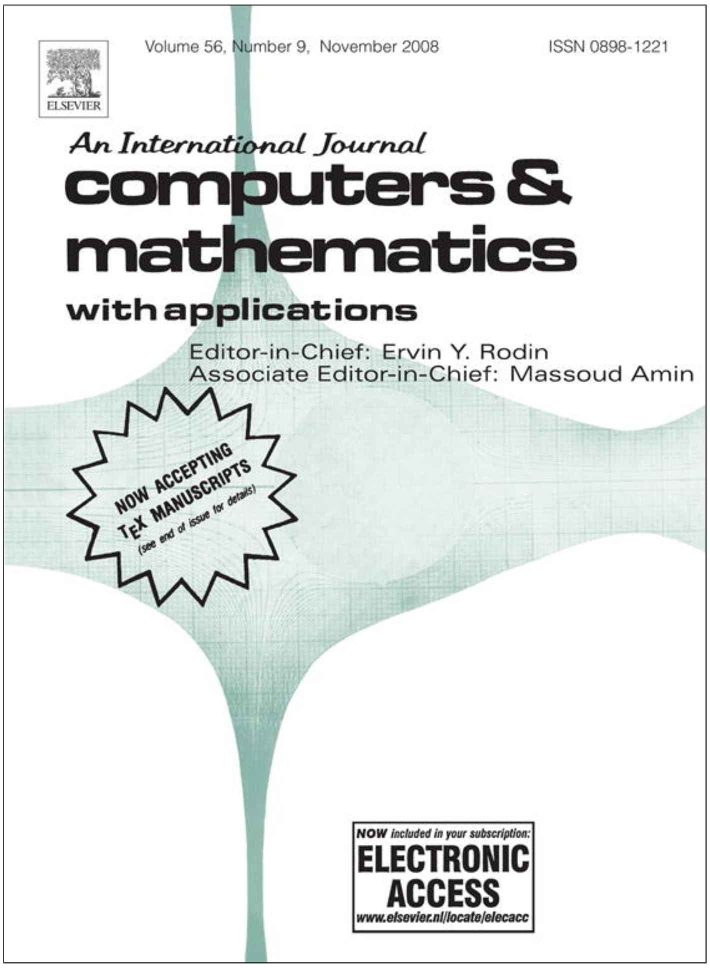
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Fixed point solutions of variational inequalities for a finite family of asymptotically nonexpansive mappings without common fixed point assumption

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ABSTRACT

Let E be a real Banach space with a uniformly Gâteaux differentiable norm and which possesses uniform normal structure, K a nonempty bounded closed convex subset of E, $\{T_i\}_{i=1}^N$ a finite family of asymptotically nonexpansive self-mappings on K with common sequence $\{k_n\}_{n=1}^\infty \subset [1, \infty)$, $\{t_n\}$, $\{s_n\}$ be two sequences in $(0, 1)$ such that $s_n + t_n = 1$ $(n \geq 1)$ and f be a contraction on K. Under suitable conditions on the sequences $\{s_n\}$, $\{t_n\}$, we show the existence of a sequence $\{x_n\}$ satisfying the relation $x_n = (1-\frac{1}{k_n})x_n + \frac{s_n}{k_n} f(x_n) + \frac{t_n}{k_n} T_{r_n}^n x_n$ where $n = l_n N + r_n$ for some unique integers $l_n \geq 0$ and $1 \leq r_n \leq N$. Further we prove that $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$, which solves some variational inequality, provided $\|x_n - T_i x_n\| \to 0$ as $n \to \infty$ for $i = 1, 2, \ldots, N$. As an application, we prove that the iterative process defined by $z_0 \in K$, $z_{n+1} = (1 - \frac{1}{k_n})z_n + \frac{s_n}{k_n} f(z_n) + \frac{t_n}{k_n} T_{r_n}^n z_n$, converges strongly to the same common fixed point of $\{T_i\}_{i=1}^N$.

1. Introduction

Throughout this paper, we assume that E is a real Banach space with dual E^* and K a nonempty closed convex subset of E. Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in E.$$

A mapping $f : K \to K$ is called a contraction if there exists a constant $\alpha \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\| \forall x, y \in K$. A mapping $T : K \to K$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\| \forall x, y \in K$, and is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $n \geq 1$ and all $x, y \in K$. It is easy to see that every contraction is nonexpansive, and every nonexpansive mapping is asymptotically nonexpansive. The converse is not valid. In 1972, Goebel and Kirk [1] proved that if the space E is assumed to be uniformly convex, then every asymptotically nonexpansive self-mapping T on a bounded closed convex subset $K \subset E$ has a fixed point. Subsequently, Lim and Xu [2] also proved another existence result which is similar to the existence theorem in [1]. Moreover, Lim and Xu [2] introduced an implicit iterative scheme as follows:

Suppose that K is a bounded closed convex subset of a Banach space E and $T : K \to K$ is an asymptotically nonexpansive mapping. Fix a u in K and define for each $n \geq 1$ the contraction $S_n : K \to K$ by

$$S_n(x) = \left(1 - \frac{t_n}{k_n}\right) u + \frac{t_n}{k_n} T^n x, \tag{1}$$

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where $\{t_n\} \subset [0,1)$ is any sequence such that $t_n \to 1$ as $n \to \infty$. Then the Banach Contraction Principle yields a unique point x_n fixed by S_n. Now the question gives rise to whether the sequence $\{x_n\}$ converges strongly to a fixed point of T. The following is a partial answer.

Theorem 1.1 ([2]). *Suppose E is uniformly smooth and $\{t_n\}$ is chosen so that*

$$\lim_{n\to\infty}(k_n-1)/(k_n-t_n)=0.$$

(Such a sequence $\{t_n\}$ always exists, for example, take $t_n=\min\{1-(k_n-1)^{1/2}, 1-n^{-1}\}$.) Suppose in addition the condition $\lim_{n\to\infty}\|x_n-Tx_n\|=0$ holds. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

On the other hand, Moudafi [3] proposed a viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces. If H is a Hilbert space, $T : K \to K$ is a nonexpansive self-mapping of a nonempty closed convex subset K of H, and $f : K \to K$ is a contraction, he proved the strong convergence of both the implicit and explicit methods:

$$x_n=\frac{1}{1+\varepsilon_n}Tx_n+\frac{\varepsilon_n}{1+\varepsilon_n}f(x_n), \tag{2}$$

and

$$x_{n+1}=\frac{1}{1+\varepsilon_n}Tx_n+\frac{\varepsilon_n}{1+\varepsilon_n}f(x_n), \tag{3}$$

where $\lim_{n\to\infty}\varepsilon_n=0$. Motivated by Moudafi [3], Xu [4] studied the viscosity approximation methods for a nonexpansive mapping in a uniformly smooth Banach space. For a contraction f on K and $t\in(0,1)$, let $x_t\in K$ be the unique fixed point of the contraction $x\mapsto tf(x)+(1-t)Tx$. Consider also the iteration process $\{x_n\}$, where $x_0\in K$ is arbitrary and $x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)Tx_n$ for $n\geq 1$, where $\{\alpha_n\}\subset(0,1)$. Xu [4] proved that $\{x_t\}$ and, under certain appropriate conditions on $\{\alpha_n\}$, $\{x_n\}$ converge strongly to a fixed point of T which solves some variational inequality.

