TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 5, pp. 2357-2376, October 2011
This paper is available online at http://tjm.math.ntu.edu.tw/index.php/TJM

CHARACTERIZATIONS AND UNIQUENESS OF BEST SIMULTANEOUS τ_C-APPROXIMATIONS

X. F. Luo, L. H. Peng, C. Li and N. C. Wong*

Abstract. This paper is concerned with the problem of best simultaneous τ_C-approximations in terms of the Minkowski functional. The notions of simultaneous sun and simultaneous regular set are extended to suit the case of simultaneous τ_C-approximation problems. Characterization and uniqueness results are established for simultaneous τ_C-suns.

1. INTRODUCTION

Let X be a normed linear space and C be a bounded closed convex subset of X having the origin as set interior point. Recall that the Minkowski function $p_C : X \to \mathbb{R}$ with respect to set C is defined by

$$p_C(x) := \inf\{t > 0 : \ x \in tC\}, \quad \forall\, x \in X. \tag{1.1}$$

Let G be a nonempty subset of X and F a bounded set in X. Write

$$\tau_C(F; G) := \inf_{g\in G} \sup_{x\in F} p_C(g - x).$$

If there exists an element $g_0 \in G$ such that

$$\sup_{x\in F} p_C(g_0 - x) = \tau_C(F; G),$$

then g_0 is called a best simultaneous τ_C-approximation to F form G. The set of all best simultaneous τ_C-approximations to F from G is denoted by $\mathrm{P}^C_G(F)$. In particular, we write $\tau_C(x; G)$ and $\mathrm{P}^C_G(x)$ for $\tau_C(\{x\}; G)$ and $\mathrm{P}^C_G(\{x\})$, respectively.

Received January 10, 2010, accepted February 26, 2010.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: Primary 41A65.
Key words and phrases: Minkowski function, Best simultaneous τ_C-approximation, Simultaneous τ_C-sun, Characterization, Uniqueness.
The first author is supported in part by the NNSF of China (Grant No. 90818020) and the NSF of Zhejiang Province of China (Grant No. Y7080235); the second author is supported in part by the NNSF of China (Grant No. 10926035); the third author is supported in part by NNSF of China (Grant No. 10731060); and the fourth author is supported in part by the Grant NSC 99-2115-M-110-007-MY3.
*Corresponding author.

Note that in the case when C is the closed unit ball of X, the best simultaneous τ_C-approximation is reduced to the classical best simultaneous approximation, which has a long history and continues to generate much interest; see, e.g., [7, 10, 16, 12, 17] and references therein. In particular, Freilich and Maclaughlin established in [7] the Kolomogorov type characterization of best simultaneous approximation from a convex set, which was further extended in [16] to the case of nonlinear settings. Some uniqueness results of the best simultaneous approximation from a subspace and from a simultaneous sun were given in [1] and [9], respectively. While in the case when F is a singleton, the best simultaneous τ_C-approximation is reduced to the generalized best approximation following [5] or the best τ_C-approximation following [14]. This has been extensively studied; see, e.g., [5, 11, 13, 14] and references therein. In particular, the generic well-posedness of the generalized best approximation problem in terms of the Baire category was investigated in [5, 11]; the relationships between the existence of a generalized best approximation and the directional derivative of the function $\tau_C(\cdot; G)$ were studied in [13], while the characterizations of the generalized best approximation from a general set were given in [14].

We study in the present paper the problem of best simultaneous τ_C-approximation from subsets, which are not necessarily convex. Our main results are on twofolds: one is on the characterization and the other is on the uniqueness. For the first one, we introduce the notions of the simultaneous τ_C-sun and the simultaneous regular set, and establish the equivalence between the above notions and the Kolomogorov type condition for the best simultaneous τ_C-approximation, which are extensions of the corresponding ones due to [14, 16, 17]; while for the second one, we use the strict convexity with respect to the approximating set, together with the uniform convexity in every direction in the approximating set, to characterize the uniqueness of the best simultaneous τ_C-approximation. It should be remarked that the uniqueness problem for the general case are different from the special case when C is the closed unit ball. Recall from [17], for a simultaneous sun G, that the best simultaneous approximation to F from G is unique for any compact set F (resp. for any bounded set F) if and only if X is strictly convex with respect to G (resp. uniformly convex in every direction in G). However these results are no longer true for the general case; see Example 2 of section 4. For convex sets G we provide a complete characterization result for the uniqueness of the best simultaneous τ_C-approximation for all compact sets in terms of the strict convexity with respect to G, which seems new even for the case when C is the closed unit ball.

2. Preliminaries

Throughout this paper, unless otherwise stated, we always assume that X is a real or complex normed linear space with its Banach dual X^*. Also, let C be a bounded closed convex subset of X having the origin as an interior point. The pole of C is denoted by C° and is defined by

$$C^\circ := \{x^* \in X^* : \text{ Re } x^*(x) \le 1, \ \forall\, x \in C\}$$

(cf., [15, Section 1]). Then C° is a nonempty weakly*-compact convex subset of X. Moreover, we endow C° with the restricted weak*-topology. Then C° is a compact Hausdorff space. Let A be a nonempty subset of X. We use $\mathrm{bd}A$, $\mathrm{int}A$, $\mathrm{ext}A$ and $\mathrm{cone}A$ to denote the boundary of A, the interior of A, the set of all extreme points of A and the cone generated by A, respectively. Let $x \in X$ and $\delta > 0$. $\mathbf{B}(x,\delta)$ and $\mathrm{U}(x,\delta)$ stand for the closed and open ball with center x and radius δ; in particular, $\mathbf{B}$ denotes the closed unit ball of X. Furthermore, for $x, y \in X$, we use $[x, y]$ and (x, y) to denote the closed and open interval with ends x and y, respectively.

For the convenience of the reader, we first list some known and useful properties of the Minkowski function which will be used in the remainder of this paper; see, e.g., [15, Section 1]. Recall that the Minkowski function with respect to C is defined by (1.1).

Proposition 1. *Let $x, y \in X$. Then*

(i) $p_C(x) \ge 0$ *and* $p_C(x) = 0 \Leftrightarrow x = 0$.

(ii) $p_C(tx) = tp_C(x)$ *for each* $t \ge 0$.

(iii) $p_C(x+y) \le p_C(x) + p_C(y)$ *and* $p_C(x-y) \ge p_C(x) - p_C(y)$.

(iv) $p_C(x) < 1 \Leftrightarrow x \in \mathrm{int}C$ *and* $p_C(x) = 1 \Leftrightarrow x \in \mathrm{bd}C$.

(v) $\mu\|x\| \le p_C(x) \le \nu\|x\|$, *where*

$$\mu := \inf_{y \in \mathrm{bd}\mathbf{B}} p_C(y) \quad \text{and} \quad \nu := \sup_{y \in \mathrm{bd}\mathbf{B}} p_C(y). \tag{2.1}$$

(vi) $p_C(x) = \sup_{x^* \in C^\circ} \text{Re } x^*(x)$ *and* $p_{C^\circ}(x^*) = \sup_{x \in C} \text{Re } x^*(x)$.

