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Linear disjointness preservers of W*-algebras

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Abstract In this paper, we give a complete description of the structure of zero product and orthogonality preserving linear maps between W*-algebras. In particular, two W*-algebras are *-isomorphic if and only if there is a bijective linear map between them preserving their zero product or orthogonality structure in two directions. It is also the case when they have equivalent linear and left (right) ideal structures.

Keywords Zero product preservers · Orthogonality preservers · Automatic continuity · Left ideal preservers · W*-algebras

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1 Introduction

Recall that a W*-algebra M is a C*-algebra with a predual. So M carries many different structures, including the geometric (i.e., norm) structure, the $*$-algebraic structure, and the normal structure (i.e., weak* topology). As the norm of an element a in M is equal to the square root of the spectral radius of a^*a, the geometric structure of M can be recovered from its $*$-algebraic structure. It is further showed by Gardner [15] that two W*-algebras

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are $*$-algebraic isomorphic if and only if they are algebraic isomorphic, and all algebraic isomorphisms between W*-algebras are norm and σ-weakly bi-continuous. Indeed, every algebra isomorphism $\theta : M \to N$ between W*-algebras carries the form $\theta(a) = \pi(hah^{-1})$ for some invertible positive element h in M and some $*$-isomorphism π from M onto N (See, e.g., Sakai [24, Section 4.1]). Therefore, W*-algebras are completely determined by their linear and product structures. In this paper, we show that the linear and disjointness structures also suffice.

In the context of operator algebras (on Hilbert spaces) there are at least three versions of disjointness: zero product ($ab = 0$), range orthogonality ($a^*b = 0$), and domain orthogonality ($ab^* = 0$). Of course, the latter two are symmetric. If the algebra is abelian, then all three concepts coincide. In [14,17,18], it is shown that two abelian C*-algebras are *-isomorphic if and only if there is a bijective linear map θ between them such that

$$ab = 0 \quad \text{implies} \quad \theta(a)\theta(b) = 0.$$

On the other hand, it is shown in [2,28] that every surjective linear map $\theta : A \to B$ between two standard operator algebras preserving zero products, or range/domain orthogonality in two directions is basically an inner automorphism, and thus it is automatically bounded as well. Recall that standard operator algebras are those containing all finite rank operators.

Bounded linear zero product and orthogonality preservers $\theta : A \to B$ between general C*-algebras were studied in [8,19,25,29,31]. Assume that θ is bijective and norm continuous. Suppose θ sends self-adjoint elements with zero products to (not necessarily self-adjoint) elements with zero products, i.e.,

$$ab = 0 \quad \text{implies} \quad \theta(a)\theta(b) = 0, \quad \forall a, b \in A_{sa}.$$

Then A and B are isomorphic as Jordan algebras. If θ preserves zero products of arbitrary elements in A, then A and B are isomorphic as $*$-algebras [8,31].

Without assuming continuity, we cannot make use of any tools from functional calculus, which is used heavily in previous literature. Anyway, a few partial results follow. If the C*-algebra A is linearly generated by idempotents (e.g., properly infinite unital C*-algebras [22, Corollary 2.2]) *and* $\theta(1) = 1$ then θ is again an algebra homomorphism [8]. Else, suppose A and B are CCR C*-algebras with Hausdorff spectrum (for the definition, see, e.g. [12, Section 4.2] or [13]). If $\theta : A \to B$ is linear, bijective, and preserves zero products in two directions, then θ gives rise to an algebra isomorphism, and thus A and B are *-isomorphic by the result of Gardner [15]. In fact, we have

Proposition 1.1 ([21, Theorem 3.3]) *Let A and B be CCR C*-algebras with Hausdorff spectrum. Let $\theta : A \to B$ be a bijective linear map such that*

$$ab = 0 \text{ in } A \quad \text{if and only if} \quad \theta(a)\theta(b) = 0 \text{ in } B.$$

Then θ is automatically bounded. More precisely, $\theta = m\Psi$ where $m = \theta^{**}(1)$ is an invertible central multiplier of B and Ψ is an algebra isomorphism from A onto B.