Very recently, the viscosity approximation methods are extended by Shahzad and Udomene [5] to develop new iterative schemes for an asymptotically nonexpansive mapping. They proved the following theorems.

Theorem 1.2 ([5, Theorem 3.1]). *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K a nonempty closed convex and bounded subset of E, $T : K \to K$ an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1,\infty)$ and $f : K \to K$ a contraction with constant $\alpha \in [0,1)$. Let $\{t_n\}_n \subset (0, \frac{(1-\alpha)k_n}{k_n-\alpha})$ be such that $\lim_{n\to\infty}t_n=1$ and $\lim_{n\to\infty}\frac{k_n-1}{k_n-t_n}=0$. Then,*

(i) *for each $n\geq 0$, there is a unique $x_n\in K$ such that*

$$x_n=\left(1-\frac{t_n}{k_n}\right)f(x_n)+\frac{t_n}{k_n}T^nx_n; \tag{4}$$

and, if in addition, $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, then,

(ii) *the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality:*

$$p\in F(T) \text{ such that } \langle (I-f)p, j(p-x^*)\rangle \leq 0 \quad \forall x^*\in F(T).$$

Theorem 1.3 ([5, Theorem 3.3]). *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of E, $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1,\infty)$ and $f : K\to K$ be a contraction with constant $\alpha\in[0,1)$. Let $\{t_n\}_n\subset(0,\xi_n)$ be such that $\lim_{n\to\infty}t_n=1$, $\sum_{n=1}^{\infty}t_n(1-t_n)=\infty$ and $\lim_{n\to\infty}\frac{k_n-1}{k_n-t_n}=0$, where $\xi_n=\min\{\frac{(1-\alpha)k_n}{k_n-\alpha},\frac{1}{k_n}\}$. For an arbitrary $z_0\in K$ let the sequence $\{z_n\}_n$ be iteratively defined by*

$$z_{n+1}:=\left(1-\frac{t_n}{k_n}\right)f(z_n)+\frac{t_n}{k_n}T^nz_n, \quad n\geq 1. \tag{5}$$

Then,

(i) *for each $n\geq 0$, there is a unique $x_n\in K$ such that*

$$x_n=\left(1-\frac{t_n}{k_n}\right)f(x_n)+\frac{t_n}{k_n}T^nx_n;$$

and, if in addition, $\lim_{n\to\infty}\|x_n-Tx_n\|=0$ and $\lim_{n\to\infty}\|z_n-Tz_n\|=0$, then

(ii) *the sequence $\{z_n\}_n$ converges strongly to the unique solution of the variational inequality:*

$$p\in F(T) \text{ such that } \langle (I-f)p, j(p-x^*)\rangle \leq 0 \quad \forall x^*\in F(T).$$

Furthermore, Chang et al. [6] studied the weak and strong convergence of implicit iteration process

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{r_n}^{l_n+1} x_n, \quad n \geq 1, \tag{6}$$

for a finite family $\{T_i\}_{i=1}^N$ of asymptotically nonexpansive self-mappings on a nonempty closed convex subset K of a uniformly convex Banach space satisfying Opial condition, where $n = l_n N + r_n$ for some unique integers $l_n \geq 0$ and $1 \leq r_n \leq N$. In the proof of the main results of Chang et al. [6], the following proposition is crucial.

Proposition 1.1 *([6, Proposition 1]). Let K be a nonempty subset of E, and $\{T_i\}_{i=1}^N$ be N asymptotically nonexpansive self-mappings on K. Then,*

(i) *there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that*

$$\|T_i^n x - T_i^n y\| \leq k_n \|x - y\| \quad \forall n \geq 1, x, y \in K, i = 1, 2, \ldots, N; \tag{7}$$

(ii) *$\{T_i^n\}_{i=1}^N$ is uniformly Lipschitzian with a Lipschitzian constant $L \geq 1$, i.e., there exists a constant $L \geq 1$ such that*

$$\|T_i^n x - T_i^n y\| \leq L \|x - y\| \quad \forall n \geq 1, x, y \in K, i = 1, 2, \ldots, N. \tag{8}$$

We call the sequence $\{k_n\}_n$ a common sequence of a finite family $\{T_i\}_{i=1}^N$ of asymptotically nonexpansive self-mappings. Meantime, the authors [7] introduced and studied the implicit iteration scheme with perturbed mappings for common fixed points of a finite family of nonexpansive mappings, as a special case of asymptotically nonexpansive mappings, in a Hilbert space.

The main aim of this paper is to obtain fixed point solutions of variational inequalities for a finite family of asymptotically nonexpansive mappings defined on a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure. We prove, under appropriate conditions on K, T and $\{s_n\}, \{t_n\} \subset (0, 1)$, that the sequence $\{z_n\}$ defined iteratively by: $z_0 \in K$,

$$z_{n+1} := \left(1 - \frac{1}{k_n}\right) z_n + \frac{s_n}{k_n} f(z_n) + \frac{t_n}{k_n} T_{r_n}^n z_n, \tag{9}$$

where $s_n + t_n = 1$, and $n = l_n N + r_n$ for some unique integers $l_n \geq 0$ and $1 \leq r_n \leq N$, converges strongly to the unique solution of the above variational inequality. We remark that Shahzad and Udomene's theorems [5] extend Theorems 4.1 and 4.2 of [4] to the more general class of asymptotically nonexpansive self-mappings and to the much more general class of Banach spaces (see Theorems 1.1 and 1.2) and the corresponding results of [8] (hence of [9]) follow as immediate corollaries of their theorems. Now, our results extend Theorems 3.1 and 3.3 of [5] to new viscosity iterative schemes and to the case of a finite family of asymptotically nonexpansive self-mappings. Therefore our results are the improvements and extension of the corresponding ones in [3–10].

