LINEAR ORTHOGONALITY PRESERVERS OF STANDARD OPERATOR ALGEBRAS

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To the memory of our beloved late Professor Sen-Yen Shaw

Abstract. In 2003, Araujo and Jarosz showed that every bijective linear map $\theta : A \to B$ between unital standard operator algebras preserving zero products in two ways is a scalar multiple of an inner automorphism. Later in 2007, Zhao and Hou showed that similar results hold if both A, B are unital standard algebras on Hilbert spaces and θ preserves range or domain orthogonality. In particular, such maps are automatically bounded. In this paper, we will study linear orthogonality preservers in a unified way. We will show that every surjective linear map between standard operator algebras preserving range/domain orthogonality carries a standard form, and is thus automatically bounded.

1. Introduction

An algebra A of bounded linear operators on a Banach space M is called *standard* if A contains the algebra $\mathcal{F}(M)$ of all bounded finite rank operators on M. Assume that $\theta : A \to B$ is a bijective linear map between two unital standard operator algebras on Banach spaces M, N, *preserving zero products* in two ways, i.e., $ab = 0$ in A if and only if $\theta(a)\theta(b) = 0$ in B. Araujo and Jarosz [1] showed that in this case there exist a nonzero scalar λ and a bounded invertible linear map $S : M \to N$ such that

$$\theta(a) = \lambda S a S^{-1}, \quad \forall a \in A.$$

It was pointed out in [3] that the same result holds also when A, B are non-unital.

On the other hand, let A, B be unital standard operator algebras on (real or complex) infinite dimensional Hilbert spaces H, K, respectively. Assume that $\theta : A \to B$ is a surjective additive map *preserving range orthogonality* in two ways, i.e., $a^*b = 0$ in A if and only if $\theta(a)^*\theta(b) = 0$ in B. Zhao and Hou [6] showed that in this case there exist a unitary (or conjugate unitary) operator $U : H \to K$ and a bounded linear (or conjugate linear) invertible operator $V : K \to H$ such that

$$\theta(a) = UaV, \quad \forall a \in A.$$

Zhao and Hou [6] also obtained a similar version for surjective additive maps *preserving domain orthogonality* in two ways, i.e., the ones with $\theta(a)\theta(b)^* = 0$ exactly when $ab^* = 0$.

2000 *Mathematics Subject Classification.* 46J10, 46L05.

This work is partially supported by Taiwan National Science Council grants 96-2115-M-110-004-MY3.

In this paper, we will give a unified approach, with new proofs, to different linear orthogonality preservers. We will show that every surjective linear map $\theta : A \to B$ between two standard operator algebras on (real or complex) Hilbert spaces preserving range/domain orthogonality in two ways carries a standard form, and is thus automatically bounded as well. The following table summarizes our results.

The structures θ preserves	The form of θ carries
$ab = 0 \Leftrightarrow \theta(a)\theta(b) = 0$	$\lambda S a S^{-1}$
$a^*b = 0 \Leftrightarrow \theta(a)^*\theta(b) = 0$	UaT
$ab^* = 0 \Leftrightarrow \theta(a)\theta(b)^* = 0$	SaV
$a^*b = 0 \Leftrightarrow \theta(a)\theta(b)^* = 0$	Sa^tV
$ab^* = 0 \Leftrightarrow \theta(a)^*\theta(b) = 0$	Ua^tT
$a^*b = ab^* = 0 \Leftrightarrow \theta(a)^*\theta(b) = \theta(a)\theta(b)^* = 0$	λUaV or λUa^tV

θ : a surjective linear map between standard operator algebras
λ : a nonzero scalar
S, T : bounded invertible linear operators
U, V : unitary operators

We note that we need to assume A is unital or A contains all trace class operators in the second to the fifth cases. Without this assumption, θ can be unbounded. For example, let H be an infinite dimensional Hilbert space and let T be any unbounded bijective linear operator on H. Then $x \otimes y \mapsto x \otimes Ty$ (resp. $x \otimes y \mapsto Tx \otimes y$) defines an unbounded bijective range (resp. domain) orthogonality preserving linear map from $\mathcal{F}(H)$ onto $\mathcal{F}(H)$. However, in the first case of zero product preservers and in the last case of doubly orthogonality preservers, this assumption can be dropped.