There is also a similar result for linear orthogonality preservers.

Proposition 1.2 [27, Theorem 3] *Let $\mathcal{A}$, $\mathcal{B}$ be two C*-algebras with continuous traces. Let $\theta : \mathcal{A} \to \mathcal{B}$ be a bijective linear map preserving orthogonality in two directions. Then θ is automatically bounded. More precisely, we have the following implication table.*

	The structures θ preserves	*implies the form θ assumes*
Case 1:	$a^*b = 0 \Leftrightarrow \theta(a)^*\theta(b) = 0,$	$\Psi r;$
Case 2:	$ab^* = 0 \Leftrightarrow \theta(a)\theta(b)^* = 0,$	$l\Psi;$
Case 3:	$a^*b = 0 \Leftrightarrow \theta(a)\theta(b)^* = 0,$	$l\Phi;$
Case 4:	$ab^* = 0 \Leftrightarrow \theta(a)^*\theta(b) = 0,$	$\Phi r.$

Here,

r: *invertible right multiplier of $\mathcal{B}$,*
l: *invertible left multiplier of $\mathcal{B}$,*
Ψ: *$*$-algebra isomorphism from $\mathcal{A}$ onto $\mathcal{B}$,*
Φ: *anti-$*$-algebra isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

In this paper, we obtain a complete characterization for linear zero product/orthogonality preservers of general W*-algebras. Our main result states

Theorem 1.3 *Let M, N be two W*-algebras. Let $\theta : M \to N$ be a bijective linear map. Then M, N are isomorphic as W*-algebras, provided that any of the following conditions holds.*

(A) θ preserves zero products in two directions, i.e.,

$$ab = 0 \text{ in } M \quad \text{if and only if} \quad \theta(a)\theta(b) = 0 \text{ in } N.$$

In this case, $\theta(1)$ is a central invertible element and $\theta(1)^{-1}\theta(\cdot)$ is an algebra isomorphism.

(B) θ preserves range orthogonality in two directions, i.e.,

$$a^*b = 0 \text{ in } M \quad \text{if and only if} \quad \theta(a)^*\theta(b) = 0 \text{ in } N.$$

In this case, $\theta(1)$ is an invertible element and $\theta(\cdot)\theta(1)^{-1}$ is a $*$-algebra isomorphism.

(C) θ preserves domain orthogonality in two directions, i.e.,

$$ab^* = 0 \text{ in } M \quad \text{if and only if} \quad \theta(a)\theta(b)^* = 0 \text{ in } N.$$

In this case, $\theta(1)$ is an invertible element and $\theta(1)^{-1}\theta(\cdot)$ is a $*$-algebra isomorphism.

(D) θ preserves reverse orthogonality in two directions, i.e.,

$$a^*b = 0 \text{ in } M \quad \text{if and only if} \quad \theta(a)\theta(b)^* = 0 \text{ in } N.$$

In this case, $\theta(1)$ is an invertible element and $\theta(\cdot)^{tr}\theta(1)^{tr\,-1}$ is a $*$-algebra isomorphism. Here, T^{tr} is the transpose of an operator T in $N \subseteq B(H)$ with respect to an arbitrary but fixed orthonormal basis of the underlying Hilbert space H of the universal representation of the W-algebra N.*

It is clear that Cases (C) and (D) follow easily from Case (B) in Theorem 1.3. In the next section, we will provide the proofs for Cases (A) and (B).

Finally, let us mention that some other kinds of disjointness in a W*-algebra can be defined by doubly orthogonality (see, e.g., [5,6,31]), and by its left (or right) ideals (see, e.g., [1,10,20,23]). We will also discuss them at the end of the paper. As a variance of Theorem 1.3, for example, Theorem 2.5 ensures that if there is a linear bijective map θ between two W*-algebras M, N preserving left (or right) ideals in both directions then M, N are *-isomorphic, too. Indeed, Theorem 2.5 says that $\theta(\cdot)\theta(1)^{-1}$ is an algebra isomorphism.