(vii) *If C is symmetry (i.e., $-x \in C$ if $x \in C$), then p_C is a norm equivalent to the original one of X.*

Let $F \subset X$ be bounded. The associated function V_F to F on C°, defined by

$$V_F(x^*) := \inf_{x \in F} \mathrm{Re}\, x^*(x), \quad \forall\, x^* \in C^\circ, \tag{2.2}$$

plays an important role in our study. Clearly, V_F is bounded on C°, and in the case when F is totally bounded, V_F is continuous on C°. Let V_F^- be a lower envelope of V_F, that is

$$V_F^-(x^*) := \sup_{O \in N_{x^*}} \inf_{w \in O} V_F(w), \quad \forall\, x^* \in C^\circ,$$

where N_{x^*} stands for the set of all open neighborhoods of x^* in C°. Then V_F^- is lower semicontinuous on C°, and

(2.3) $$\min_{x^*\in C^\circ} V_F^-(x^*) = \inf_{x^*\in C^\circ} V_F(x^*), \quad \forall\, x^* \in C^\circ.$$

The following result will be used in the next section. For brevity, we sometime write $p_C(F)$ for $\sup_{x\in F} p_C(x)$ and set for any $g \in X$

(2.4) $$V_g(x^*) := V_{\{g\}}(x^*) = \operatorname{Re} x^*(g), \quad \forall x^* \in C^\circ.$$

Lemma 1. *Let $g \in X$ and $F \subset X$ be a bounded set. Then*

$$\max_{x^*\in C^\circ} [V_g(x^*) - V_F^-(x^*)] = p_C(g - F).$$

Proof. Noting that V_g is continuous on C°, one has that

$$V_{F-g}^-(x^*) = V_F^-(x^*) - V_g(x^*), \quad \forall\, x^* \in C^\circ.$$

Thus, by definitions together with (2.4) and Proposition 1(vi) (applied to $g - x$ in place of x), we have that

$$\max_{x^*\in C^\circ} [V_g(x^*) - V_F^-(x^*)] = \sup_{x^*\in C^\circ} (-V_{F-g}^-(x^*)) = \sup_{x\in F} \sup_{x^*\in C^\circ} \operatorname{Re} x^*(g-x) = p_C(g-F).$$

The proof is complete. ∎

3. Characterizations of Best Simultaneous τ_C-Approximations

The notion of suns introduced by Efimov and Stechkin (cf. [6]) has played important roles in nonlinear approximation theory in Banach spaces; see, e.g., [2, 3, 4, 6, 17] and references therein, and an extension to the case of the best simultaneous approximation was done in [16]. Recently, it was extended in [14] to the setting of the best τ_C-approximation. Below we further extend this notion to the case of the best simultaneous τ_C-approximation. In what follows, we always assume that G is a nonempty subset of X. Let $F \subset X$ be bounded and let $g_0 \in G$. We write

$$F_\alpha := g_0 + \alpha(F - g_0), \quad \forall \alpha \geq 0.$$

Then the following implication is direct by definition:

(3.1) $$g_0 \in \mathrm{P}_G^C(F) \Rightarrow g_0 \in \mathrm{P}_G^C(F_\alpha), \quad \forall\, \alpha \in [0, 1].$$

Definition 1. Let $g_0 \in G$ and let $F \subset X$ be bounded. The element g_0 is called

(a) a simultaneous τ_C-solar point of G with respect to F if $g_0 \in \mathrm{P}_G^C(F)$ implies that $g_0 \in \mathrm{P}_G^C(F_\alpha)$ for each $\alpha > 0$.

(b) a simultaneous τ_C-solar point of G if g_0 is a simultaneous τ_C-solar point of G with respect to each bounded set.

We say G is a simultaneous τ_C-sun of X if each point of G is a simultaneous τ_C-solar point of G.

Clearly, we see by definition that any convex set in X is a simultaneous τ_C-sun. To provide some examples of nonconvex simultaneous τ_C-suns , we recall from [17] that a subset G of X is called

(a) quasi-convex if for each pair of $g_1, g_2 \in G$, $[g_1, g_2] \cap G$ is dense in $[g_1, g_2]$.

(b) pseudo-convex if there exist a convex set D_1 and a closed set D_2 in X such that $G = D_1 \setminus D_2$.

In the following example, we show that both quasi-convex sets and pseudo-convex sets are simultaneous τ_C-suns of X.

Example 1. Let $G \subset X$ be quasi-convex or pseudo-convex. Then G is a simultaneous τ_C-sun. To show this fact, let F be a bounded set in X and $g_0 \in \mathrm{P}_G^C(F)$. Let $g \in G$ and write $g_{\frac{1}{\alpha}} \coloneqq \left(1 - \frac{1}{\alpha}\right) g_0 + \frac{1}{\alpha} g$ for each $\alpha > 0$. Then, for each $\alpha > 1$, $p_C(g_0 - F_\alpha) \le p_C(g - F_\alpha)$ if and only if

$$p_C(g_0 - F) \le p_C(g_{\frac{1}{\alpha}} - F). \tag{3.2}$$

This together with (3.1), it suffices to show that (3.2) holds for all $\alpha \in (1, +\infty)$. To do this, let $\alpha > 1$. Then $g_{\frac{1}{\alpha}} \in [g_0, g]$. We first consider the case when G is quasi-convex. Thus by the definition there exists a sequence $\{v_n\} \subset G$ such that $\lim_{n\to\infty} v_n = g_{\frac{1}{\alpha}}$. This implies that

$$p_C(g_0 - F) \le p_C(v_n - F), \quad \forall\, n \in \mathbb{N}.$$

Letting $n \to \infty$ yields (3.2). We then consider the case when G is a pseudo-convex set. Assume $G = D_1 \setminus D_2$ for some convex set D_1 and closed set D_2 in X. Since $g_0 \in G$, one has that $g_0 \notin D_2$. By the closedness of D_2, there exists $\delta > 0$ such that $\mathbf{B}(g_0, \delta) \cap D_2 = \emptyset$. Let $\alpha_0 = \max\{2, \|g - g_0\|/\delta\}$. If $\alpha \ge \alpha_0$, then $\|g_{\frac{1}{\alpha}} - g_0\| = \frac{1}{\alpha}\|g - g_0\| \le \delta$ and $g_{\frac{1}{\alpha}} \in G$; hence (3.2) holds as $g_0 \in \mathrm{P}_G^C(F)$. It remains to consider the case when $\alpha \in (1, \alpha_0)$. In this case, we have that $g_{\frac{1}{\alpha_0}} = (1-\lambda) g_0 + \lambda g_{\frac{1}{\alpha}}$ where $\lambda := \frac{\alpha}{\alpha_0} \in (0, 1)$. This implies that

$$p_C(g_0 - F) \le p_C(g_{\frac{1}{\alpha_0}} - F) \le (1-\lambda) p_C(g_0 - F) + \lambda p(g_{\frac{1}{\alpha}} - F)$$

(noting that (3.2) is true for $\alpha = \alpha_0$) and (3.2) is seen to hold. The proof is complete.

Definition 2. Let $g_0 \in G$ and $F \subset X$ be bounded. The element g_0 is called a local best simultaneous τ_C-approximation to F from G if there exists $\delta > 0$ such that $g_0 \in \mathrm{P}^C_{G\cap \mathrm{U}(g_0,\delta)}(F)$.

Obviously, if $g_0 \in \mathrm{P}^C_G(F)$, then g_0 is a local best simultaneous τ_C-approximation to F from G. The following theorem shows that the converse remains true if g_0 is a simultaneous τ_C-solar point of G.