2. Preliminaries

In the following, A and B are standard operator algebras on (real or complex) Hilbert spaces H, K, respectively, and θ is a surjective linear map from A onto B. As pointed out in [6], if θ preserves any kind of orthogonality in two ways, then θ is injective. For example, if $\theta(x) = 0$ and θ preserves zero products in two ways, then $\theta(x)\theta(y) = 0$ implies $xy = 0$ for all y in A. Thus $x = 0$ as well.

Recall that by the Fundamental Theorem of Affine Geometry any bijective linear map $\theta : \mathcal{F}(H) \to \mathcal{F}(K)$ sending exactly rank one operators onto rank one operators must be in one of the following forms.

(1) $\theta(x \otimes y) = Sx \otimes Ry$, where $S, R\colon H \to K$ are invertible linear maps.
(2) $\theta(x \otimes y) = Ry \otimes Sx$, where $S, R\colon H \to K$ are invertible conjugate linear maps.

Here, $x \otimes y(z) = \langle z, y\rangle x$ is the rank at most one operator, and $\langle \cdot, \cdot \rangle$ is the inner product of the Hilbert space H or K. Note that for any scalar α we have $\alpha(x \otimes y) = (\alpha x) \otimes y = x \otimes (\bar{\alpha} y)$. See, e.g., [4, 6].

Fix an orthonormal basis $\{e_j\}$ of a Hilbert space H. For all $x = \sum \langle x, e_j\rangle e_j$ in H, we set $\overline{x} = \sum \langle e_j, x\rangle e_j$. Let T be a bounded linear operator on H. The transpose operator T^t of T with respect to $\{e_j\}$ is the bounded linear operator satisfying the condition

$$\langle Te_i, e_j\rangle = \langle e_i, T^* e_j\rangle = \langle T^t e_j, e_i\rangle, \quad \forall i, j.$$

The transpose operator is well-defined and $\|T\| = \|T^*\| = \|T^t\|$. Here T^* is the adjoint operator of T. Note that the definition of $\overline{x}$ and T^t depend on the choice of the orthonormal basis. However, they are unique up to unitarily equivalence.

Some properties of the transpose operators are given below. For all $x, y \in H$ we have

(1) $\langle \overline{x}, \overline{y}\rangle = \langle y, x\rangle$.
(2) $(x \otimes y)^t = \overline{y} \otimes \overline{x}$.
(3) $(T^t)^* = (T^*)^t$.
(4) $T^t x = \overline{T^* \overline{x}}$.

3. Results

We first give, with a new proof, a modified version of the result of Zhao and Hou in [6] about linear range orthogonality preservers mentioned in the introduction. Note that we can allow the algebras not being unital, provided instead that they contain trace class operators.

Theorem 1. Let A, B be standard operator algebras on Hilbert spaces H, K, respectively. Suppose A is unital, or A contains all trace class operators on H. Assume that $\theta : A \to B$ is a surjective linear map such that $a^*b = 0$ if and only if $\theta(a)^*\theta(b) = 0$. Then θ is bounded, and there exist a bounded invertible linear operator $T : K \to H$ and a unitary operator $U : H \to K$ such that

$$\theta(a) = UaT, \quad \forall a \in A.$$

Proof. Note that θ is indeed bijective. Put

$$a^{\dashv} = \{c \in A : c^*a = 0\}, \quad \text{for all nonzero } a \text{ in } A.$$

For any a and b in A, it is clear that $a^{\dashv} \subseteq b^{\dashv}$ if and only if the closure of the range space of a contains that of b. We define a partial order on A by $a \leq b$ if and only if $a^{\dashv} \subseteq b^{\dashv}$. In this partial order, a is a maximum if and only if a is a rank one operator. By the two way range orthogonality preserving assumption, we see that both θ and θ^{-1} preserve this partial order, and thus send the maxima onto the maxima. In other words, θ and θ^{-1} send rank one operators onto rank one operators. It then follows from the Fundamental Theorem of Affine Geometry that there exist invertible linear or conjugate linear maps $S : H \to K$ and $R : K \to H$ such that either

$$\theta(x \otimes y) = Sx \otimes Ry, \quad \forall x, y \in H,$$

or

$$\theta(x \otimes y) = Sy \otimes Rx, \quad \forall x, y \in H.$$

However, the second case does not give us a range orthogonality preserver, and thus be ruled out.