2 The results

We need the following result of Goldstein and Paszkiewicz.

Lemma 2.1 ([16, Theorem 3(3)]) *A W*-algebra M is the linear span, with integer coefficients, of its projections if and only if it has no direct summand of finite type I. If this is the case, any self-adjoint operator of norm not greater than one can be represented in the form*

$$p_1 + \cdots + p_{12} - p_{13} - \cdots - p_{24}$$

for some projections $p_1, \ldots, p_{24}$ in M.

We also need the following well-known fact. If M is a finite type I W*-algebra, then for each n in $\mathbb{N}$, there exist a hyperstonean space Ω_n (could be empty) and a central projection w_n in M such that $\{w_n\}$ are orthogonal to each another, $\sum_n w_n$ weak-*-converges to 1, and $w_n A \cong C(\Omega_n) \otimes M_n$ (see e.g. [24, Section 2.2]). Here we use the convention that $C(\Omega_n) = \{0\}$ if $\Omega_n = \emptyset$. In particular, M is a CCR C*-algebra with Hausdorff spectrum and continuous trace.

Proof of Theorem 1.3(A) Let z be a central projection in M such that the ideal $M_1 = (1-z)M$ is of finite type I, and the ideal $M_2 = zM$ contains no finite type I summand. Similarly, we write $N = N_1 + N_2$.

As $M_1 M_2 = M_2 M_1 = \{0\}$, we have $\theta(M_1)\theta(M_2) = \theta(M_2)\theta(M_1) = 0$. Let L_i, R_i be the weak* closed left and right ideals of N generated by $\theta(M_i)$, for $i = 1, 2$, respectively. It is clear that $L_1 R_2 = L_2 R_1 = \{0\}$. As θ^{-1} also preserves zero products, we have $\theta^{-1}(L_1)M_2 = M_1\theta^{-1}(R_2) = M_2\theta^{-1}(R_1) = \theta^{-1}(L_2)M_1 = 0$. Therefore, $\theta^{-1}(L_i), \theta^{-1}(R_i) \subseteq M_i$ for $i = 1, 2$, respectively. It follows that $\theta(M_i) = L_i = R_i$ is a weak* closed two-sided ideal of N, for $i = 1, 2$. Since $N = \theta(M) = \theta(M_1) + \theta(M_2)$, there is a central projection q in N such that $\theta(M_1) = (1-q)N$ and $\theta(M_2) = qN$.

Let $b, e \in M_2$ with $e^2 = e$. As $(z-e)eb = e(z-e)b = 0$, we have

$$0 = \theta(z-e)\theta(eb) = (\theta(z) - \theta(e))\theta(eb),$$

and

$$0 = \theta(e)\theta(b - eb) = \theta(e)(\theta(b) - \theta(eb)).$$

It follows

$$\theta(z)\theta(eb) = \theta(e)\theta(eb) = \theta(e)\theta(b).$$

By Lemma 2.1, every element a in M_2 is a linear sum of at most 48 projections. As a result, we have

$$\theta(z)\theta(ab) = \theta(a)\theta(b), \quad \forall a, b \in M_2. \tag{2.1}$$

Putting $b = z$ in (2.1) we have

$$\theta(z)\theta(a) = \theta(a)\theta(z), \quad \forall a \in M_2.$$

So $\theta(z)$ is a central element in the ideal qN, and thus in N. Let $d \in M_2$ such that $\theta(d) = q$. Then (2.1) gives

$$\theta(z)\theta(d^2) = \theta(d)^2 = q.$$

It follows $\theta(z)$ is invertible in qN, and its inverse $\theta(d^2)$ is also a central element.