Proposition 2. *Let g_0 be a simultaneous τ_C-solar point. Then $g_0 \in \mathrm{P}^C_G(F)$ if and only if g_0 is a local best simultaneous τ_C-approximation to F from G.*

Proof. We only prove the sufficiency part because the necessity part is obvious. To this end, Suppose that g_0 is a local best simultaneous τ_C-approximation to F from G. Then there is a positive number δ such that

$$p_C(g_0 - F) \le p_C(g - F), \quad \forall\, g \in G \cap \mathrm{U}(g_0, \delta). \tag{3.3}$$

Let $\lambda := \min\{1, \mu\}$ and $\delta' := \frac{\lambda\delta}{\mu}$, where μ is defined by (2.1). Then $\delta' \le \delta$. Without loss of generality, we may assume that $F \ne \{g_0\}$. Let

$$\alpha := \min\left\{1,\ \frac{\lambda\delta'}{\inf_{x\in F} p_C(x - g_0) + p_C(g_0 - F)}\right\}. \tag{3.4}$$

We assert that $g_0 \in \mathrm{P}^C_G(F_\alpha)$. To show this assertion, let $g \in G$. We first assume that $g \in G \setminus \mathrm{U}(g_0, \delta')$. Then $\|g - g_0\| \ge \delta'$. By Proposition 1(iii) and (3.4), one has that

$$\begin{aligned}
p_C(g - F_\alpha) &\ge \sup_{x\in F_\alpha} [p_C(g - g_0) - p_C(x - g_0)] \\
&= p_C(g - g_0) - \inf_{x\in F_\alpha} p_C(x - g_0) \\
&\ge \mu\|g - g_0\| - \alpha \inf_{x\in F} p_C(x - g_0) \\
&\ge \lambda\delta' - \alpha \inf_{x\in F} p_C(x - g_0) \\
&\ge \alpha p_C(g_0 - F) \\
&= p_C(g_0 - F_\alpha),
\end{aligned}$$

where the third inequality holds because $\mu \ge \lambda$. Now we consider the case when $g \in G \cap \mathrm{U}(g_0, \delta')$, and suppose on the contrary that there exists $\bar g \in G \cap \mathrm{U}(g_0, \delta')$ such that $p_C(\bar g - F_\alpha) < p_C(g_0 - F_\alpha)$. Then

$$\begin{aligned}
p_C(\bar g - F) &= \sup_{x\in F} p_C((1-\alpha)(g_0 - x) + \bar g - (g_0 + \alpha(x - g_0))) \\
&\le (1-\alpha)p_C(g_0 - F) + p_C(\bar g - F_\alpha) \\
&< (1-\alpha)p_C(g_0 - F) + p_C(g_0 - F_\alpha) \\
&= p_C(g_0 - F),
\end{aligned}$$

which contradicts (3.3) since $\bar{g} \in G \cap \mathrm{U}(g_0, \delta') \subset G \cap \mathrm{U}(g_0, \delta)$. Therefore the assertion is proved. Since $F = g_0 + \frac{1}{\alpha}(F_\alpha - g_0)$ and since g_0 is a simultaneous τ_C-solar point, it follows that $g_0 \in \mathrm{P}_G^C(F)$ and the proof is complete. ■

The notion of the simultaneous τ_C-regular point in the following definition is an extension of the corresponding one in [16] for the case of simultaneous approximations. Write

$$M_{g_0-F} := \{x^* \in C^\circ : \ V_{g_0}(x^*) - V_F^-(x^*) = p_C(g_0 - F)\}. \tag{3.5}$$

Then M_{g_0-F} is a nonempty compact subset of C°.

Definition 3. Let $g_0 \in G$ and $F \subset X$ be bounded. Then g_0 is called

(a) a simultaneous τ_C-regular point of G with respect to F if, for each closed subset A of C° satisfying the condition

$$M_{g_0-F} \subset A \subset C^\circ \quad \text{and} \quad \min_{x^* \in A} \mathrm{Re}\ x^*(g_0 - g) > 0 \tag{3.6}$$

for some $g \in G$, there exists a sequence $\{g_n\} \subset G$ such that $\lim_{n\to\infty} g_n = g_0$ and

$$\mathrm{Re}\, x^*(g_0 - g_n) > \mathrm{Re}\, x^*(g_0) - V_F^-(x^*) - p_C(g_0 - F), \quad \forall\, x^* \in A, \quad \forall\, n \in \mathbb{N}. \tag{3.7}$$

(b) a simultaneous τ_C-regular point if g_0 is a simultaneous τ_C-regular point of G with respect to each bounded set.

We say that G is a simultaneous τ_C-regular set if each point of G is a simultaneous τ_C-regular point of G.

The relationships among the simultaneous τ_C-solar point with respect to F, the simultaneous τ_C-regular point with respect to F and the Kolmogorov type condition for F are described in the following proposition.

Proposition 3. *Let $g_0 \in G$ and let $F \subset X$ be bounded. Consider the following assertions.*

(i) *g_0 is a simultaneous τ_C-regular point of G with respect to F.*

(ii) $g_0 \in \mathrm{P}_G^C(F) \Leftrightarrow \max\{\mathrm{Re}\ x^*(g - g_0) : \ x^* \in M_{g_0-F}\} \geq 0, \quad \forall\, g \in G.$

(iii) *g_0 is a simultaneous τ_C-solar point of G with respect to F.*

Then (i)⇒(ii)⇔(iii).

Proof. (i)⇒(ii) Suppose that (i) holds. Since the proof for the sufficiency part in (ii) is straightforward, below we only prove the necessity part of (ii). To do this, assume that $g_0 \in \mathrm{P}_G^C(F)$ and, on the contrary, that there exists $\bar{g} \in G$ such that

$$\max\{\mathrm{Re}\, x^*(\bar{g} - g_0) : \ x^* \in M_{g_0-F}\} = -\epsilon < 0. \tag{3.8}$$

Let

$$U := \left\{x^* \in C^\circ : \text{ Re } x^*(\bar{g} - g_0) < -\frac{\epsilon}{2}\right\} \quad \text{and} \quad A := \overline{U}^*. \tag{3.9}$$

Then U is an open subset of C° containing M_{g_0-F}. Moreover, (3.6) holds with $\bar{g}$ in place of g. In view of Definition 3, there exists a sequence $\{g_n\} \subset G$ such that $\lim_{n\to\infty} g_n = g_0$ and (3.7) holds. It follows from (3.7) that

$$\text{Re } x^*(g_n) - V_F^-(x^*) < p_C(g_0 - F), \quad \forall\ x^* \in A. \quad \forall\ n \in \mathbb{N};$$

hence by (2.4),

$$\sup_{x^*\in U} [V_{g_n}(x^*) - V_F^-(x^*)] < p_C(g_0 - F), \quad \forall\ n \in \mathbb{N} \tag{3.10}$$

because $A = \overline{U}^*$ is compact. On the other hand, since $(C^\circ \setminus U) \cap M_{g_0-F} = \emptyset$, it follows from (3.5) and Lemma 1 that

$$V_{g_0}(x^*) - V_F^-(x^*) < p_C(g_0 - F), \quad \forall\ x^* \in C^\circ \setminus U.$$

Thus there is a $\delta > 0$ such that

$$\max_{x^*\in C^\circ\setminus U} [V_{g_0}(x^*) - V_F^-(x^*)] \le p_C(g_0 - F) - \delta \tag{3.11}$$

because $C^\circ \setminus U$ is a compact subset of C°. Noting that $\lim_{n\to\infty} g_n = g_0$, we have that $\lim_{n\to\infty} p_C(g_n - g_0) = 0$. Take a positive integer n_0 such that $p_C(g_{n_0} - g_0) < \delta$. It follows from (3.11) and Proposition 1(vi) that

$$\begin{aligned}\sup_{x^*\in C^\circ\setminus U} [V_{g_{n_0}}(x^*) - V_F^-(x^*)] &= \sup_{x^*\in C^\circ\setminus U} [V_{g_0}(x^*) - V_F^-(x^*) + V_{g_{n_0}-g_0}(x^*)] \\ &< p_C(g_0 - F) - \delta + p_C(g_{n_0} - g_0) \\ &< p_C(g_0 - F).\end{aligned}$$

This together with Lemma 1 and (3.10) implies that

$$p_C(g_{n_0} - F) = \max_{x^*\in C^\circ} [V_{g_{n_0}}(x^*) - V_F^-(x^*)] < p_C(g_0 - F),$$

which contradicts that $g_0 \in \mathrm{P}_G^C(F)$ and the necessity part of (ii) holds.