Observe that

$$
\begin{aligned}
&\langle x_1, x_2\rangle = 0 \\
\text{implies}\quad & (x_2\otimes y_2)^*(x_1\otimes y_1) = 0, \quad \forall y_1, y_2 \in H \\
\text{implies}\quad & \theta(x_2\otimes y_2)^*\theta(x_1\otimes y_1) = 0, \quad \forall y_1, y_2 \in H \\
\text{implies}\quad & (Sx_2\otimes Ry_2)^*(Sx_1\otimes Ry_1) = 0, \quad \forall y_1, y_2 \in H \\
\text{implies}\quad & \langle Sx_1, Sx_2\rangle = 0.
\end{aligned}
$$

For any two orthogonal norm one elements x, y in H, we have $\langle x, y\rangle = \langle x+y, x-y\rangle = 0$. This gives $\langle Sx, Sy\rangle = \langle Sx+Sy, Sx-Sy\rangle = 0$, and therefore $\|Sx\| = \|Sy\|$. It follows that $S = \lambda U$ for a nonzero scalar λ and a unitary operator U from H onto K. Renaming λR by R, we will have

$$\theta(x\otimes y) = Ux\otimes Ry, \quad \forall x, y \in H.$$

To get the boundedness of R we need to utilize the extra assumptions on A now. Suppose first that A is unital. For any norm one element e in H, as $(e\otimes e)(1-e\otimes e) = 0$, we have $\theta(e\otimes e)^*(\theta(1)-\theta(e\otimes e)) = 0$. It follows $Re\otimes\theta(1)^*Ue = \langle Ue, Ue\rangle Re\otimes Re = Re\otimes Re$, and consequently, $Re = \theta(1)^*e$. So $R = \theta(1)^*U$ is bounded.

Suppose then that A contains all trace class operators on H and H is of infinite dimension. Suppose on contrary that there were an orthonormal sequence $\{x_n\}$ in H such that $\|Rx_n\| \geq n^3$ for $n = 1, 2, 3, \ldots$. Define a trace class operator W on H by $W = \sum_n x_n\otimes x_n/n^2$. Since $(x_n\otimes x_n)(n^2W - x_n\otimes x_n) = 0$, we have $\theta(x_n\otimes x_n)^*(n^2\theta(W) - \theta(x_n\otimes x_n)) = 0$. It follows $n^2Rx_n\otimes\theta(W)^*Ux_n = \langle Ux_n, Ux_n\rangle Rx_n\otimes Rx_n = Rx_n\otimes Rx_n$. As a result, $\|\theta(W)^*\| \geq \|\theta(W)^*Ux_n\| = \|Rx_n\|/n^2 \geq n$ for all $n = 1, 2, 3, \ldots$. This contradiction ensures again that R is bounded.

Let $a \in A$. For any $x \neq 0$ in H, let $y \in H$ such that $\langle x, y\rangle = 1$. Set $b = a - (y\otimes a^*x)$. Observe $b^*(x\otimes y) = 0$. Thus,

$$
\begin{aligned}
0 = \theta(b)^*\theta(x\otimes y) &= (\theta(b)^*Ux)\otimes Ry \\
&= ([\theta(a)^* - \theta(y\otimes a^*x)^*]Ux)\otimes Ry \\
&= (\theta(a)^*Ux - (Ra^*x\otimes Uy)Ux)\otimes Ry.
\end{aligned}
$$

This implies

$$\theta(a)^*Ux = (Ra^*x\otimes Uy)Ux = Ra^*x, \quad \forall x \in H.$$

Hence,

$$\theta(a) = UaR^*, \quad \forall a \in A.$$

Setting $T = R^*$, we are done, as the boundedness of θ is now clear. □

Next, we consider the other cases θ transforming the domain/range orthogonality to the domain/range orthogonality.

Theorem 2. Let A, B be standard operator algebras on Hilbert spaces H, K, respectively. Suppose A is unital, or A contains all trace class operators on H. Let $\theta : A \to B$ be a surjective linear map.