2. Preliminaries

Let E be a Banach space. Let $S_E := \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. Recall that E is said to have a Gâteaux differentiable norm if for each $x \in S_E$ the limit

$$\lim_{t \to 0} (\|x + ty\| - \|x\|)/t \tag{10}$$

exists for all $y \in S_E$, and we call E smooth. In this case, it is known [11] that the normalized duality mapping J on E is single-valued. E is said to have a uniform Gâteaux differentiable norm if for each $y \in S_E$ the limit (10) is attained uniformly for $x \in S_E$. Further, E is said to be uniformly smooth if the limit (10) exists uniformly for $(x, y) \in S_E \times S_E$. It is known [11] that if E has a uniform Gâteaux differentiable norm then the normalized duality mapping J on E is single-valued and norm-to-weak* uniformly continuous on any bounded subset of E.

Let K be a nonempty closed convex and bounded subset of E and let the diameter of K be defined by $d(K) := \sup\{\|x-y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The normal structure coefficient $N(E)$ of E (cf. [12]) is defined by

$$N(E) := \inf \left\{ \frac{d(K)}{r(K)} : K \text{ is a closed convex and bounded subset of } E \text{ with } d(K) > 0 \right\}.$$

A space E such that $N(E) > 1$ is said to have uniform normal structure. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniformly normal structure (see e.g., [2,13]).

Let LIM be a Banach limit. Recall that $\mathrm{LIM} \in (\ell^\infty)^*$ such that $\|\mathrm{LIM}\| = 1$, $\liminf_{n\to\infty} a_n \leq \mathrm{LIM}_n a_n \leq \limsup_{n\to\infty} a_n$, and $\mathrm{LIM}_n a_n = \mathrm{LIM}_n a_{n+1}$ for all $\{a_n\}_n \in \ell^\infty$.

The following lemmas will be needed in what follows. Lemma 2.1 is well known.

Lemma 2.1. *Let E be an arbitrary real Banach space. Then*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle \tag{11}$$

for all $x, y \in E$ and all $j(x+y) \in J(x+y)$.

Lemma 2.2 (*Kim and Xu [4]*). *Let E be a Banach space with uniform normal structure, K a nonempty closed convex and bounded subset of E, and $T : K \to K$ an asymptotically nonexpansive mapping. Then T has a fixed point.*

Lemma 2.3 (*Chidume et al. [8], Xu [4,9]*). *Let $\{a_n\}_n$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\lambda_n)a_n + \lambda_n\gamma_n, \quad n = 0, 1, 2, \dots,$$

where $\{\lambda_n\}_n$ is a sequence in $(0,1)$ and $\{\gamma_n\}_n$ is a sequence in $\mathcal{R}$ such that

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\lambda_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 (*See Lim and Xu [2, Theorem 1]*). *Suppose that E is a Banach space with uniform normal structure, C is a nonempty bounded subset of E, and $T : C \to C$ is an asymptotically nonexpansive mapping with $\{k_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\ge1} k_n < \sqrt{N(E)}$. Suppose also that there exists a nonempty bounded closed convex subset D of C with the following property (P):*

$$x \in D \Rightarrow \omega_w(x) \subset D, \tag{P}$$

where $\omega_w(x)$ is the weak ω-limit set of T at x, i.e., the set

$$\{y \in E : y = \text{weak} - \lim_{j\to\infty} T^{n_j}x \text{ for some } n_j \uparrow \infty\}.$$

Then T has a fixed point in D.

3. Main results

Theorem 3.1. *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of E, $\{T_1, T_2, \dots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings with common sequence $\{k_n\}_n \subset [1,\infty)$ such that $\sup_{n\ge1} k_n < \sqrt{N(E)}$, and let $f : K \to K$ be a contraction with constant $\alpha \in [0,1)$. Let $\{s_n\}, \{t_n\}$ be two sequences in $(0,1)$ such that* (a) $s_n + t_n = 1$ *for all* $n \ge 1$, *and* (b) $\{t_n\}_n \subset (0, \frac{1-\alpha}{k_n-\alpha})$, $\lim_{n\to\infty} t_n = 1$ *and* $\lim_{n\to\infty} \frac{k_n-1}{1-t_n} = 0$. *Then*

(i) *for each $n \ge 1$, there is a unique $x_n \in K$ such that*

$$x_n = \left(1 - \frac{1}{k_n}\right)x_n + \frac{s_n}{k_n}f(x_n) + \frac{t_n}{k_n}T_{r_n}^n x_n \tag{12}$$

where $n = l_nN + r_n$ for some unique integers $l_n \ge 0$ and $1 \le r_n \le N$; and if in addition, $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$, $T_iT_j = T_jT_i$ and $F(T_i)$ is convex for $1 \le i, j \le N$, then

(ii) *the sequence $\{x_n\}_n$ converges strongly to the unique solution of the variational inequality:*

$$p \in F \text{ such that } \langle (I-f)p, j(p-x^*)\rangle \le 0 \quad \forall x^* \in F, \tag{13}$$

where $F = \cap_{i=1}^N F(T_i)$.

Proof. First, using Lemma 2.2 we know that each $F(T_i)$ is nonempty, bounded, and closed for $1 \le i \le N$. By the condition on $\{t_n\}$, for each $n \ge 1$ the mapping $S_n : K \to K$ defined for each $x \in K$ by $S_nx := (1-\frac{1}{k_n})x + \frac{s_n}{k_n}f(x) + \frac{t_n}{k_n}T_{r_n}^n x$ is a contraction. Indeed, observe that for all $x, y \in K$

$$\begin{aligned}
\|S_nx - S_ny\| &\le \left(1-\frac{1}{k_n}\right)\|x-y\| + \frac{s_n}{k_n}\|f(x)-f(y)\| + \frac{t_n}{k_n}\|T_{r_n}^n x - T_{r_n}^n y\| \\
&\le \left(1-\frac{1}{k_n}\right)\|x-y\| + \frac{s_n\alpha}{k_n}\|x-y\| + \frac{t_nk_n}{k_n}\|x-y\| \\
&= \left\{\left(1-\frac{1}{k_n}\right) + \frac{s_n\alpha}{k_n} + t_n\right\}\|x-y\| \\
&= \theta_n\|x-y\|,
\end{aligned}$$

where $\theta_n = (1 - \frac{1}{k_n}) + \frac{s_n \alpha}{k_n} + t_n$. Observe that

$$\theta_n < 1 \Leftrightarrow \left(1 - \frac{1}{k_n}\right) + \frac{s_n \alpha}{k_n} + t_n < 1$$
$$\Leftrightarrow t_n < \frac{1-\alpha}{k_n - \alpha}.$$

It follows that there exists a unique $x_n \in K$ such that $S_n x_n = x_n$. Now we define $\phi : K \to [0, \infty)$ by

$$\phi(z) = \mathrm{LIM}_n \|x_n - z\|^2.$$

Since ϕ is continuous and convex, $\phi(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, ϕ attains its infimum over K. Hence, the set

$$D = \{x \in K : \phi(x) = \min_{z \in K} \phi(z)\}$$

is nonempty, closed and convex.

