n x_n + (1-\alpha_n) J_r^{B_n} t_n, \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0, 1, ...,$ where $J_r^{B_n} = J_r^{B_{n \mathrm{mod} N}}$, and the following hold:

(i) $\{\mu_n\} \subset (0, 1]$ and $\lim_{n\to\infty} \mu_n = 1$;

(ii) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$;

(iii) $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_{\bigcap_{i=1}^{N} B_i^{-1}0 \cap A^{-1}0} x$.

**Proof.** We know that $J_r^{B_i}$ is nonexpansive for every $i = 1, 2, ..., N$. We also have $A^{-1}0 = VI(H, A)$ and $F(J_r^{B_i}) = B_i^{-1}0$ for every $i = 1, 2, ..., N$. Putting $P_H = I$, by Theorem 3.1 we obtain the desired result. □

We also know one more definition of a pseudocontractive mapping, which is equivalent to the definition given in the introduction. A mapping $T$ of $C$ into itself is called pseudocontractive if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2$$

for all $x, y \in C$; see [6]. Obviously, the class of pseudocontractive mappings is more general than the class of nonexpansive mappings. For the class of pseudocontractive mappings there are some nontrivial examples; see [21, p.1239] for more details. In the following theorem we introduce an iterative process that converges strongly to a common fixed point of $N + 1$ mappings, one of which is Lipschitz-continuous and pseudocontractive, and the rest $N$ mappings are nonexpansive.

**Theorem 4.5.** Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $T$ be a pseudocontractive and $m$-Lipschitz-continuous mapping of $C$ into itself , and let $\{S_i\}_{i=1}^{N}$ be $N$ nonexpansive mappings of $C$ into itself such that $\bigcap_{i=1}^{N} F(S_i) \cap F(T) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n \mu_n A x_n - \lambda_n (1-\mu_n) A y_n), \\ t_n = P_C(x_n - \lambda_n A y_n - \lambda_n (1-\mu_n) A t_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) S_n t_n, \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases} \tag{3.1}$$

for every $n = 0, 1, ...$, where $A = I - T$, $S_n = S_{n\mathrm{mod}N}$, and the following hold:

(i) $\{\mu_n\} \subset (0, 1]$ and $\lim_{n\to\infty} \mu_n = 1$;

(ii) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$;

(iii) $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_{\bigcap_{i=1}^N F(S_i)\cap F(T)} x$.

**Proof.** Let $A = I - T$. Let us show the mapping $A$ is monotone and $(m+1)$-Lipschitz-continuous. Indeed, observe that

$$\langle Ax - Ay, x - y\rangle = \|x - y\|^2 - \langle Tx - Ty, x - y\rangle \geq 0,$$

and

$$\|Ax - Ay\| = \|x - y - (Tx - Ty)\| \leq \|x - y\| + \|Tx - Ty\| \leq (m + 1)\|x - y\|.$$

Now let us show $F(T) = VI(C, A)$. Indeed, we have, for fixed $\lambda_0 \in (0, 1)$,

$$Tu = u \Leftrightarrow u = u - \lambda_0 Au = P_C(u - \lambda_0 Au) \Leftrightarrow \langle Au, y - u\rangle \geq 0 \;\forall y \in C.$$

By Theorem 3.1 we obtain the desired result. □

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