(a) Assume that $ab^* = 0$ if and only if $\theta(a)\theta(b)^* = 0$. Then θ is bounded, and there exists a bounded invertible linear operator $S : H \to K$ and a unitary operator $V : K \to H$ such that
$$\theta(a) = SaV, \quad \forall a \in A.$$
(b) Assume that $a^*b = 0$ if and only if $\theta(a)\theta(b)^* = 0$. Then θ is bounded, and there exist a bounded invertible linear operator $S : H \to K$ and a unitary operator $V : K \to H$ such that
$$\theta(a) = Sa^tV, \quad \forall a \in A.$$
(c) Assume that $ab^* = 0$ if and only if $\theta(a)^*\theta(b) = 0$. Then there exist a unitary operator $U : H \to K$ and a bounded invertible linear operator operator $T : K \to H$ such that
$$\theta(a) = Ua^tT, \quad \forall a \in A.$$

Proof. For a fixed orthonormal basis, we can define three range orthogonality preserving surjective linear maps respectively by setting
$$a \mapsto \theta(a^t)^t, \quad a \mapsto \theta(a)^t, \quad \text{and } a \mapsto \theta(a^t).$$
Then Theorem 1 applies. □

Finally, we will investigate the doubly orthogonality preservers. A map θ is called a *doubly orthogonality preserver* if $\theta(a)^*\theta(b) = \theta(a)\theta(b)^* = 0$ whenever $a^*b = ab^* = 0$. Bounded doubly orthogonality preservers between C*-algebras and JB*-algebras are studied in [5, 2]. Note also that like the case of the zero product preservers, we do not need to assume A is unital or A contains any trace class operator on H in this case.

Theorem 3. Let $\theta : A \to B$ be a surjective linear map between standard operator algebras on Hilbert space H, K, respectively, such that $a^*b = ab^* = 0$ if and only if $\theta(a)^*\theta(b) = \theta(a)\theta(b)^* = 0$. Then θ is bounded, and there exist a nonzero scalar λ and unitary operators $U : H \to K$ and $V : K \to H$ such that either
$$\theta(a) = \lambda UaV, \quad \forall a \in A,$$
or
$$\theta(a) = \lambda Ua^tV, \quad \forall a \in A.$$

Proof. Put for all nonzero a in A that
$$a^{\dashv} = \{c \in A : c^*a = 0\} \quad \text{and} \quad a^{\vdash} = \{c \in A : ac^* = 0\}.$$
Set $a^+ = a^{\dashv} \cap a^{\vdash}$. For any a and b in A, it is not difficult to see that that $a^+ \subseteq b^+$ if and only if the closure of the range space of a contains that of b, and the initial space of a contains that of b. Define a partial order on A by saying $a \leq b$ if and only if $a^+ \subseteq b^+$. In this partial order, a is a maximum if and only if a is of rank one. By the doubly orthogonality preserving property of θ and θ^{-1}, we see that both of them preserves this partial order, and thus sends the maxima onto the maxima. In other words, both

θ and θ^{-1} send rank one operators onto rank one operators. It then follows from the Fundamental Theorem of Affine Geometry that there exist invertible linear or conjugate linear maps $S : H \to K$ and $R : K \to H$ such that either

$$\theta(x \otimes y) = Sx \otimes Ry, \quad \forall x, y \in H,$$

or

$$\theta(x \otimes y) = Ry \otimes Sx, \quad \forall x, y \in H.$$

By replacing θ with the map $a \mapsto \theta(a)^t$ if necessary, we can assume that the first case happens.

Arguing as in the proof of Theorem 1, we will see that there exist nonzero scalars λ_1, λ_2 such that $U = \lambda_1^{-1} S$ is a unitary operator from H onto K, and $W = \lambda_2^{-1} R$ is a unitary operator from K onto H. Put $\lambda = \lambda_1 \lambda_2$ and $V = W^*$, we will have

$$\theta(a) = \lambda U a V, \quad \forall a \in \mathcal{F}(H).$$

In general, let $a \in A$. For any x in H with $a^* x \neq 0$, let $y \in H$ such that $\langle x, ay \rangle = 1$. Set $b = a - (ay \otimes a^* x)$. Observe $b^*(x \otimes y) = b(x \otimes y)^* = 0$. Thus,

$$\begin{aligned} 0 = \theta(b)^* \theta(x \otimes y) &= \lambda(\theta(b)^* U x) \otimes V^* y \\ &= \lambda([\theta(a)^* - \theta(ay \otimes a^* x)^*] U x) \otimes V^* y \\ &= \lambda(\theta(a)^* U x - \bar{\lambda}(V^* a^* x \otimes U a y) U x) \otimes V^* y. \end{aligned}$$

This implies

$$\theta(a)^* U x = \bar{\lambda}(V^* a^* x \otimes U a y) U x = \lambda V^* a^* x, \quad \forall x \in H.$$

Hence,

$$\theta(a) = \lambda U a V, \quad \forall a \in A.$$

The map θ is clearly bounded. □

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