Define $\pi_2 : M_2 \to qN$ by

$$\pi_2(a) = \theta(d^2)\theta(a), \quad \forall a \in M_2.$$

Then π_2 is linear and bijective. Moreover,

$$\pi_2(z) = \theta(d^2)\theta(z) = q,$$

and

$$\pi_2(ab) = \theta(d^2)\theta(z)\theta(ab)\theta(d^2) = \theta(d^2)\theta(a)\theta(b)\theta(d^2) = \pi_2(a)\pi_2(b), \quad \forall a, b \in M_2.$$

Therefore, π_2 is an algebra isomorphism from M_2 onto qN. It follows from a result of Gradner [15] that M_2 and qN are indeed isomorphic as W*-algebras. Inheriting from M_2, the W*-algebra qN contains no finite type I summand either. In particular, $\theta(M_2) = qN \subseteq N_2$. Applying the same arguments to θ^{-1}, we see that $\theta^{-1}(N_2) \subseteq M_2$. Consequently, $\theta(M_2) = N_2$, and hence $\theta(M_1) = N_1$.

We have already seen that π_2 is an algebra isomorphism from M_2 onto N_2. On the other hand, Proposition 1.1 says that $\theta(1 - z)$ is a central invertible element in N_1, and there is an algebra isomorphism $\pi_1 : M_1 \to N_1$ such that $\theta(a) = \theta(1 - z)\pi_1(a)$ for all a in M_1. This gives in turn that $\theta(1)$ is a central invertible element in N and the map $\pi : M \to N$ defined by $\pi(a) = \theta(1)^{-1}\theta(a)$ is an algebra isomorphism. □

Proof of Theorem 1.3(B) Let z be a central projection in M such that the ideal $M_1 = (1-z)M$ is of finite type I, and the ideal $M_2 = zM$ contains no finite type I summand. Similarly, we write $N = N_1 + N_2$ with $N_1 = (1 - z')N$ and $N_2 = z'N$.

As $M_1^* M_2 = \{0\}$, we have $\theta(M_1)^*\theta(M_2) = 0$. Let R_1, R_2 be the weak* closed right ideals of N generated by $\theta(M_1), \theta(M_2)$, respectively. It is clear that $R_1^* R_2 = \{0\}$. Moreover, the identity $N = \theta(M) = \theta(M_1) + \theta(M_2)$ forces $R_1 = \theta(M_1)$ and $R_2 = \theta(M_2)$, respectively. Let q be the projection in N such that $\theta(M_1) = (1 - q)N$ and $\theta(M_2) = qN$.

Consider an projection p in $M_2 = zM$, and any arbitrary element b in M_2. Since

$$(z - p)pb = p(z - p)b = 0,$$

we have

$$(\theta(z)^* - \theta(p)^*)\theta(pb) = \theta(p)^*(\theta(b) - \theta(pb)) = 0.$$

It follows

$$\theta(z)^*\theta(pb) = \theta(p)^*\theta(pb) = \theta(p)^*\theta(b).$$

By Lemma 2.1, we have

$$\theta(z)^*\theta(a^*b) = \theta(a)^*\theta(b), \quad \forall a, b \in M_2. \tag{2.2}$$

Let $d \in M_2$ such that $\theta(d) = q$. By (2.2), we have

$$\theta(z)^*\theta(d^*d) = \theta(d)^*\theta(d) = q. \tag{2.3}$$

Setting $b = z$ in (2.2), we have

$$\theta(z)^*\theta(a^*) = \theta(a)^*\theta(z), \quad \forall a \in M_2. \tag{2.4}$$

In particular, as $\theta(z) = q\theta(z) \in \theta(M_2) = qN$, it follows

$$\theta(z)^*N = \theta(z)^*qN = Nq\theta(z) = N\theta(z)$$

is a two-sided self-adjoint ideal of N. Let w be the central projection in N such that wN is the weak* closure of $N\theta(z)$. Let $s_l(\theta(z))$ and $s_r(\theta(z))$ be the left and right support projections of $\theta(z)$ in N, respectively. Putting $a = d^*$ in (2.4), we have

$$\theta(z)^* = \theta(z)^* q = \theta(d^*)^* \theta(z).$$

Observe

$$\theta(a)^*\theta(z) = \theta(a)^*\theta(z)s_r(\theta(z)), \quad \forall a \in M_2.$$