We claim that for any $l \geq 1$, $\bigcap_{i=1}^{l} F(T_i) \cap D \neq \emptyset$ if $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, $T_i T_j = T_j T_i$ and $F(T_i)$ is convex for $1 \leq i, j \leq l$. Indeed, whenever $l = 1$, we set $T_1 = T$ for convenience. Then in terms of Lemma 2.2 we have $F(T) \neq \emptyset$. We follow the line of argument in Lim and Xu [2, Theorem 2]. Though D is not necessarily invariant under T, it does have the property (P). In fact, if x is in D and $y = w\text{-}\lim_{j\to\infty} T^{m_j} x$ belongs to the weak ω-limit set $\omega_w(x)$ of T at x, then from the w-l.s.c. of ϕ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, we deduce that

$$\begin{aligned}
\phi(y) &\leq \liminf_{j\to\infty} \phi(T^{n_j} x) \leq \limsup_{m\to\infty} \phi(T^m x) \\
&= \limsup_{m\to\infty} (\mathrm{LIM}_n \|x_n - T^m x\|^2) \\
&= \limsup_{m\to\infty} (\mathrm{LIM}_n \|T^m x_n - T^m x\|^2) \\
&\leq \limsup_{m\to\infty} k_m^2 \mathrm{LIM}_n \|x_n - x\|^2 = \mathrm{LIM}_n \|x_n - x\|^2 \\
&= \min_{z \in K} \phi(z).
\end{aligned}$$

This shows that y belongs to D and hence D satisfies the property (P). It follows from Lemma 2.4 that T has a fixed point in D, i.e., $F(T) \cap D \neq \emptyset$.

For $l \geq 1$, assume that $\bigcap_{i=1}^{l} F(T_i) \cap D \neq \emptyset$ whenever $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, $T_i T_j = T_j T_i$ and $F(T_i)$ is convex for $1 \leq i, j \leq l$. Let us show that $\bigcap_{i=1}^{l+1} F(T_i) \cap D \neq \emptyset$ if $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, $T_i T_j = T_j T_i$ and $F(T_i)$ is convex for $1 \leq i, j \leq l+1$. In this case, it is clear that $\bigcap_{i=1}^{l} F(T_i) \cap D$ is nonempty, bounded, closed and convex. Then define a subset W of C as

$$W = \left\{ x \in \bigcap_{i=1}^{l} F(T_i) \cap D : \phi(x) = \min_{z \in \bigcap_{i=1}^{l} F(T_i) \cap D} \phi(z) \right\}.$$

Since ϕ is continuous and convex, $\phi(z) \to \infty$ as $\|z\| \to \infty$ and E is reflexive, ϕ attains its infimum over $\bigcap_{i=1}^{l} F(T_i) \cap D$. Hence the subset W is nonempty, bounded, closed and convex. Now we prove that W has the property (P), i.e., $x \in W \Rightarrow \omega_w(x) \subset W$ where

$$\{y \in X : y = \text{weak} - \lim_{j\to\infty} T_{l+1}^{n_j} x \text{ for some } n_j \uparrow \infty\}.$$

Observe that $T_{l+1}(\bigcap_{i=1}^{l} F(T_i)) \subset \bigcap_{i=1}^{l} F(T_i)$ since $T_i T_{l+1} = T_{l+1} T_i$ for each $i = 1, 2, \ldots, l$, implies that for each $u \in \bigcap_{i=1}^{l} F(T_i)$

$$T_{l+1} u = T_{l+1} T_i u = T_i T_{l+1} u, \quad 1 \leq i \leq l,$$

that is, $T_{l+1} u \in \bigcap_{i=1}^{l} F(T_i)$. Suppose that x is in W and $y = w\text{-}\lim_{j\to\infty} T_{l+1}^{m_j} x$ belongs to the weak ω-limit set $\omega_w(x)$ of T_{l+1} at x. Then $x \in \bigcap_{i=1}^{l} F(T_i) \cap D$ and $\phi(x) = \min_{z \in \bigcap_{i=1}^{l} F(T_i) \cap D} \phi(z)$. From $x \in \bigcap_{i=1}^{l} F(T_i)$ and $T_{l+1} : \bigcap_{i=1}^{l} F(T_i) \to \bigcap_{i=1}^{l} F(T_i)$, we have $\{T_{l+1}^m x\} \subset \bigcap_{i=1}^{l} F(T_i)$. Again from the closedness and convexity of $\bigcap_{i=1}^{l} F(T_i)$, we have $y \in \bigcap_{i=1}^{l} F(T_i)$. Note that from the w-l.s.c. of ϕ and $\lim_{n\to\infty} \|x_n - T_{l+1} x_n\| = 0$, we derive

$$\begin{aligned}
\phi(y) &\leq \liminf_{j\to\infty} \phi(T_{l+1}^{m_j} x) \leq \limsup_{m\to\infty} \phi(T_{l+1}^m x) \\
&= \limsup_{m\to\infty} (\mathrm{LIM}_n \|x_n - T_{l+1}^m x\|^2)
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{m\to\infty}(\mathrm{LIM}_n\|T_{l+1}^m x_n - T_{l+1}^m x\|^2) \\
&\le \limsup_{m\to\infty} k_m^2 \mathrm{LIM}_n\|x_n - x\|^2 = \mathrm{LIM}_n\|x_n - x\|^2 \\
&= \min_{z\in K}\phi(z)
\end{aligned}
$$

due to $x \in D$. This shows that y belongs to D and hence $y \in \bigcap_{i=1}^l F(T_i) \cap D$. Since $x \in W$, i.e., $x \in \bigcap_{i=1}^l F(T_i) \cap D$ and $\phi(x) = \min_{z\in\bigcap_{i=1}^l F(T_i)\cap D}\phi(z)$, from the last inequality it follows that

$$
\phi(y) \le \mathrm{LIM}_n\|x_n - x\|^2 = \min_{z\in\bigcap_{i=1}^{l} F(T_i)\cap D}\phi(z).
$$

Thus $y \in W$. This implies that W has the property (P) for T_{l+1}. Consequently, all conditions in Lemma 2.4 are fulfilled. According to Lemma 2.4, T_{l+1} has a fixed point in W, i.e., $F(T_{l+1}) \cap W \neq \emptyset$. This shows that $\bigcap_{i=1}^{l+1} F(T_i) \cap D \neq \emptyset$. So $D \cap F \neq \emptyset$ where $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$.