(ii)⇒(iii) Suppose that (ii) holds and that $g_0 \in \mathrm{P}_G^C(F)$. Then, by (ii),

$$\max\{\text{Re } x^*(g - g_0) : x^* \in M_{g_0-F}\} \ge 0, \quad \forall\ g \in G. \tag{3.12}$$

Let $\alpha > 0$ and $x^* \in C^\circ$. Since V_g is continuous on C°, one has that

$$V_{F_\alpha}^-(x^*) = (1 - \alpha)V_{g_0}(x^*) + \alpha V_F^-(x^*). \tag{3.13}$$

It is easy to see that

$$V_{g_0}(x^*) - V_F^-(x^*) = p_C(g_0 - F) \Leftrightarrow V_{g_0}(x^*) - V_{F_\alpha}^-(x^*) = p_C(g_0 - F_\alpha);$$

hence $M_{g_0-F} = M_{g_0-F_\alpha}$. This and (3.12) imply that

$$\max\{\operatorname{Re} x^*(g - g_0) : \ x^* \in M_{g_0-F_\alpha}\} \geq 0, \quad \forall \ g \in G;$$

hence $g_0 \in \mathrm{P}_G^C(F_\alpha)$ by (ii). This means that g_0 is a simultaneous τ_C-solar point of G with respect to F and (iii) holds.

(iii)$\Rightarrow$(ii) Suppose that (iii) holds. We only prove the necessity part of (ii). Let $g_0 \in \mathrm{P}_G^C(F)$ and suppose on the contrary that there exists a $\bar{g} \in G$ such that (3.8) holds. Below we prove that the inequality

$$p_C(\bar{g} - F_\alpha) < p_C(g_0 - F_\alpha) \tag{3.14}$$

holds for all α large enough. Granting this, $g_0 \notin \mathrm{P}_G^C(F_\alpha)$ and hence g_0 is not a simultaneous τ_C-solar point of G with respect to F, which contradicts (iii) and proves the implication.

To show (3.14), define the sets U and A as in (3.9). Then, for each $x^* \in U$ and each $\alpha > 0$, by (3.13), (2.4), (3.9) and Lemma 1, we obtain that

$$\begin{aligned} V_{\bar{g}}(x^*) - V_{F_\alpha}^-(x^*) &= \alpha(V_{g_0}(x^*) - V_F^-(x^*)) + V_{\bar{g}}(x^*) - V_{g_0}(x^*) \\ &\leq \alpha p_C(g_0 - F) - \frac{\epsilon}{2} \\ &= p_C(g_0 - F_\alpha) - \frac{\epsilon}{2}. \end{aligned}$$

Hence,

$$\sup_{x^* \in U} [V_{\bar{g}}(x^*) - V_{F_\alpha}^-(x^*)] < p_C(g_0 - F_\alpha), \quad \forall \ x^* \in U, \ \forall \ \alpha > 0. \tag{3.15}$$

On the other hand, as in the proof of (i)$\Rightarrow$(ii), there exists a $\delta > 0$ such that (3.11) holds. Let $\alpha > p_C(\bar{g} - g_0)/\delta$. It follows from (2.4), (3.13), (3.11) and Proposition 1(vi) that

$$\begin{aligned} \sup_{x^* \in C^\circ \setminus U} [V_{\bar{g}}(x^*) - V_{F_\alpha}^-(x^*)] &= \max_{x^* \in C^\circ \setminus U} [V_{g_0}(x^*) - V_{F_\alpha}^-(x^*) + V_{\{\bar{g}-g_0\}}(x^*)] \\ &\leq \alpha \max_{x^* \in C^\circ \setminus U} [V_{g_0}(x^*) - V_F^-(x^*)] + p_C(\bar{g} - g_0) \\ &\leq \alpha p_C(g_0 - F) - \alpha\delta + p_C(\bar{g} - g_0) \\ &< p_C(g_0 - F_\alpha). \end{aligned}$$

This together with (3.15) and Lemma 1 (applied respectively to $\bar{g}$ and F_α in place of g and F) implies that (3.14) holds and the proof is complete. ■

The following corollary provides a characterization for g_0 to be a best simultaneous τ_C-approximation from G.

Corollary 1. *Let F be a bounded subset of X and let g_0 be a simultaneous τ_C-solar point of G with respect to F. Then*

$$g_0 \in \mathrm{P}_G^C(F) \Leftrightarrow \max\{\mathrm{Re}\, x^*(g-g_0) :\ x^* \in M_{g_0-F}\} \geq 0, \quad \forall\, g \in G. \tag{3.16}$$

If F is additionally totally bounded, then the set M_{g_0-F} can be replaced by the set E_{g_0-F} defined by

$$E_{g_0-F} := \{x^* \in \mathrm{ext}\, C^\circ :\ V_{g_0}(x^*) - V_F(x^*) = p_C(g_0 - F)\}.$$

Proof. Equivalence (3.16) is a direct consequence of the equivalence of (ii) and (iii) in Proposition 3. Furthermore, suppose that F is additionally totally bounded. We have to verify the following equivalence:

$$g_0 \in \mathrm{P}_G^C(F) \Leftrightarrow \max\{\mathrm{Re}\, x^*(g-g_0) :\ x^* \in E_{g_0-F}\} \geq 0, \quad \forall\, g \in G. \tag{3.17}$$

Since $E_{g_0-F} \subset M_{g_0-F}$, the sufficiency part of equivalence (3.17) is trivial. Below we prove the necessity part of (3.17). To do this, let $g \in G \setminus \{g_0\}$. Then, by (3.16), we have that

$$\max\{V_{g-g_0}(x^*) :\ x^* \in M_{g_0-F}\} = \max\{\mathrm{Re}\, x^*(g-g_0) :\ x^* \in M_{g_0-F}\} \geq 0. \tag{3.18}$$

Since F is totally bounded, one has that V_F is continuous on C°; hence $V_F^-(x^*) = V_F(x^*)$ for each $x^* \in C^\circ$ and

$$M_{g_0-F} = \{x^* \in C^\circ :\ V_{g_0}(x^*) - V_F(x^*) = p_C(g_0 - F)\}.$$

This implies that M_{g_0-F} is an extremal subset of C° and so

$$\mathrm{ext} M_{g_0-F} = M_{g_0-F} \cap \mathrm{ext} C^\circ = E_{g_0-F}, \tag{3.19}$$

thanks to [8, Lemma (d), p.32]. Note that M_{g_0-F} is compact and that the function V_{g-g_0} is continuous linear on X^*. By [8, Corollary, p.74], the maximum in (3.18) is attainable at a point of $\mathrm{ext} M_{g_0-F}$. This together with (3.19) and (3.18) completes the proof of the necessity part of (3.17). The proof is complete. ■

The following theorem gives a complete characterization for the equivalence among the simultaneous τ_C-regular point, simultaneous τ_C-solar point and the Kolmogorov type condition.