Consequently,

$$w = ws_r(\theta(z)) \leq s_r(\theta(z)). \tag{2.5}$$

Since $\theta(z) = q\theta(z)$ and $q = \theta(z)^*\theta(d^*d) \in wN$, we also have

$$s_l(\theta(z)) \leq q \leq w. \tag{2.6}$$

Because $s_r(\theta(z))$ is equivalent to $s_l(\theta(z))$, they have the same central support. It then follows from (2.5) and (2.6) that

$$w = s_r(\theta(z)) \geq q \geq s_l(\theta(z)).$$

Let $q_1 = (1 - z')q \in N_1$ and $w_1 = (1 - z')w \in N_1$. Since N_1 is of finite type I, we have $q_1 = w_1$ is a central projection in N. Note that the weak* closed two-sided ideal $q_1N \subseteq qN_1 = \theta(M_2) \cap N_1$, and $q_1\theta(z) = \theta(z)q_1$.

Argue similarly with $\Psi = \theta^{-1} : N \to M$, we have

$$\Psi(1)^*\Psi(ry) = \Psi(r)^*\Psi(y)$$

for every projection r and for every element y in N. Putting $y = \theta(z)$, we get

$$\Psi(1)^*\Psi(r\theta(z)) = \Psi(r)^*z.$$

If r is a projection in N with $r \leq q$ then $r \in qN = \theta(M_2)$, and thus

$$\Psi(1)^*\Psi(r\theta(z)) = \Psi(r)^*.$$

Since Ψ is one-to-one, $r\theta(z) = 0$ implies $r = 0$. Now, let $x \in N$ such that $x\theta(z) = xq\theta(z) = 0$. Then, $\theta(z)^*qx^*xq\theta(z) = 0$. This implies $\theta(z)^*r\theta(z) = 0$, and hence $r = 0$, for every spectral projection r of qx^*xq. Thus, $xq = 0$. As a result, the right multiplication operator $R_{\theta(z)} : q_1N \to q_1N$, sending xq_1 to $xq_1\theta(z)$, is one-to-one.

Moreover, $q_1N\theta(z) = N\theta(z)q_1 \supseteq N\theta(d^*d)^*\theta(z)q_1 = q_1N$ by (2.3). So $R_{\theta(z)}$ is a bounded bijective linear map from q_1N onto itself. Consider also the right multiplication operator $R_{\theta(d^*d)^*} : q_1N \to q_1N$ sending xq_1 to $xq_1\theta(d^*d)^*$. The identity (2.3) says that

$$R_{\theta(z)}R_{\theta(d^*d)^*} = R_{q_1}.$$

Here, R_{q_1} is the identity map from q_1N onto q_1N. Since $R_{\theta(z)}$ is bijective, we have

$$R_{\theta(d^*d)^*}R_{\theta(z)} = R_{q_1}.$$

In particular,

$$q_1\theta(z)\theta(d^*d)^* = q_1. \tag{2.7}$$

Define $\pi_{21} : M_2 \to q_1N$ by

$$\pi_{21}(a) = q_1\theta(a)\theta(d^*d)^*, \quad \forall a \in M_2.$$

It is easy to see that π_{21} is onto, $\pi_2(z) = q_1$, the identity of the W*-algebra q_1N, and by (2.7),

$$\begin{aligned}\pi_{21}(a^*b) &= q_1\theta(a^*b)\theta(d^*d)^* \\ &= q_1\theta(d^*d)\theta(z)^*\theta(a^*b)\theta(d^*d)^* \\ &= q_1\theta(d^*d)\theta(a)^*\theta(b)\theta(d^*d)^* \\ &= \pi_{21}(a)^*\pi_{21}(b), \quad \forall a, b \in M_2.\end{aligned}$$