According to $D\cap F \neq \emptyset$, we take $p \in D\cap F$ and $t \in (0, 1)$. Then $(1-t)p+tx \in K$ for any $x \in K$. Thus, $\phi(p) \le \phi((1-t)p+tx)$, and using Lemma 2.1 we have that

$$
\begin{aligned}
0 &\le \frac{\phi((1-t)p+tx) - \phi(p)}{t} = \frac{\mathrm{LIM}_n\|x_n - p + t(p-x)\|^2 - \mathrm{LIM}_n\|x_n - p\|^2}{t} \\
&\le \frac{\mathrm{LIM}_n[\|x_n - p\|^2 + 2t\langle p - x, j(x_n - p + t(p-x))\rangle] - \mathrm{LIM}_n\|x_n - p\|^2}{t} \\
&= -2\mathrm{LIM}_n\langle x - p, j(x_n - p - t(x-p))\rangle.
\end{aligned}
$$

This implies that

$$
\mathrm{LIM}_n\langle x - p, j(x_n - p - t(x-p))\rangle \le 0.
$$

Since K is bounded and j is norm-to-weak* uniformly continuous on any bounded subset of E, letting $t \to 0$ we have that

$$
\mathrm{LIM}_n\langle x - p, j(x_n - p)\rangle \le 0 \quad \forall x \in K.
$$

In particular,

$$
\mathrm{LIM}_n\langle f(p) - p, j(x_n - p)\rangle \le 0. \tag{14}
$$

Now, since $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings with common sequence $\{k_n\}_n \subset [1, \infty)$, we conclude that for all $x^* \in F := \bigcap_{i=1}^N F(T_i)$

$$
\begin{aligned}
\langle x_n - T_i^n x_n, j(x_n - x^*)\rangle &= \langle x_n - x^* - (T_i^n x_n - T_i^n x^*), j(x_n - x^*)\rangle \\
&\ge -(k_n - 1)\|x_n - x^*\|^2.
\end{aligned} \tag{15}
$$

By the definition of the sequence $\{x_n\}_n$, we have that

$$
x_n = \left(1 - \frac{1}{k_n}\right)x_n + \frac{s_n}{k_n}f(x_n) + \frac{t_n}{k_n}T_{r_n}^n x_n,
$$

which implies that

$$
x_n - T_{r_n}^n x_n = -\frac{s_n}{t_n}(x_n - f(x_n)) = -\frac{1-t_n}{t_n}(x_n - f(x_n)).
$$

Hence from (15) we obtain for all $x^* \in F$

$$
\begin{aligned}
\langle x_n - f(x_n), j(x_n - x^*)\rangle &= -\frac{t_n}{1-t_n}\langle x_n - T_{r_n}^n x_n, j(x_n - x^*)\rangle \\
&\le \frac{t_n(k_n - 1)}{1-t_n}\|x_n - x^*\|^2.
\end{aligned}
$$

Since K is bounded, it follows that

$$
\limsup_{n\to\infty}\langle x_n - f(x_n), j(x_n - x^*)\rangle \le 0 \quad \forall x^* \in F. \tag{16}
$$

Observe that

$$
\begin{aligned}
(1-\alpha)\|x_n - p\|^2 &\le \langle x_n - p, j(x_n - p)\rangle - \langle f(x_n) - f(p), j(x_n - p)\rangle \\
&= \langle x_n - f(x_n), j(x_n - p)\rangle + \langle f(p) - p, j(x_n - p)\rangle.
\end{aligned}
$$

Thus using (14) and (16) we derive $\text{LIM}_n\|x_n - p\| = 0$. Consequently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ as $k \to \infty$. To fulfil the proof, suppose that there is another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ which converges strongly to (say) $q \in K$. Then q is a common fixed point of $\{T_1, T_2, \ldots, T_N\}$ by the hypothesis that $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for each $i = 1, 2, \ldots, N$. Noticing $x_{n_k} \to p$ and setting $x^* = q$, we infer from (16) that

$$\langle p - f(p), j(p - q)\rangle \le 0. \tag{17}$$

Also, noticing $x_{n_l} \to q$ and setting $x^* = p$, we infer from (16) that

$$\langle q - f(q), j(q - p)\rangle \le 0. \tag{18}$$

Combining (17) with (18) yields that

$$\|p - q\|^2 \le \langle f(p) - f(q),\ j(p - q)\rangle \le \alpha\|p - q\|^2,$$

which implies that $p = q$ due to $\alpha \in [0, 1)$. Therefore, $x_n \to p$ as $n \to \infty$ and $p \in F$ is unique. Again, using (16), we can readily see that

$$\langle p - f(p), j(p - x^*)\rangle \le 0 \quad \forall x^* \in F. \tag{19}$$

Thus p is the unique solution of the variational inequality (13). This completes the proof. □

Corollary 3.2. *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of E, and $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings with common sequence $\{k_n\}_n \subset [1, \infty)$ such that $\sup_{n\ge1} k_n < \sqrt{N(E)}$. Let $u \in K$ be fixed, $\{s_n\}, \{t_n\}$ be two sequences in $(0, 1)$ such that (a) $s_n + t_n = 1$ for all $n \ge 1$, and (b) $\{t_n\}_n \subset (0, \frac{1}{k_n})$, $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{1-t_n} = 0$. Then*

(i) *for each $n \ge 1$, there is a unique $x_n \in K$ such that*

$$x_n = \left(1 - \frac{1}{k_n}\right) x_n + \frac{s_n}{k_n} u + \frac{t_n}{k_n} T_{r_n}^n x_n$$

where $n = l_n N + r_n$ for some unique integers $l_n \ge 0$ and $1 \le r_n \le N$; and if in addition, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, $T_iT_j = T_jT_i$ and $F(T_i)$ is convex for $1 \le i, j \le N$, then

(ii) *the sequence $\{x_n\}_n$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.*

Proof. In this case the map $f : K \to K$ defined by $f(x) = u \forall x \in K$ is a strict contraction with constant $\alpha = 0$. The proof follows immediately from Theorem 3.1. □

Remark 3.1. For the case of $N = 1$, in the proof of Theorem 3.1 we have proven $F(T_1) \cap D \ne \emptyset$ where $F(T_1)$ is not necessarily convex. Hence by the careful analysis of the proof of Theorem 3.1 we can see that the following consequence is valid.