Theorem 1. *Let $g_0 \in G$. Then the following statements are equivalent.*

(*i*) *g_0 is a simultaneous τ_C-regular point of G.*

(*ii*) (3.16) *holds for each bounded set F in X.*

(*iii*) *g_0 is a simultaneous τ_C-solar point of G.*

Proof. By Proposition 3, it suffices to show (ii)⇒(i). For this purpose, suppose that (ii) holds. Let F be any bounded subset of X and let A be any closed subset of C° satisfying (3.6) for some $g \in G$. Then

$$\max_{x^* \in M_{g_0 - F}} \operatorname{Re} x^*(g - g_0) \le \max_{x^* \in A} \operatorname{Re} x^*(g - g_0) < 0.$$

Hence $g_0 \notin \mathrm{P}_G^C(F)$ due to (3.16). By the equivalence of (ii) and (iii) just proved and Proposition 2, g_0 is not a local best simultaneous τ_C-approximation to F from G. This implies that $g_0 \notin \mathrm{P}^C_{G \cap \mathrm{U}(g_0, \frac{1}{n})}(F_{\frac{1}{n}})$ for each $n \in \mathbb{N}$, and therefore there exists $g_n \in G \cap \mathrm{U}\left(g_0, \frac{1}{n}\right)$ such that $p_C(g_n - F) < p_C(g_0 - F)$. It follows that $\lim_{n \to \infty} g_n = g_0$ and

$$p_C(g_0 - F) > \operatorname{Re} x^*(g_n) - V_F^-(x^*), \quad \forall\, x^* \in A, \quad \forall\, n \in \mathbb{N},$$

thanks to Lemma 1 and (2.4). This implies that (3.7) holds and g_0 is a simultaneous τ_C-regular point of G with respect to F. The proof is complete. ∎

The global version of Theorem 1 is as follows.

Corollary 2. *The following statements are equivalent.*

(*i*) *G is a simultaneous τ_C-regular set.*

(*ii*) (3.16) *holds for each $g_0 \in G$ and each bounded set F in X.*

(*iii*) *G is a simultaneous τ_C-sun.*

4. Uniqueness of Best Simultaneous τ_C-Approximations

This section is devoted to the study of the uniqueness problems of best simultaneous τ_C-approximations. For this purpose, we need to extend in the following definition the notions of the strictly convex subsets with respect to a subset G. Note that, in the case when C is the unit ball, these notions were introduced by Amir and Ziegler in [1] for the case of linear subspaces G, and by Li in [9] for the case of general subsets G.

Definition 4. Let G be a nonempty subset of X. The convex set C is said to be (a) strictly convex with respect to G if for each pair of distinct elements $x, y \in X$,

$$x,\, y \in C,\ x - y \in \operatorname{cone}(G - G) \Rightarrow \frac{x + y}{2} \in \operatorname{int} C.$$

(b) uniformly convex in every direction in G if, for any sequences $\{x_n\}, \{y_n\} \subset C$ satisfying $\lim_{n \to \infty} p_C(x_n + y_n) = 2$ and $x_n - y_n = \lambda_n z$ for some element $z \in G - G$ and some sequence $\{\lambda_n\} \subset \mathbb{R}$, we have that $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Clearly, by definitions, one sees that, for any nonempty subset G of X, a uniformly convex set with respect to every direction in G is strictly convex with respect to G.

Theorems 2 and 3 below show that the strict convexity of C with respect to G and the uniform convexity of C with respect to every direction in G are sufficient conditions ensuring the unicity of the best simultaneous τ_C-approximation to compact subsets F and to bounded subsets F, respectively.

Theorem 2. *Let G be a simultaneous τ_C-sun. Consider the following assertions:*

(i) *C is uniformly convex in every direction in G.*

(ii) *For each bounded subset F of X, $\mathrm{P}_G^C(F)$ contains at most an element.*

(iii) *For each pair of elements $x, y \in X$, $\mathrm{P}_G^C(\{x, y\})$ contains at most an element.*

Then (i)⇒(ii)⇒(iii). *If C is in addition, symmetric, then assertions* (i)-(iii) *are equivalent.*

Proof. (ii)⇒(iii) is trivial. Below we prove (i)⇒(ii). Suppose that (i) holds and, on the contrary, that there exists a bounded subset F of X such that $\mathrm{P}_G^C(F)$ contains two points g_1, g_2. Then

$$p_C(g_1 - F) = p_C(g_2 - F) = \tau_C(F; G). \tag{4.1}$$

Since $g_1 \in \mathrm{P}_G^C(F)$ and g_1 is a simultaneous τ_C-solar point of G, one has that $g_1 \in \mathrm{P}_G^C(g_1 + 2(F - g_1))$. This and (4.1) imply that

$$\begin{aligned} 2\tau_C(F; G) &= 2p_C(g_1 - F) \le p_C(g_2 - g_1 - 2(F - g_1)) \\ &\le p_C(g_1 - F) + p_C(g_2 - F) = 2\tau_C(F; G); \end{aligned}$$

hence

$$p_C(g_1 + g_2 - 2F) = 2\tau_C(F; G). \tag{4.2}$$

Take a sequence $\{x_n\} \subset F$ such that $\lim_{n\to\infty} p_C(g_1+g_2-2x_n) = p_C(g_1+g_2-2F)$. This together with (4.1) and (4.2) implies that

$$\begin{aligned} 2\tau_C(F; G) &= \lim_{n\to\infty} p_C(g_1 + g_2 - 2x_n) \\ &\le \liminf_{n\to\infty} [p_C(g_1 - x_n) + p_C(g_2 - x_n)] \\ &\le \limsup_{n\to\infty} [p_C(g_1 - x_n) + p_C(g_2 - x_n)] \\ &\le \limsup_{n\to\infty} p_C(g_1 - x_n) + \limsup_{n\to\infty} p_C(g_2 - x_n) \\ &\le p_C(g_1 - F) + p_C(g_2 - F) \\ &= 2\tau_C(F; G). \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} p_C(g_1 + g_2 - 2x_n) = 2\tau_C(F;G). \tag{4.3}$$

Let $\bar{x}_n := \frac{g_1 - x_n}{\tau_C(F;G)}$ and $\bar{y}_n := \frac{g_2 - x_n}{\tau_C(F;G)}$ for each $n \in \mathbb{N}$. Then $p_C(\bar{x}_n) \leq 1$ and $p_C(\bar{y}_n) \leq 1$ for each $n \in \mathbb{N}$ by (4.1). This with Proposition 1(iv) implies that $\{\bar{x}_n\}, \{\bar{x}_n\} \subset C$. Note that $\lim_{n\to\infty} p_C(\bar{x} + \bar{y}_n) = 2$ by (4.3) and that $\bar{x}_n - \bar{y}_n = \frac{g_1 - g_2}{\tau_C(F;G)}$. One has that $g_1 - g_2 = 0$ by the uniformly convexity of C in every direction in G, and hence $g_1 = g_2$, which is a contradiction. The shows (i)⇒(ii).