In other words, π_{21} is a surjective $*$-homomorphism. It then follows from Lemma 2.1 that the W*-algebra $q_1N \subseteq N_1$ contains no finite type I summand, as M_2 does not either. This forces the finite and discrete central projection $q_1 = 0$ and thus $\theta(M_2) = qN \subseteq N_2$. Dealing with $\Psi = \theta^{-1}$, we see also that $\Psi(N_2) \subseteq M_2$. It follows $\theta(M_2) = N_2$, and thus $\theta(M_1) = N_1$.

At this stage, one have already seen that $q = z'$ is a central projection in N. Repeating some of the above arguments with q_1 replaced by q, one can see that

$$\theta(z)^*\theta(d^*d) = \theta(z)\theta(d^*d)^* = q.$$

Similarly, the map $\pi_2 : M_2 \to N_2$ sending a to $\theta(a)\theta(d^*d)^*$ is a *-isomorphism. On the other hand, it follows from Proposition 1.2 that $\theta(1 - z)$ is invertible in $N_1 = (1 - q)N$ and there is a $*$-isomorphism $\pi_1 : M_1 \to N_1$ such that $\theta(a) = \pi_1(a)\theta(1 - z), \forall a \in M_1$. In conclusion, $\theta(1)$ is invertible in N and the map $\pi : M \to N$ defined by $\pi(a) = \theta(a)\theta(1)^{-1}$ is a $*$-isomorphism. □

The following two results supplement Propositions 1.1 and 1.2, and Theorem 1.3(A,B). Other similar variances of Theorem 1.3 can also be derived easily.

Proposition 2.2 *Let M be a W*-algebras containing no finite type I summand. Let N be a unital algebra. Let $\theta : M \to N$ be a linear map satisfying the condition:*

$$ab = 0 \text{ in } M \implies \theta(a)\theta(b) = 0 \text{ in } N. \tag{2.8}$$

Consider the following conditions. We have (1) $\implies$ (2) $\implies$ (3).

(1) *θ is surjective.*
(2) *$\theta(1)$ is a central invertible element in N.*
(3) *There exists an algebra homomorphism π from M into N such that*

$$\theta(a) = \theta(1)\pi(a) = \pi(a)\theta(1), \quad \forall a \in M.$$

Proof Using some arguments in the proof for the Case (A) of Theorem 1.3, we will establish

$$\theta(1)\theta(ab) = \theta(a)\theta(b), \quad \forall a, b \in M. \tag{2.9}$$

If θ is surjective then we will also see that $\theta(1)$ is a central invertible element in N.

Now, suppose $\theta(1)$ is central and invertible. Define $\pi : M \to N$ by

$$\pi(a) = \theta(1)^{-1}\theta(a), \quad \forall a \in M.$$

It follows from (2.9) that π is an algebra homomorphism. □

Proposition 2.3 *Let M, N be two W*-algebras. Suppose M contains no finite type I summand. Let $\theta : M \to N$ be a linear map satisfying the condition:*

$$a^*b = 0 \text{ in } M \quad \implies \quad \theta(a)^*\theta(b) = 0 \text{ in } N. \tag{2.10}$$

Consider the following conditions. We have (1) $\implies$ (2) $\implies$ (3).

(1) *θ is bijective, and the reverse implication in (2.10) also holds.*
(2) *$\theta(1)$ is invertible.*
(3) *There exists a $*$-homomorphism π from M into N such that*

$$\theta(a) = \pi(a)\theta(1), \quad \forall a \in M.$$

Proof Arguing as in the proof for the Case (B) of Theorem 1.3, we have

$$\theta(1)^*\theta(a^*b) = \theta(a)^*\theta(b), \quad \forall a, b \in M.$$

Suppose first that θ is bijective and (2.10) is satisfied in two directions. Using again the proof for the Case (B) of Theorem 1.3, without assuming that N has no finite type I summand though, we see that there is a d in M such that $\theta(d) = 1$ in N and $\theta(1)^{-1} = \theta(d^*d)^*$ exists in N.