Corollary 3.3. *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of E, $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$ such that $\sup_{n\ge1} k_n < \sqrt{N(E)}$, and let $f : K \to K$ be a contraction with constant $\alpha \in [0, 1)$. Let $\{s_n\}, \{t_n\}$ be two sequences in $(0, 1)$ such that (a) $s_n + t_n = 1$ for all $n \ge 1$, and (b) $\{t_n\}_n \subset (0, \frac{1-\alpha}{k_n-\alpha})$, $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{1-t_n} = 0$. Then*

(i) *for each $n \ge 1$, there is a unique $x_n \in K$ such that*

$$x_n = \left(1 - \frac{1}{k_n}\right) x_n + \frac{s_n}{k_n} f(x_n) + \frac{t_n}{k_n} T^n x_n;$$

and if in addition, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then

(ii) *the sequence $\{x_n\}_n$ converges strongly to the unique solution of the variational inequality:*

$$p \in F(T) \text{ such that } \langle (I - f)p, j(p - x^*)\rangle \le 0 \quad \forall x^* \in F(T).$$

Theorem 3.4. *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of E, $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings with common sequence $\{k_n\}_n \subset [1, \infty)$ such that $\sup_{n\ge1} k_n < \sqrt{N(E)}$, and let $f : K \to K$ be a contraction with constant $\alpha \in [0, 1)$. Let $\{s_n\}, \{t_n\}$ be two sequences in (0,1) such that (a) $s_n + t_n = 1$ for all $n \ge 1$, and (b) $\{t_n\}_n \subset (0, \xi_n)$, $\lim_{n\to\infty} t_n = 1$, $\sum_{n=1}^{\infty}(1 - t_n) = \infty$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\lim_{n\to\infty} \frac{k_n-1}{1-t_n} = 0$, where $\xi_n = \min\{\frac{1-\alpha}{k_n-\alpha}, \frac{1}{k_n}\}$. For an arbitrary $z_0 \in K$ let the sequence $\{z_n\}_n$ be iteratively defined by (9). Then*

(i) *for each $n \ge 1$, there is a unique $x_n \in K$ such that*

$$x_n = \left(1 - \frac{1}{k_n}\right) x_n + \frac{s_n}{k_n} f(x_n) + \frac{t_n}{k_n} T_{r_n}^n x_n$$

where $n = l_nN + r_n$ *for some unique integers* $l_n \geq 0$ *and* $1 \leq r_n \leq N$*; and if in addition,* $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0$, $\lim_{n\to\infty} \|z_n - T_iz_n\| = 0$, $T_iT_j = T_jT_i$ *and* $F(T_i)$ *is convex for* $1 \leq i, j \leq N$*, then*

(ii) *the sequence* $\{z_n\}_n$ *converges strongly to the unique solution of the variational inequality:*

$$p \in F \text{ such that } \langle (I - f)p, j(p - x^*)\rangle \leq 0 \quad \forall x^* \in F,$$

where $F = \cap_{i=1}^{N} F(T_i)$.

Proof. Part (i) has already been proved in Theorem 3.1. Assume that $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0$ and $\lim_{n\to\infty} \|z_n - T_iz_n\| = 0$ for each $i = 1, 2, \ldots, N$, and $F \neq \emptyset$. We proceed to prove part (ii). Let $n > m$. Then, from (12) we get

$$x_m - z_n = \left(1 - \frac{1}{k_m}\right)(x_m - z_n) + \frac{s_m}{k_m}(f(x_m) - z_n) + \frac{t_m}{k_m}(T_{r_m}^m x_m - z_n).$$

We follow the line of the argument in [8]. Applying inequality (11), we estimate as follows:

$$\begin{aligned}
\|x_m - z_n\|^2 &\leq \left\| \left(1 - \frac{1}{k_m}\right)(x_m - z_n) + \frac{t_m}{k_m}(T_{r_m}^m x_m - z_n) \right\|^2 + 2\frac{s_m}{k_m}\langle f(x_m) - z_n, j(x_m - z_n)\rangle \\
&= \left\| \left(1 - \frac{1}{k_m}\right)(x_m - z_n) + \frac{t_m}{k_m}(T_{r_m}^m x_m - T_{r_m}^m z_n) + \frac{t_m}{k_m}(T_{r_m}^m z_n - z_n) \right\|^2 + 2\frac{s_m}{k_m}\langle f(x_m) - z_n, j(x_m - z_n)\rangle \\
&\leq \left[\left(1 - \frac{1}{k_m} + t_m\right)\|x_m - z_n\| + \frac{t_m}{k_m}\|T_{r_m}^m z_n - z_n\| \right]^2 + 2\frac{s_m}{k_m}\langle f(x_m) - z_n, j(x_m - z_n)\rangle \\
&= \left(1 - \frac{1}{k_m} + t_m\right)^2 \|x_m - z_n\|^2 + 2\left(1 - \frac{1}{k_m} + t_m\right)\frac{t_m}{k_m}\|x_m - z_n\|\|T_{r_m}^m z_n - z_n\| \\
&\quad + \frac{t_m^2}{k_m^2}\|T_{r_m}^m z_n - z_n\|^2 + 2\frac{s_m}{k_m}[\langle f(x_m) - x_m, j(x_m - z_n)\rangle + \|x_m - z_n\|^2] \\
&\leq \left[2\frac{s_m}{k_m} + \left(1 - \frac{1}{k_m} + t_m\right)^2 \right]\|x_m - z_n\|^2 + \|T_{r_m}^m z_n - z_n\| \left\{ 2\left(1 - \frac{1}{k_m} + t_m\right)\frac{t_m}{k_m}\|x_m - z_n\| \right. \\
&\quad \left. + \frac{t_m^2}{k_m^2}\|T_{r_m}^m z_n - z_n\| \right\} + 2\frac{s_m}{k_m}\langle f(x_m) - x_m, j(x_m - z_n)\rangle.
\end{aligned}$$

Since K is bounded, for some constant $M > 0$, it follows that

$$\langle f(x_m) - x_m, j(z_n - x_m)\rangle \leq \frac{\left(1 - \frac{1}{k_m} + t_m\right)^2 - (1 - 2\frac{s_m}{k_m})}{2\frac{s_m}{k_m}} M + \frac{M\|z_n - T_{r_m}^m z_n\|}{2\frac{s_m}{k_m}}.$$

Observe that

$$\begin{aligned}
\lim_{m\to\infty} \frac{\left(1 - \frac{1}{k_m} + t_m\right)^2 - (1 - 2\frac{s_m}{k_m})}{2\frac{s_m}{k_m}} &= \lim_{m\to\infty} \left\{ \frac{\left(1 - \frac{1}{k_m} + t_m\right)^2 - \left(1 - \frac{s_m}{k_m}\right)^2}{2\frac{s_m}{k_m}} + \frac{s_m}{2k_m} \right\} \\
&= \lim_{m\to\infty} \left\{ \frac{k_m}{2s_m}\left(2 - \frac{1}{k_m} - \frac{s_m}{k_m} + t_m\right)\left(-\frac{1}{k_m} + t_m + \frac{s_m}{k_m}\right) + \frac{s_m}{2k_m} \right\} \\
&= \lim_{m\to\infty} \left\{ \frac{t_m(k_m - 1)}{2(1 - t_m)}\left(2 - \frac{2 - t_m}{k_m} + t_m\right) + \frac{1 - t_m}{2k_m} \right\} \\
&= 0,
\end{aligned}$$