Now we assume that C is additionally symmetric. Then p_C is a norm equivalent to the original one of X and implication (iii)⇒(i) is exactly the necessity part of [17, Theorem 3.5, p.267-269]. However, for completeness, we include the proof for this implication here. To do this, suppose that (i) does not hold, that is, C is not uniformly convex in every direction in G. Then there exist nonzero element $z := g_1 - g_2 \in G - G$ with $g_1, g_2 \in G$, sequences $\{x_n\}, \{y_n\} \subset C$, and a sequence $\{\lambda_n\}$ satisfying

$$\lim_{n\to\infty} p_C(x_n + y_n) = 2 \quad \text{and} \quad \inf_{n\geq 1} \lambda_n = \lambda > 0 \tag{4.4}$$

and

$$x_n - y_n = \lambda_n(g_1 - g_2) \quad \text{for each } n \in \mathbb{N}. \tag{4.5}$$

Let $g_0 := \frac{1}{2}(g_1 + g_2)$ and $u_n := \frac{1}{2\lambda}(x_n + y_n)$ for each $n \in \mathbb{N}$. Then

$$g_1 - (g_0 \pm u_n) = \left(\frac{1}{2\lambda_n} \mp \frac{1}{2\lambda}\right) x_n - \left(\frac{1}{2\lambda_n} \pm \frac{1}{2\lambda}\right) y_n. \tag{4.6}$$

Consider the bounded set F defined by

$$F := \{g_0 \pm u_n : \ n \in \mathbb{N}\},$$

Then, by (4.6), we have that

$$p_C(g_1 - (g_0 \pm u_n)) \leq \left|\frac{1}{2\lambda_n} \mp \frac{1}{2\lambda}\right| + \left|\frac{1}{2\lambda_n} \pm \frac{1}{2\lambda}\right| = \frac{1}{\lambda}$$

(noting that each $\lambda_n \geq \lambda$ by (4.4)). This together with the definition of F implies that $p_C(g_1 - F) \leq \frac{1}{\lambda}$. Similarly, we also have that $p_C(g_2 - F) \leq \frac{1}{\lambda}$. Below we prove that $\tau_C(F;G) = \frac{1}{\lambda}$, which is equivalent to

$$\tau_C(F;G) \geq \frac{1}{\lambda}. \tag{4.7}$$

Granting this, one sees that $\{g_1, g_2\} \subset \mathrm{P}_G^C(F)$ and so (iii) does not hold. Therefore, implication (iii)⇒(i) is proved.

To show (4.7), let $n \in \mathbb{N}$ and $g \in G$. Then

$$p_C(g-(g_0+u_n)) = p_C(-2u_n+(g-g_0+u_n)) \geq 2p_C(u_n) - p_C(g-(g_0-u_n)),$$

that is,

$$p_C(g-(g_0+u_n)) + p_C(g-(g_0-u_n)) \geq 2p_C(u_n).$$

This implies that

$$\max\{p_C(g-(g_0+u_n)), p_C(g-(g_0-u_n))\} \geq p_C(u_n).$$

Recalling the definitions of F and $p_C(g-F)$, we see that

$$p_C(g-F) \geq \limsup_{n\to\infty} \max\{p_C(g-(g_0+u_n)), p_C(g-(g_0-u_n))\} \geq \lim_{n\to\infty} p_C(u_n) = \frac{1}{\lambda},$$

where the last equality is due to (4.4). Hence (4.7) is showed and the proof is complete. ■

The proof of the following theorem is similar but simpler to the one for the above theorem and so we omit it here.

Theorem 3. *Let G be a simultaneous τ_C-sun. Consider the following assertions:*

(*i*) *C is strictly convex with respect to G.*

(*ii*) *For each compact subset F of X, $\mathrm{P}_G^C(F)$ contains at most an element.*

(*iii*) *For each pair of elements $x, y \in X$, $\mathrm{P}_G^C(\{x, y\})$ contains at most an element.*

Then (i)⇒(ii)⇒(iii). *If C is in addition, symmetric, then assertions* (i)-(iii) *are equivalent.*

The following example shows that the condition that C is symmetric can not be dropped for the equivalence of assertions (i)-(iii) in Theorems 2 and 3.

Example 2. Let $X := \mathbb{R}^2$ be the 2-dimensional Euclidean space. Consider the convex subset C to be the equilateral triangle with vertexes $A(-\sqrt{3}, -1)$, $B(\sqrt{3}, -1)$ and $C(0, 2)$; see Figure 1. Then $0 \in \mathrm{int}C$. Let $G := X$. Clearly, C is not strictly convex with respect to G.

Below we show that the best simultaneous τ_C-approximation to any two elements of X from X is unique. To do this, let $F := \{x, y\} \subset X$ with $x \neq y$ and let $g_0 \in \mathrm{P}_G^C(F)$. Without loss of generality, we assume that $g_0 = 0$ and $\tau_C(F; X) = 1$. Then

$$1 = \max\{p_C(-x), p_C(-y)\} \leq \max\{p_C(g-x), p_C(g-y)\}, \quad \forall\, g \in X. \tag{4.8}$$

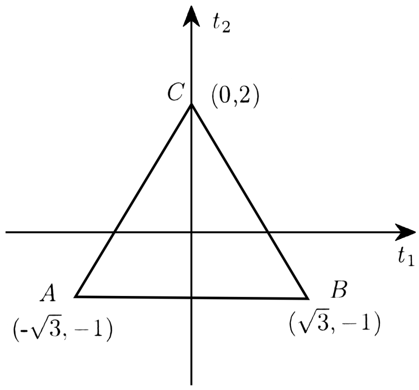

Fig. 1. The closed convex set C.

We assert that $p_C(-x) = p_C(-y) = 1$. Indeed, otherwise, we may assume that $p_C(-y) < p_C(-x) = 1$. Take $g := t(x-y)$ with $t \in \left(0, \frac{1-p_C(-y)}{p_C(y-x)}\right)$. Then

$$p_C(g-x) \leq 1-t(1-p_C(-y)) < 1 \quad \text{and} \quad p_C(g-y) \leq p_C(-y)+tp_C(y-x) < 1,$$

which contradicts (4.8) and the assertion is proved. To proceed, we express the Minkowski function p_C as:

$$p_C(x) = \begin{cases} \frac{1}{2}(\sqrt{3}|t_1| + t_2), & t_2 \geq -\dfrac{\sqrt{3}}{3}|t_1|, \\ -t_2, & t_2 \leq -\dfrac{\sqrt{3}}{3}|t_1|, \end{cases} \qquad \forall x = (t_1, t_2) \in X. \tag{4.9}$$