Now, we assume that $\theta(1)$ is invertible. Define a bijective linear map $\pi : M \to N$ by $\pi(a) = \theta(a)\theta(1)^{-1}$. It is then easy to see that π is a $*$-homomorphism. □

There is yet another disjointness structure attracting attention from people. A linear map $\theta : A \to B$ between two C*-algebras is said to be *doubly orthogonality preserving* if $\theta(a)^*\theta(b) = \theta(a)\theta(b)^* = 0$ in B whenever $a^*b = ab^* = 0$ in A. In [30], it is shown that every bounded linear doubly orthogonality preserver θ between C*-algebra preserves the triple products $\{a, b, c\} := ab^*c + cb^*a$ whenever $\theta^{**}(1)$ is a partial isometry. It is further investigated in [5,6] to extend this concept to JB*-algebras and JB*-triples. In [7], the following theorem is proved. We remark that it might be possible to get an alternative proof by applying similar arguments as in the proof of Theorem 1.3 above and [27, Theorem 10], which states that two CCR C*-algebras with Hausdorff spectrum are isomorphic as JB*-algebras if and only if they carries equivalent linear and doubly orthogonality structures.

Theorem 2.4 (Burgos, Garcès and Peralta [7]) *Every linear surjection between W*-algebras preserving doubly orthogonality in two directions is automatically continuous. Consequently, two W*-algebras are isomorphic as JB*-triples if and only if they carry equivalent linear and doubly orthogonality structures.*

Finally, we show that the linear and the left (or right) ideal structures of a W*-algebra M also completely determine M. The following result supplements those in [10,11,20,23,26].

Theorem 2.5 *Let $\theta : M \to N$ be a linear bijection between W*-algebras. Suppose that both θ and θ^{-1} send left (resp. right) ideals to left (resp. right) ideals. Then $\theta(1)$ is invertible in N and $\pi(\cdot) = \theta(\cdot)\theta(1)^{-1}$ is an algebra isomorphism from M onto N.*

Consequently, two W-algebras are *-isomorphic if and only if they carries equivalent linear and left (resp. right) ideal structures.*

Proof We assume θ preserves left ideals in two directions, and the case for linear right ideal preserving maps is similar.

Observe that an element in M is not left invertible exactly when it is contained in a proper left ideal of M. In other words, the set of all left invertible elements in M is the complement to the union of all proper left ideals of M. As θ preserves left ideals in two directions, it preserves left invertible elements in two directions as well. In particular, $x = \theta^{-1}(1)$ has a left inverse y in M. Thus,

$$M = Myx \subseteq Mx \subseteq M. \tag{2.11}$$

Let $\pi : M \to N$ be defined by $\pi(a) = \theta(ax)$. It follows from (2.11) that π is a linear surjection such that $\pi(1) = 1$. For any proper left ideal J in N, we see that $I = \theta^{-1}(J)$ is a proper left ideal in M, and thus $\pi^{-1}(J) = \{a \in M : ax \in I\}$ is also a proper left ideal in M. We claim that the unital linear surjection π is left spectrum compressing, and thus does not increase the spectral radius. Indeed, suppose λ is in the left spectrum of $\pi(a)$, i.e., $\pi(a - \lambda)$ is not left invertible in B. Then $B\pi(a - \lambda)$ is a proper left ideal of B, and hence, $\pi^{-1}(B\pi(a - \lambda))$ is a proper left ideal of A. In particular, $a - \lambda$ is not left invertible in A, i.e., λ is in the left spectrum of a. By [3, Theorem 5.5.2], we see that π is bounded, and by [10, Lemma 2], π sends idempotent elements to idempotent elements.