and hence

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(x_m) - x_m, j(z_n - x_m)\rangle &\leq \frac{\left(1 - \frac{1}{k_m} + t_m\right)^2 - \left(1 - 2\frac{s_m}{k_m}\right)}{2\frac{s_m}{k_m}} M + \limsup_{n\to\infty} \frac{M\|z_n - T_{r_m}^m z_n\|}{2\frac{s_m}{k_m}} \\
&= \frac{\left(1 - \frac{1}{k_m} + t_m\right)^2 - \left(1 - 2\frac{s_m}{k_m}\right)}{2\frac{s_m}{k_m}} M
\end{aligned}$$

since $\lim_{n\to\infty}\|z_n - T_i z_n\| = 0$ for each $i = 1, 2, \ldots, N$, implies that

$$\lim_{n\to\infty}\|z_n - T_{r_m}^m z_n\| \le \lim_{n\to\infty}\{\|z_n - T_{r_m}z_n\| + \|T_{r_m}z_n - T_{r_m}^2 z_n\| + \cdots + \|T_{r_m}^{m-1}z_n - T_{r_m}^m z_n\|\} = 0.$$

In terms of Theorem 3.1, $x_m \to p \in F$, which solves the variational inequality (13). Since j is norm-to-weak* uniformly continuous on any bounded subset of E, in the limit as $m \to \infty$, we obtain that

$$\limsup_{n\to\infty}\langle f(p) - p, j(z_n - p)\rangle \le 0. \tag{20}$$

Now from the iterative process (9) and Lemma 2.1, we estimate as follows:

$$\begin{aligned}
\|z_{n+1} - p\|^2 &\le \left\|\left(1 - \frac{1}{k_n}\right)(z_n - p) + \frac{t_n}{k_n}(T_{r_n}^n z_n - p)\right\|^2 + 2\frac{s_n}{k_n}\langle f(z_n) - p, j(z_{n+1} - p)\rangle \\
&\le \left[\left(1 - \frac{1}{k_n}\right)\|z_n - p\| + \frac{t_n}{k_n}\|T_{r_n}^n z_n - p\|\right]^2 + 2\frac{s_n}{k_n}\|f(z_n) - f(p)\|\|z_{n+1} - p\| \\
&\quad + 2\frac{s_n}{k_n}\langle f(p) - p, j(z_{n+1} - p)\rangle \\
&\le \left(1 - \frac{1}{k_n} + t_n\right)^2\|z_n - p\|^2 + 2\frac{s_n\alpha}{k_n}\|z_n - p\|\|z_{n+1} - p\| + 2\frac{s_n}{k_n}\langle f(p) - p, j(z_{n+1} - p)\rangle \\
&\le \left(1 - \frac{1}{k_n} + t_n\right)^2\|z_n - p\|^2 + \frac{s_n\alpha}{k_n}(\|z_n - p\|^2 + \|z_{n+1} - p\|^2) + 2\frac{s_n}{k_n}\langle f(p) - p, j(z_{n+1} - p)\rangle,
\end{aligned}$$

so that

$$\begin{aligned}
\|z_{n+1} - p\|^2 &\le \frac{\left(1 - \frac{1}{k_n} + t_n\right)^2 + \frac{s_n\alpha}{k_n}}{1 - \frac{s_n\alpha}{k_n}}\|z_n - p\|^2 + 2\frac{\frac{s_n}{k_n}}{1 - \frac{s_n\alpha}{k_n}}\langle f(p) - p, j(z_{n+1} - p)\rangle \\
&= \left(1 - \frac{1 - 2\frac{s_n\alpha}{k_n} - \left(1 - \frac{1}{k_n} + t_n\right)^2}{1 - \frac{s_n\alpha}{k_n}}\right)\|z_n - p\|^2 + 2\frac{\frac{s_n}{k_n}}{1 - \frac{s_n\alpha}{k_n}}\langle f(p) - p, j(z_{n+1} - p)\rangle.
\end{aligned} \tag{21}$$

Observe that

$$\begin{aligned}
\frac{1 - 2\frac{s_n\alpha}{k_n} - \left(1 - \frac{1}{k_n} + t_n\right)^2}{1 - \frac{s_n\alpha}{k_n}} &= \frac{\left(1 - \frac{s_n\alpha}{k_n}\right)^2 - \left(1 - \frac{1}{k_n} + t_n\right)^2 - \frac{s_n^2\alpha^2}{k_n^2}}{1 - \frac{s_n\alpha}{k_n}} \\
&= \frac{\left(-\frac{s_n\alpha}{k_n} + \frac{1}{k_n} - t_n\right)\left(2 - \frac{s_n\alpha}{k_n} - \frac{1}{k_n} + t_n\right)}{1 - \frac{s_n\alpha}{k_n}} - \frac{\frac{s_n^2\alpha^2}{k_n^2}}{1 - \frac{s_n\alpha}{k_n}} \\
&= \frac{\left(\frac{s_n}{k_n}(1 - \alpha) - \frac{t_n(k_n - 1)}{k_n}\right)\left(2 - \frac{s_n\alpha}{k_n} - \frac{1}{k_n} + t_n\right)}{1 - \frac{s_n\alpha}{k_n}} - \frac{\frac{s_n^2\alpha^2}{k_n^2}}{1 - \frac{s_n\alpha}{k_n}}
\end{aligned}$$

and by (21) for some constant $M > 0$

$$\begin{aligned}
\|z_{n+1} - p\|^2 &\le \left(1 - \frac{\left(\frac{s_n}{k_n}(1 - \alpha) - \frac{t_n(k_n - 1)}{k_n}\right)\left(2 - \frac{s_n\alpha}{k_n} - \frac{1}{k_n} + t_n\right)}{1 - \frac{s_n\alpha}{k_n}}\right)\|z_n - p\|^2 + \frac{\frac{s_n^2\alpha^2}{k_n^2}}{1 - \frac{s_n\alpha}{k_n}}\|z_n - p\|^2 \\
&\quad + 2\frac{\frac{s_n}{k_n}}{1 - \frac{s_n\alpha}{k_n}}\langle f(p) - p, j(z_{n+1} - p)\rangle \\
&\le \left(1 - \frac{\left(\frac{s_n}{k_n}(1 - \alpha) - \frac{t_n(k_n - 1)}{k_n}\right)\left(2 - \frac{s_n\alpha}{k_n} - \frac{1}{k_n} + t_n\right)}{1 - \frac{s_n\alpha}{k_n}}\right)\|z_n - p\|^2 + \frac{s_n^2\alpha^2}{k_n(k_n - s_n\alpha)}M \\
&\quad + 2\frac{s_n}{k_n - s_n\alpha}\langle f(p) - p, j(z_{n+1} - p)\rangle
\end{aligned} \tag{22}$$