We claim that one of the two points $-x$ and $-y$ must be a vertex of the equilateral triangle above, while the other must lie on the edge opposite to this vertex. To show this claim, we suppose on the contrary that it is not the case. Then, without loss of generality, we may assume that $-x \in [A, C)$ and $-y \in (A, C)$, or $-x \in (A, C)$ and $-y \in (B, C)$. For the former case, we express $-x = (s_1, \sqrt{3}s_1 + 2)$ with $s_1 \in [-\sqrt{3}, 0)$ and $-y = (s_2, \sqrt{3}s_2 + 2)$ with $s_2 \in (-\sqrt{3}, 0)$ and $s_2 > s_1$. Let $a \in (0, 1)$ be such that $s_2 + \sqrt{3}a < 0$ and let $g_1 = (\sqrt{3}a, a)$. Then

$$g_1 - x = (t_1, t_2) := (\sqrt{3}a + s_1, a + \sqrt{3}s_1 + 2).$$

Since $t_1 < \sqrt{3}a + s_2 < 0$ and $t_2 \geq \frac{\sqrt{3}}{3}t_1 = -\frac{\sqrt{3}}{3}|t_1|$, it follows from (4.9) that

$$p_C(g_1 - x) = \frac{1}{2}[-\sqrt{3}(\sqrt{3}a + s_1) + (a + \sqrt{3}s_1 + 2)] = 1 - a < 1. \tag{4.10}$$

Similarly, $p_C(g_1 - y) < 1 - a < 1$. This with (4.10) is in contradiction to (4.8). For the latter case, we assume that $-x := (s_1, \sqrt{3}s_1 + 2)$ with $s_1 \in (-\sqrt{3}, 0)$ and

$-y := (s_2, 2-\sqrt{3}s_2)$ with $s_2 \in (0, \sqrt{3})$. Noting that $\min\{\sqrt{3}s_1+2,\ 2-\sqrt{3}s_2\} > -1$, we can take $a \in (0,1)$ such that $\min\{\sqrt{3}s_1+2-a,\ 2-\sqrt{3}s_2-a\} > -1$ and let $g_2 := (0, -a)$. Then, one can check similarly that $\max\{p_C(g_2-x), p_C(g_2-y)\} < 1$, which again contradicts (4.8). Therefore, the claim holds.

Thus, without loss of generality, we may assume that $-x := A = (-\sqrt{3}, -1)$ and $-y := (s_1, -\sqrt{3}s_1+2) \in [B, C]$ for some $s_1 \in [0, \sqrt{3}]$. Let $g := (v_1, v_2) \neq 0$. To complete the proof, it suffices to show that

$$\max\{p_C(g-x), p_C(g-y)\} > 1. \tag{4.11}$$

Below we divide our consideration into two cases: (a): $v_2 > -\frac{\sqrt{3}}{3}|v_1|$ and (b): $v_2 \le -\frac{\sqrt{3}}{3}|v_1|$.

(a) Let $v_2 > -\frac{\sqrt{3}}{3}|v_1|$. Then

$$g-y=(t_1,t_2) := (v_1+s_1, v_2-\sqrt{3}s_1+2) \quad \text{and} \quad g-x=(\bar{t}_1,\bar{t}_2) := (v_1-\sqrt{3}, v_2-1).$$

If $v_1 \ge 0$, then $t_1 \ge 0$ and

$$t_2 = v_2 - \sqrt{3}s_1 + 2 > -\frac{\sqrt{3}}{3}v_1 - \sqrt{3}s_1 + 2 \ge -\frac{\sqrt{3}}{3}(v_1+s_1) = -\frac{\sqrt{3}}{3}|t_1|$$

(as $s_1 \in [0, \sqrt{3}]$), which together with (4.9) implies that

$$\begin{aligned} p_C(g-y) &= \frac{1}{2}(\sqrt{3}|t_1| + t_2) \\ &> \frac{1}{2}\left[\sqrt{3}(v_1+s_1) + \left(-\frac{\sqrt{3}}{3}v_1 - \sqrt{3}s_1 + 2\right)\right] \\ &= 1 + \frac{\sqrt{3}}{3}v_1 \ge 1. \end{aligned} \tag{4.12}$$

If $v_1 < 0$, then $\bar{t}_1 = v_1 - \sqrt{3} \le -\sqrt{3} < 0$ and

$$\bar{t}_2 = v_2 - 1 > \frac{\sqrt{3}}{3}v_1 - 1 = \frac{\sqrt{3}}{3}\bar{t}_1,$$

which together with (4.9) implies that $p_C(g-x) = \frac{1}{2}(-\sqrt{3}\bar{t}_1 + \bar{t}_2) > -\frac{\sqrt{3}}{3}\bar{t}_1 > 1$. Combining this estimate and estimate (4.12) gives that assertion (4.11).

(b) Let $v_2 \le -\frac{\sqrt{3}}{3}|v_1|$. Then $g - x = (t_1, t_2) := (v_1 - \sqrt{3}, v_2 - 1)$ and

$$t_2 = v_2 - 1 \le -\frac{\sqrt{3}}{3}|v_1| - 1 \le -\left|\frac{\sqrt{3}}{3}v_1 - 1\right| = -\frac{\sqrt{3}}{3}|t_1|.$$

It follows from (4.9) that

$$\max\{p_C(g-x), p_C(g-y)\} \ge p_C(g-x) = -t_2 = 1 - v_2 > 1$$

because $v_2 < 0$ (noting that $g \neq 0$) and (4.11) is also proved in this case.

The following theorem gives a complete characterization for C to be strictly convex with respect to G in terms of the uniqueness of the best simultaneous τ_C-approximation from G, which seems new even for the case when C is the closed unit ball.

Theorem 4. *Let G be a convex subset of X. Then the following statements are equivalent.*

(i) *C is strictly convex with respect to G.*

(ii) *For each simultaneous τ_C-sun $G_1 \subset G$ and each compact subset F of X, $\mathrm{P}^C_{G_1}(F)$ contains at most an element.*

(iii) *For each simultaneous τ_C-sun $G_1 \subset G$ and each $x, y \in X$, $\mathrm{P}^C_{G_1}(\{x, y\})$ contains at most an element.*

(iv) *For each pair of elements $g_1, g_2 \in G$ with $g_1 \neq g_2$ and each $x, y \in X$, $\mathrm{P}^C_{[g_1,g_2]}(\{x, y\})$ is a singleton.*

(v) *For each pair of elements $g_1, g_2 \in G$ with $g_1 \neq g_2$ and each $x \in X$, $\mathrm{P}^C_{[g_1,g_2]}(x)$ is a singleton.*

Proof. Note that if C is strictly convex with respect to G then it is strictly convex with respect to each subset of G. Thus implication (i)⇒(ii) follows from Theorem 3. Implications (ii)⇒(iii)⇒(iv)⇒(v) hold trivially. Below we prove (v)⇒(i). To do this, we suppose that (v) holds and, on the contrary, that C is not strictly convex with respect to G. Then there exist two distinct elements $x_1, y_1 \in C$ such that

$$x_1 - y_1 \in \mathrm{cone}(G - G) \quad \text{and} \quad \frac{1}{2}(x_1 + y_1) \in \mathrm{bd}C. \tag{4.13}$$