Note that two idempotents e_1, e_2 are orthogonal, i.e., $e_1e_2 = e_2e_1 = 0$ exactly when e_1+e_2 is an idempotent. It follows that π sends orthogonal idempotents to orthogonal idempotents. By spectral theory, every self-adjoint element a in M can be approximated in norm by finite linear sums of orthogonal projections. Accordingly, $\pi(a)$ can be approximated in norm by finite linear sums of orthogonal idempotents. Taking squares, we see that $\pi(a^2) = \pi(a)^2$ for all self-adjoint, and thus all, elements a in M. As a consequence, π is a surjective Jordan homomorphism. The kernel $I = \pi^{-1}(0)$ of π is a norm closed Jordan ideal, and thus a two-sided ideal by [9], of M. Let p be the central projection in M such that the weak* closure of I in Mp. It follows from $\pi(I) = \theta(Ix) = 0$ that $Ix = 0$, and thus $Mpx = 0$, or $xp = px = 0$. As a result, $p = yxp = 0$. Therefore, π is a Jordan isomorphism.

By a result of Brešar ([4, Lemma 2.1 and Corollary 5.4]), there are central projections z in M and z' in N such that $\pi\mid_{zM}$ gives rise to an algebra isomorphism from zM onto $z'N$, and $\pi\mid_{(1-z)M}$ gives rise to an anti-algebra isomorphism from $(1-z)M$ onto $(1-z')N$. We verify that $(1-z')N$ is abelian, and thus π is an algebra isomorphism. To this end, let q be a projection in $(1-z')N$. Since $\pi\mid_{(1-z)M}$ gives rise to an anti-algebra isomorphism from $(1-z)M$ onto $(1-z')N$, the pre-image $\pi^{-1}(qN)$ of the right ideal qN is a left ideal in M. However, π sends left ideals to left ideals. Therefore, $qN = \pi(\pi^{-1}(qN))$ is also a left, and thus a two-sided, ideal in N. This forces q to be a central projection. Now we see that every projection in $(1-z')N$ is central. By spectral theory, every self-adjoint element in $(1-z')N$ is central, and thus $(1-z')N$ is abelian, as asserted.

At this stage we have proved that π is a continuous algebra isomorphism from M onto N. By a result of Gardner [15], we see that the W*-algebras M, N are *-isomorphic. Finally, we check $\pi(\cdot) = \theta(\cdot)\theta(1)^{-1}$. Observe that if $a \in M$ such that $ax = 0$ then $\pi(a) = \theta(ax) = 0$ forces $a = 0$. Now, the right multiplication $R_x : M \to M$ defined by $R_x(a) = ax$ is a bijective bounded linear operator on M with a right inverse R_y, the right multiplication by y. By the open mapping theorem, R_y is the inverse of R_x. Since M is unital, we have $xy = R_yR_x(1) = 1$. In other words, $x = y^{-1}$, and thus $\pi(x) = \pi(y)^{-1} = \theta(yx)^{-1} = \theta(1)^{-1}$. It follows $\theta(a)\theta(1)^{-1} = \theta(ayx)\theta(1)^{-1} = \pi(ay)\pi(x) = \pi(a)\pi(y)\pi(x) = \pi(a)$ for all a in M. This completes the proof. □

Note that a bijective linear map θ sending left ideals to left ideals might not send orthogonal ones to orthogonal ones. In other words, if p is a projection in M then the images of Mp and $M(1-p)$ are left ideals $L_1 = \theta(Mp)$, $L_2 = \theta(M(1-p))$ of N such that $L_1 \cap L_2 = \{0\}$ and $N = L_1 + L_2$, but $L_1L_2^*$ might not be zero. For example, consider the right multiplication $\theta(a) = ax$ of M by a non-central invertible element x in M. Then θ does not send orthogonal left ideals to orthogonal left ideals, and hence it is not domain orthogonality preserving. From this we see that Theorem 2.5 is not a direct consequence of Theorem 1.3.

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