since K is bounded. Now putting

$$\lambda_n = \frac{\left(\frac{s_n}{k_n}(1-\alpha) - \frac{t_n(k_n-1)}{k_n}\right)\left(2 - \frac{s_n\alpha}{k_n} - \frac{1}{k_n} + t_n\right)}{1 - \frac{s_n\alpha}{k_n}},$$

and

$$\gamma_n = \left(\frac{s_n}{k_n}(1-\alpha) - \frac{t_n(k_n-1)}{k_n}\right)^{-1}\left(2 - \frac{s_n\alpha}{k_n} - \frac{1}{k_n} + t_n\right)^{-1}\left\{\frac{s_n^2\alpha^2}{k_n^2}M + 2\frac{s_n}{k_n}\langle f(p) - p, j(z_{n+1} - p)\rangle\right\},$$

we rewrite (22) as follows:

$$\|z_{n+1} - p\|^2 \leq (1-\lambda_n)\|z_n - p\|^2 + \lambda_n\gamma_n.$$

Since $\lim_{n\to\infty} t_n = 1$, $\sum_{n=1}^{\infty}(1-t_n) = \infty$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\lim_{n\to\infty}\frac{k_n-1}{1-t_n} = 0$, we deduce that $\sum_{n=1}^{\infty}(\frac{s_n}{k_n}(1-\alpha) - \frac{t_n(k_n-1)}{k_n}) = \infty$ and hence $\sum_{n=1}^{\infty}\lambda_n = \infty$. Furthermore, it is easy to see that

$$\lim_{n\to\infty} s_n\left(\frac{s_n}{k_n}(1-\alpha) - \frac{t_n(k_n-1)}{k_n}\right)^{-1}\left(2 - \frac{s_n\alpha}{k_n} - \frac{1}{k_n} + t_n\right)^{-1} = \frac{1}{2(1-\alpha)},$$

and hence $\limsup_{n\to\infty}\gamma_n \leq 0$. Consequently, it follows from Lemma 2.3 that $z_n \to p$ as $n \to \infty$. This completes the proof. □

Corollary 3.5. *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of E, and $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings with common sequence $\{k_n\}_n \subset [1,\infty)$ such that $\sup_{n\geq 1} k_n < \sqrt{N(E)}$. Let $u \in K$ be fixed, $\{s_n\}$, $\{t_n\}$ be two sequences in $(0, 1)$ such that* (a) *$s_n + t_n = 1$ for all $n \geq 1$, and* (b) *$\lim_{n\to\infty} t_n = 1$, $\{t_n\}_n \subset (0, \frac{1}{k_n})$, $\sum_{n=1}^{\infty}(1-t_n) = \infty$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\lim_{n\to\infty}\frac{k_n-1}{1-t_n} = 0$. Define the sequence $\{z_n\}_n$ iteratively by $z_0 \in K$,*

$$z_{n+1} = \left(1 - \frac{1}{k_n}\right)z_n + \frac{s_n}{k_n}u + \frac{t_n}{k_n}T_{r_n}^n z_n,$$

where $n = l_nN + r_n$ for some unique integers $l_n \geq 0$ and $1 \leq r_n \leq N$. Then

(i) *for each $n \geq 1$, there is a unique $x_n \in K$ such that*

$$x_n = \left(1 - \frac{1}{k_n}\right)x_n + \frac{s_n}{k_n}u + \frac{t_n}{k_n}T_{r_n}^n x_n;$$

and, if in addition, $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$, $\lim_{n\to\infty}\|z_n - T_iz_n\| = 0$, $T_iT_j = T_jT_i$ and $F(T_i)$ is convex for $1 \leq i, j \leq N$, then

(ii) *$\{z_n\}_n$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.*

If $N = 1$ then the following corollary follows immediately from Remark 3.1 and Theorem 3.4.

Corollary 3.6. *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of E, $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1,\infty)$ such that $\sup_{n\geq 1} k_n < \sqrt{N(E)}$, and let $f : K \to K$ be a contraction with constant $\alpha \in [0, 1)$. Let $\{s_n\}$, $\{t_n\}$ be two sequences in $(0, 1)$ such that* (a) *$s_n + t_n = 1$ for all $n \geq 1$, and* (b) *$\{t_n\}_n \subset (0, \xi_n)$, $\lim_{n\to\infty} t_n = 1$, $\sum_{n=1}^{\infty}(1-t_n) = \infty$ $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\lim_{n\to\infty}\frac{k_n-1}{1-t_n} = 0$, where $\xi_n = \min\{\frac{1-\alpha}{k_n-\alpha}, \frac{1}{k_n}\}$. For an arbitrary $z_0 \in K$ let the sequence $\{z_n\}_n$ be iteratively defined by*

$$z_{n+1} = \left(1 - \frac{1}{k_n}\right)z_n + \frac{s_n}{k_n}f(z_n) + \frac{t_n}{k_n}T^n z_n.$$

Then

(i) *for each $n \geq 1$, there is a unique $x_n \in K$ such that*

$$x_n = \left(1 - \frac{1}{k_n}\right)x_n + \frac{s_n}{k_n}f(x_n) + \frac{t_n}{k_n}T^n x_n;$$

and, if in addition, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$, then

(ii) *the sequence $\{z_n\}_n$ converges strongly to the unique solution of the variational inequality:*

$$p \in F(T) \text{ such that } \langle (I - f)p, j(p - x^*)\rangle \leq 0 \quad \forall x^* \in F(T).$$

Remark 3.2. (i) Since every nonexpansive mapping is asymptotically nonexpansive, our Corollaries 3.3 and 3.6 hold for the case when T is simply nonexpansive. In this case, $k_n = 1\ \forall n \geq 1$, our viscosity iterative schemes coincide essentially with Shahzad and Udomene's viscosity iterative schemes in [5]. As pointed out in [5, p. 566, Remarks (B)], the boundedness requirement on K can be removed from the above Corollaries 3.3 and 3.6 (see [4]); $k_n = 1\ \forall n \geq 1$ and the conditions: $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ are satisfied. The choice of t_n is as follows: $t_n = 1 - \frac{1}{n}$.

(ii) Since every uniformly smooth Banach space has a uniformly Gâteaux differentiable norm and possesses uniform normal structure (see e.g., [2,10,11,14,15]), our Theorems 3.1 and 3.4, proved for the more general class of asymptotically nonexpansive mappings and in the more general real Banach spaces considered here are significant improvements on the results of [4], and hence of [3]. Meantime, our Theorems 3.1 and 3.4 extend Theorem 3.1 and 3.3 of [5] to new viscosity iterative schemes and to the case of a finite family of asymptotically nonexpansive mappings.

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