It follows from Proposition 1(iii) that $p_C(x_1) = p_C(y_1) = p_C\left(\frac{x_1+y_1}{2}\right) = 1$. We further assert that

$$p_C(tx_1 + (1-t)y_1) = 1, \quad \forall\, t \in [0, 1]. \tag{4.14}$$

Indeed, otherwise, without loss of generality, we may assume that there is $t_0 \in \left(0, \frac{1}{2}\right)$ such that $p_C(t_0x_1+(1-t_0)y_1) < 1$. Write $z_0 := t_0x_1+(1-t_0)y_1$ and $\lambda_0 := \frac{1-2t_0}{2(1-t_0)}$. Then $\lambda_0 \in (0, 1)$ and $\frac{x+y}{2} = \lambda_0 x_1+(1-\lambda_0)z_0$. This together with Proposition 1(iii) implies that $p_C\left(\frac{x_1+y_1}{2}\right) \le \lambda_0 p_C(x_1) + (1-\lambda_0)p_C(z_0) < 1$, which is a contradicts. Thus assertion (4.14) is proved. By (4.13), there exist $\bar{g}_1, \bar{g}_2 \in G$ and $\lambda > 0$ such that

$$x_1 - y_1 = \lambda(\bar{g}_1 - \bar{g}_2). \tag{4.15}$$

We may assume, without loss of generality (if necessary, one can use $z_1 := \left(1 - \frac{1}{\lambda}\right) x_1 + \frac{1}{\lambda}y_1$ in place of y_1), that $0 < \lambda \le 1$. This implies that $\lambda(\bar{g}_1 - \bar{g}_2) \in G - G$

and so $\lambda(\bar{g}_1 - \bar{g}_2) = g_1 - g_2$ for some $g_1, g_2 \in G$. Consider $x := g_2 - y_1$ and $y := g_2 + \frac{1}{2}(x_1 - 3y_1)$. Then

$$tg_1 + (1-t)g_2 - x = tx_1 + (1-t)y_1, \quad \forall\, t \in [0,1],$$

since $x_1 - y_1 = g_1 - g_2$ by (4.15). This together with (4.14) implies that

$$p_C(tg_1 + (1-t)g_2 - x) = p_C(tx_1 + (1-t)y_1) = 1, \quad \forall\, t \in [0,1]. \tag{4.16}$$

This means that $\tau_C(x; [g_1, g_2]) = 1$ and $[g_1, g_2] \subset \mathrm{P}^C_{[g_1,g_2]}(x)$. This contradicts (v) and we complete the proof. ■

Equivalence between (i) and (iv) in the following corollary extends [1, Corollary 1.5] which deals with the case when C is the closed unit ball.

Corollary 3. *Let G be a subspace of X. Then the following statements are equivalent.*

(i) *C is strictly convex with respect to G.*

(ii) *For each real 1-dimensional subspace G_1 of G and each compact subset F of X, $\mathrm{P}^C_{G_1}(F)$ is a singleton.*

(iii) *For each real 1-dimensional subspace G_1 of G and each $x, y \in X$, $\mathrm{P}^C_{G_1}(\{x, y\})$ is a singleton.*

(iv) *For each real 1-dimensional subspace G_1 of G and each $x \in X$, $\mathrm{P}^C_{G_1}(x)$ is a singleton.*

Proof. The implication (i)⇒(ii) follows from Theorem 4; while implications (ii)⇒(iii)⇒(iv) are trivial. Thus we only need to prove (iv)⇒(i). By Theorem 4, it suffices to show that condition (iv) here implies the condition (v) of Theorem 4. To do this, let $x \in X$, and, we, without loss of generality, consider $[0, g] \subset G$ with $g \neq 0$. Suppose that $\mathrm{P}^C_{[0,g]}(x)$ is not a singleton. Then we may assume that $\mathrm{P}^C_{[0,g]}(x) = [0, \bar{t}g]$ for some $\bar{t} \in (0, 1]$. Take $t_0 \in (0, \bar{t})$. Since $t_0 g \in \mathrm{P}^C_{[0,g]}(x)$, it follows from Theorem 1(ii) that $\max_{x^* \in M_{t_0 g - x}} \mathrm{Re}\, x^*(tg - t_0 g) \geq 0$ for each $t \in [0, 1]$ and so for all $t \in \mathbb{R}$. Hence, we have that $t_0 g \in \mathrm{P}^C_{G_1}(x)$ by Theorem 1(ii), where $G_1 := \mathrm{span}\{g\}$ is a real 1-dimensional subspace of G. This implies that

$$\tau_C(x; [0, g]) = \tau_C(x; G_1) = p_C(t_0 g - x).$$

Therefore, $[0, \bar{t}g] = \mathrm{P}^C_{[0,g]}(x) \subset \mathrm{P}^C_{G_1}(x)$, and (iv) does not hold. Thus implication (iv)⇒[(v), Theorem 4] is proved and the proof is complete. ■

REFERENCES

1. D. Amir and Z. Ziegler, Relative Chebyshev centers in normed linear spaces, *J. Approx. Theory*, **29** (1980), 235-252.

2. D. Braess, *Nonlinear Approximation Theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1986.

3. B. Brosowski and F. Deutsch, On some geometric properties of suns, *J. Approx. Theory*, **10** (1974), 245-267.

4. B. Brosowski and R. Wegmann, Charakterisierung bester approximationen in normierten Vektorräumen, *J. Approx. Theory*, **3** (1970), 369-397.

5. F. S. De Blasi and J. Myjak, On a generalized best approximation problem, *J. Approx. Theory*, **94** (1998), 54-72.

6. N. V. Efimov and S. B. Stechkin, *Some properties of Chebyshev sets*, Dokl. Akad. Nauk SSSR, 118, 1958, pp. 17-19.

7. J. H. Freilich and H. W. Mclaughlin, Approximation of bounded sets, *J. Approx. Theory*, **34** (1982), 146-158.

8. R. B. Holmes, *Geometrical Functional Analysis and its Applications*, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

9. C. Li, Characterization and unicity for best simultaneous approximations, *J. Hangzhou Uni.*, **21** (1994), 365-373.

10. C. Li and X. H. Wang, Almost Chebyshev set with respect to bounded subsets, *Science in China Series A: Math.*, **40** (1997), 375-383.

11. C. Li, On well posed generalized best approximation problems, *J. Approx. Theory*, **107** (2000), 96-108.

12. C. Li, Strong uniqueness of the restricted Chebyshev center with respect to an RS-set in a Banach space, *J. Approx. Theory*, **135** (2005), 35-53.

13. C. Li and R. X. Ni, Derivatives of generalized distance functions and existence of generalized nearest points, *J. Approx. Theory*, **115** (2002), 44-55.

14. X. F. Luo, C. Li and J. C. Yao, Anisotropic best τ_C-approximation in normed spaces, to appear in *Optimization*.

15. R. R. Phelps, *Convex Functions, Monotonic Operators and Differentiability*, 2nd edition, Lecture Notes in Math. 1364, Spring-Verlag, Berlin, Heidelberg, New York, 1993.

16. S. Y. Xu and C. Li, Characterization of best simultaneous approximation, *Acta Math. Sinica*, **30** (1987), 528-535, in Chinese; *Approx. Theory Appl.*, **3** (1987), 190-198.

17. S. Y. Xu, C. Li and W. S. Yang, *Nonlinear Approximation Theory in Banach Space*, Science Press, Beijing, 1997, in Chinese.

X. F. Luo
Department of Mathematics
China Jiliang University
Hangzhou 310018
P. R. China
E-mail: luoxianfaseu@163.com

L. H. Peng
Department of Mathematics
Zhejiang Gongshang University
Hangzhou 310018
P. R. China
Email: penglihui790124@sohu.com

C. Li
Department of Mathematics
Zhejiang University
Hangzhou 310027
P. R. China
and
Department of Mathematics
College of Sciences
King Saud University
P. O. Box 2455
Riyadh 11451
Saudi Arabia
E-mail: cli@zju.edu.cn

N. C. Wong
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung 804, Taiwan
E-mail: wong@math.nsysu.edu.tw